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On the minimal distance of a polynomial code

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For a polynomial $f(x) \in \mathbb{Z}_2[x]$ it is natural to consider the near-ring code generated by the polynomials $f \circ x, f \circ x^2, \dots, f \circ x^k$ as a vectorspace. It is a 19 year old conjecture of Günter Pilz that for the polynomial $f(x) = x^n + x^{n-1} + \dots + x$ the minimal distance of this code is n.

The conjecture is equivalent to the following purely number theoretical problem. Let $\underline{m} = \{1, 2, \dots, m\}$ and $A \subset \mathbb{N}$ be an arbitrary finite subset of $\mathbb{N}$. Show that the number of products that occur odd many times in $\underline{n} \cdot A$ is at least n. Pilz also formulated the conjecture for the special case when $A = \underline{k}$. We show that for $A = \underline{k}$ the conjecture holds and that the minimal distance of the code is at least $n/(\log n)^{0.223}$.

While proving the case $A = \underline{k}$ we use different number theoretical methods depending on the size of k (respect to n). Furthermore, we apply several estimates on the distribution of primes.

Keywords: near-ring code, minimal distance, prime

1 Introduction

For two finite subsets of the positive integers, A and B let $A * B = \{ab \mid a \in A,\ b \in B$ and ab occurs odd many times in $A \cdot B\}$. In other words, if $A = \{a_1, \dots, a_k\}$, then $A * B = a_1 B \Delta \cdots \Delta a_k B$, where Δ denotes the symmetric difference. For a positive integer m let $\underline{m} = \{1, 2, \dots, m\}$.

Conjecture 1 *If n, k are positive integers, then $|\underline{n} * \underline{k}| \geq n$.*

For an arbitrary finite subset $A \subset \mathbb{N}$ it was proved that $|\underline{m} * A| \geq \pi(m) + 1$, where $\pi(x)$ is the prime counting function, and the following conjecture was formulated (Pilz (1992)):

Conjecture 2 *Let n be a positive integer and $K \subset \mathbb{N}$ be a finite set of integers. Then $|\underline{n} * K| \geq n$.*

These purely number theoretical problems originate in the theory of near-ring codes. A near-ring can be described as a ring, where the addition is not necessarily commutative and only one of the distributive laws is required. A typical example is the near-ring of polynomials, where the addition is the usual polynomial addition, and multiplication is the composition of the polynomials. In this example the addition

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is commutative and only the right distributive law holds. Near-rings play an important role in combinatorics: They are used to construct block designs that give rise to efficient error correcting codes. For more information on these codes see Eggetsberger (2011), Pilz (1983) and Pilz (2011). A special and very interesting near-ring code is defined in the following way: Let $f \in \mathbb{Z}_2[x]$ be a polynomial and $C(f,k)$ the code generated (as a subspace) by the polynomials $f = f \circ x, f \circ x^2, \ldots, f \circ x^k$. For $f = x + x^2 + \cdots + x^n$ a typical codeword is

$$\sum_{i \in K} f \circ x^i = \sum_{j \in K * \underline{n}} x^j,$$

where K is a finite subset of $\underline{k}$. As $C(f,k)$ is a linear code, its minimal distance is equal to the minimal weight of any nonzero codeword. Hence the minimum distance of $C(f,k)$ is the minimal value of $|\underline{n} * K|$ for some $K \subseteq \underline{k}$.

In this paper we settle Conjecture 1, and prove that for arbitrary $n \in \mathbb{N}$ and finite set $K \subset \mathbb{N}$ we have $|\underline{n} * K| \geq c \cdot \dfrac{n}{\log^{0.223} n}$ for some $c > 0$. Note that the minimal distance in $C(f,k)$ depends heavily on f. If, for example, we start with $f(x) = x + x^2 + x^4 + \cdots + x^{2^k}$, then $f \circ x + f \circ x^2 = x + x^{2^{k+1}}$, hence the minimal distance of the corresponding code is 2.

The natural logarithm will be denoted by $\log$ through the whole paper.

2 The general case

Let us denote by $g(n)$ the minimal size of the set $\underline{n} * K$, where K is a finite subset of the positive integers. In Pilz (1992) it is proved that $g(n) \geq \pi(n) + 1$. In this section we improve this lower bound and prove that $g(n) \geq c \cdot \dfrac{n}{\log^{0.223} n}$ for some $c > 0$. The proof is based on the following lemma:

Proposition 1 *For every positive integer* n

$$g(n) \geq \sum_{p \leq n} g\left(\lfloor n/p^{\alpha_p} \rfloor\right),$$

where the sum goes over the primes less than n*, and* α_p *is the largest integer such that* $p^{\alpha_p} \leq n$.

Proof: Let $p \leq n$ be a prime and $K_p \subseteq K$ the subset of K containing the elements that are divisible by the largest power of p occuring as divisor of some element of K (possibly $p^0 = 1$). Similarly, let $\underline{n}_p \subseteq \underline{n}$ be the set of elements of $\underline{n}$ that are divisible by p^{α_p}. Note that $\underline{n}_p$ is never empty. By the maximality of the exponents of p in K_p and $\underline{n}_p$, for any $a \in \underline{n}_p$, $b \in K_p$ and $c \in \underline{n}$, $d \in K$ if $ab = cd$, then $c \in \underline{n}_p$ and $d \in K_p$ hold. We prove that for $p < q \leq n$ different primes $\underline{n}_p \cdot K_p$ and $\underline{n}_q \cdot K_q$ are disjoint. If for some $a \in \underline{n}$ and $b \in K$ we have $ab \in \underline{n}_p \cdot K_p \cap \underline{n}_q \cdot K_q$, then $a \in \underline{n}_p \cap \underline{n}_q$. Thus $a = pqd'$, and $\bar{a} = p^2 d' < a$ is in $\underline{n}$. The exponent of p in $\bar{a}$ is larger than the one in a, which is contradiction. Hence, $\underline{n} * K$ contains the disjoint union of the sets $\underline{n}_p \cdot K_p$ for $p \leq n$, so

$$|\underline{n} * K| \geq \sum_{p \leq n} |\underline{n}_p * K_p|. \tag{1}$$

As $p^{\alpha_p} \le n < p^{\alpha_p+1}$, clearly, $\underline{n}_p = \{p^{\alpha_p}, 2p^{\alpha_p}, \dots, \lfloor n/p^{\alpha_p} \rfloor p^{\alpha_p}\}$, where $\lfloor n/p^{\alpha_p} \rfloor < p$. Dividing by p^{α_p}, we obtain that $|\underline{n}_p * K_p| = |\underline{\lfloor n/p^{\alpha_p} \rfloor} * K_p|$, thus by the definition of g we get

$$|\underline{n}_p * K_p| = |\underline{\lfloor n/p^{\alpha_p} \rfloor} * K_p| \ge g(\lfloor n/p^{\alpha_p} \rfloor).$$

By (1) we have

$$g(n) \ge \sum_{p \le n} g\left(\lfloor n/p^{\alpha_p} \rfloor\right),$$

and this is what we wanted to prove. □

Theorem 2 *For every* $\lambda > \lambda_0$ *there exists a* $c = c(\lambda) > 0$ *such that for every* $n > 1$

$$g(n) \ge c \cdot \frac{n}{\log^{\lambda} n},$$

where λ_0 *satisfies* $\int\limits_0^1 \left(\frac{2}{y}\right)^{\lambda_0} \frac{1}{2-y} dy = 1$. *Note that* $\lambda_0 \sim 0.2223...$

Proof: Fix $1 > \lambda > \lambda_0$. We claim that there exists some $c > 0$ such that the inequality

$$g(n) \ge c \cdot \frac{n}{\log^{\lambda} n} \tag{2}$$

holds for every $n > 1$. The proof is by induction on n. First we discuss the induction step. Assume that (2) holds for $n < m$. Now, we show that it holds for $n = m$, as well. The value of c will be chosen later. By Proposition 1 and the induction hypothesis:

$$\begin{aligned} g(m) \ge \sum_{\sqrt{m} < p \le m} g\left(\lfloor m/p \rfloor\right) \ge \sum_{\sqrt{m} < p < m/2} c \cdot \frac{\lfloor m/p \rfloor}{\log^{\lambda}(\lfloor m/p \rfloor)} \ge \\ \ge \sum_{\sqrt{m} < p < m/2} c \cdot \frac{\lfloor m/p \rfloor}{\log^{\lambda}(\lfloor m/p \rfloor)} \ge \sum_{\sqrt{m} < p < m/2} c \cdot \frac{m/p - 1}{\log^{\lambda}(\lfloor m/p \rfloor)} = \\ = \sum_{\sqrt{m} < p < m/2} c \cdot \frac{m/p}{\log^{\lambda}(\lfloor m/p \rfloor)} - \sum_{\sqrt{m} < p < m/2} c \cdot \frac{1}{\log^{\lambda}(\lfloor m/p \rfloor)}. \end{aligned} \tag{3}$$

In Rosser and Schoenfeld (1962) it is proved that $\pi(m) < \frac{1.25506m}{\log m}$ for every $m > 1$, hence $\pi(m/2) - \pi(\sqrt{m}) \le \pi(m) < 1.5 \cdot \dfrac{m}{\log m}$. For the second term of the last line of (3) we obtain:

$$\sum_{\sqrt{m} < p < m/2} c \cdot \frac{1}{\log^{\lambda}(\lfloor m/p \rfloor)} \le \sum_{\sqrt{m} < p < m/2} c \cdot \frac{1}{(\log 2)^{\lambda}} \le 1.5 \cdot \frac{m}{\log m} \cdot \frac{c}{\log 2} = o\left(\frac{m}{\log^{\lambda} m}\right), \tag{4}$$

since $\lambda < 1$.

Now we estimate the main term. By Mertens' theorem, there exists a constant M such that $\sum_{p\le x}\frac{1}{p} = \log\log x + M + o(1)$. Hence, for every $\varepsilon > 0$ there exists $B = B(\varepsilon)$ such that for $B \le a \le b$

$$\left|\sum_{a<p<b}\frac{1}{p} - \log\log b + \log\log a\right| < \varepsilon \tag{5}$$

holds. For $m > 2^{2K}$ we have $m^{\frac{1}{2}+\frac{K-1}{2K}} < m/2$. Applying (5) to the interval $I_h = (m^{\frac{1}{2}+\frac{h-1}{2K}}, m^{\frac{1}{2}+\frac{h}{2K}}]$, where h is an integer satisfying $1 \le h \le K-1$ we obtain that

$$\sum_{p\in I_h}\frac{1}{p} > \log\frac{K+h}{K+h-1} - \varepsilon. \tag{6}$$

If $p \in I_h$, then $\log^\lambda(m/p) \le \log^\lambda(m)(\frac{K-h+1}{2K})^\lambda$. Substituting into the main term of the last line of (3), omitting the integer parts and rearranging we get that

$$\begin{aligned}\sum_{\sqrt{m}<p<m/2} c\cdot\frac{m/p}{\log^\lambda(\lfloor m/p\rfloor)} &\ge cm\sum_{\sqrt{m}<p<m/2}\frac{1/p}{\log^\lambda(m/p)} \ge \\ &\ge \frac{cm}{\log^\lambda m}\sum_{h=1}^{K-1}\sum_{p\in I_h}\left(\frac{2K}{K-h+1}\right)^\lambda\cdot\frac{1}{p} \ge \\ &\ge \frac{cm}{\log^\lambda m}\left(\sum_{h=1}^{K-1}\left(\frac{2K}{K-h+1}\right)^\lambda\log\frac{K+h}{K+h-1} - \varepsilon\sum_{h=1}^{K-1}\left(\frac{2K}{K-h+1}\right)^\lambda\right).\end{aligned} \tag{7}$$

Now we show that there exists some K such that

$$S_K = \sum_{h=1}^{K-1}\left(\frac{2K}{K-h+1}\right)^\lambda\log\frac{K+h}{K+h-1} > 1. \tag{8}$$

Let $f_K(y) = \left(\frac{2}{y}\right)^\lambda K\cdot\log\left(1+\frac{1}{K(2-y)}\right)$ and $f(y) = \left(\frac{2}{y}\right)^\lambda\cdot\frac{1}{2-y}$. The sequence of functions f_K converges to f. Then

$$S_K = \frac{f_K(\frac{1}{K}) + f_K(\frac{2}{K}) + \cdots + f_K(\frac{K}{K})}{K} - \frac{f_K(\frac{1}{K})}{K}.$$

Let

$$T_K = \frac{f(\frac{1}{K}) + f(\frac{2}{K}) + \cdots + f(\frac{K}{K})}{K}.$$

As $1 > \lambda > \lambda_0$, the Riemann-sum T_k converges to $\int_0^1 f > 1$. As $f_K(\frac{1}{K})/K$ converges to 0, it is easy to see that $S_K - T_K$ converges to 0. Hence we can fix a K such that $S_K > 1$. Now, we can choose some

$\varepsilon > 0$ such that

$$\eta = \sum_{h=1}^{K-1} \left(\frac{2K}{K-h+1} \right)^{\lambda} \log \frac{K+h}{K+h-1} - 1 - \varepsilon \sum_{h=1}^{K-1} \left(\frac{2K}{K-h+1} \right)^{\lambda} > 0.$$

According to (4) there exists some R such that if $R < m$, then

$$\sum_{\sqrt{m}<p<m/2} c \cdot \frac{1}{\log^{\lambda}(\lfloor m/p \rfloor)} \leq \eta \cdot c \cdot \frac{m}{\log^{\lambda} m}.$$

By (3) and (7) we obtain that $g(m) \geq c \cdot \dfrac{m}{\log^{\lambda} m}$ holds. If we choose $c > 0$ such that (2) holds for $n \leq \max(2^{2K}, B^2(\varepsilon), R)$, then (3) is gained. □

3 The case $K = \underline{k}$

In this section we prove Conjecture 1. We distinguish cases according to how large is k according to n. The conjecture is true for $k \leq 8$. (Pilz (1992))

Case 1: $9 \leq k \leq 1.34 \cdot \log n$

We show that in this case the number of elements that occur exactly once in the product $\underline{n} \cdot \underline{k}$ is at least n. We shall need the following two observations.

Lemma 3 *Let $n/2 < a \leq n$ and $b \in \underline{k}$ such that a is relatively prime to every number less than k. Then ab occurs once in $\underline{n} \cdot \underline{k}$.*

Proof: Let us assume that $a_1, a_2 \in \underline{n}$ and $b_1, b_2 \in \underline{k}$ satisfy the conditions of the lemma, and $a_1 b_1 = a_2 b_2$. Now, $a_1 | a_2 b_2$ and a_1 and b_2 are relatively prime, hence $a_1 | a_2$. As $a_1 > n/2$ we have $2a_1 > n \geq a_2$, thus $a_1 = a_2$, which implies $b_1 = b_2$. □

Lemma 4 *If $k \geq 14$, then* $\displaystyle\prod_{p \leq k} \left(1 - \frac{1}{p}\right) \geq \frac{0.5}{\log k}$.

Proof: In Rosser and Schoenfeld (1962) it is shown that for $k > 1$

$$\frac{e^{-\gamma}}{\log k} \left(1 - \frac{1}{\log^2 k}\right) \leq \prod_{p \leq k} \left(1 - \frac{1}{p}\right),$$

where γ is the Euler constant. For $k > 21$ by using the monotonicity of the logarithm function and $e^{-\gamma} > 0.56$ we get that

$$\frac{e^{-\gamma}}{\log k} \left(1 - \frac{1}{\log^2 k}\right) \geq \frac{0.56}{\log k} \left(1 - \frac{1}{\log^2 22}\right) > \frac{0.5}{\log k}.$$

For $14 \leq k \leq 21$ it is enough to check the statement when $k = 14$, 17 and 19. For these numbers the values of $(\log k) \cdot \prod_{p \leq k} \left(1 - \frac{1}{p}\right)$ are 0.506, 0.511 and 0.503, respectively, hence the statement holds. $\square$

Proposition 5 *Let* $9 \leq k \leq 1.34 \cdot \log n$. *Then* $|\underline{n} * \underline{k}| \geq n$.

Proof: We show that there are at least n products satisfying the conditions of Lemma 3. For this we need to estimate the number of integers between $n/2$ and n that are not divisible by a prime less than k. This number will be denoted by D. By the inclusion-exclusion principle

$$D = n - \lfloor n/2 \rfloor + \sum_{h=1}^{r} (-1)^h \sum_{1 \leq i_1 < \ldots < i_h \leq r} \left(\left\lfloor \frac{n}{p_{i_1} \ldots p_{i_h}} \right\rfloor - \left\lfloor \frac{n/2}{p_{i_1} \ldots p_{i_h}} \right\rfloor \right), \tag{9}$$

where $\pi(k) = r$ and $p_1, \ldots, p_r$ are the primes up to k. Applying $x - 1 < \lfloor x \rfloor \leq x$ to all 2^{r+1} terms of the right side we get that

$$D \geq n - n/2 + \sum_{h=1}^{r} (-1)^h \sum_{1 \leq i_1 < \ldots < i_h \leq r} \left(\frac{n}{p_{i_1} \ldots p_{i_h}} - \frac{n/2}{p_{i_1} \ldots p_{i_h}} \right) - 2^r =$$

$$= \frac{n}{2} \prod_{p \leq k} \left(1 - \frac{1}{p}\right) - 2^r. \tag{10}$$

If $k \geq 14$, Lemma 4 applies, and

$$D \geq \frac{n}{2} \prod_{p \leq k} \left(1 - \frac{1}{p}\right) - 2^r \geq \frac{0.25n}{\log k} - 2^r$$

As $k \leq 1.34 \log n$, for $k \geq 14$ we have the estimation

$$2^r = 2^{\pi(k)} \leq 2^{k/2} \leq \frac{1}{100 \log k} \cdot e^{\frac{k}{1.34}} \leq \frac{n}{100 \log k}.$$

Hence, $D \geq \dfrac{0.24n}{\log k}$. Using Lemma 3 we obtain $|\underline{n} * \underline{k}| \geq Dk$. The function $x/\log x$ is monotone increasing on $[1, \infty)$, thus

$$|\underline{n} * \underline{k}| \geq Dk \geq \frac{0.24k}{\log k} n \geq \frac{0.24 \cdot 14}{\log 14} n > n.$$

For $9 \leq k \leq 13$ we have

$$|\underline{n} * \underline{k}| \geq Dk \geq \left(\frac{n}{2} \prod_{p \leq k} \left(1 - \frac{1}{p}\right) - 2^{\pi(k)} \right) k.$$

For $10 \leq k \leq 13$ it is obtained by calculation that the right hand side is greater than n if $n \geq e^{k/1.34}$. For $k = 9$ the inequality holds if $n > 5040$. By brute force the statement can be checked for $k = 9$ and $n \leq 5040$. Thus we obtained $|\underline{n} * \underline{k}| > n$. $\square$

Case 2:
$$1.34 \cdot \log n \le k \le n - \frac{0.22 \cdot n}{\log n} \text{ and } n \ge 1410.$$

Let $k_1 = \max(k, n/7)$ and $k_1 < p \le n$ a prime. As $k < p$, the set of elements of $\underline{n} * \underline{k}$, which are divisible by p is $\{p, 2p, \ldots, \lfloor n/p \rfloor p\} * \underline{k}$. This set has the same cardinality as the set $\underline{\lfloor n/p \rfloor} * \underline{k}$. Now, $\lfloor n/p \rfloor \le 6$, hence $|\underline{\lfloor n/p \rfloor} * \underline{k}| \ge k$. It is easy to see that for $p > q > n/7$ an element of $\underline{n} * \underline{k}$ cannot be divisible by both p and q. Hence, $|\underline{n} * \underline{k}| \ge (\pi(n) - \pi(k_1))k$.

At first, suppose that $k \le n/7$. By a theorem of Dusart Dusart (1999) for $x \ge 17$
$$\frac{x}{\log x} \le \pi(x) \le \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right)$$
holds. Hence, $\pi(n) - \pi(n/7) \ge 0.749 \cdot \dfrac{n}{\log n}$ for $n \ge 1410$. As $1.34 \cdot \log n \le k$, we have
$$|\underline{n} * \underline{k}| \ge 1.34 \cdot 0.749 \cdot n > n.$$
Secondly, let us consider the case when $n/7 < k \le n/2$. As $\pi(n) - \pi(n/2) \ge 7$,
$$|\underline{n} * \underline{k}| \ge (\pi(n) - \pi(k_1))k > 7 \cdot n/7 = n.$$
Finally, let $n/2 < k < n - \dfrac{0.22 \cdot n}{\log n}$. Then by the estimates in Dusart (1999) and Robin (1983) there are at least two primes between k and n if $n > 90000$. It can be checked that this also holds for $n > 1410$. Thus
$$|\underline{n} * \underline{k}| \ge (\pi(n) - \pi(k))k \ge 2(n/2) = n.$$

We continue with the case when k is "large", that is, $n - \frac{0.4 \cdot n}{\log n + 1.02} \le k$. By calculation we have $n - \frac{0.4 \cdot n}{\log n + 1.02} \le n - \frac{0.22 \cdot n}{\log n}$ for $n \ge 4$.

Case 3:
$$n - \frac{0.4 \cdot n}{\log n + 1,02} \le k \le n \text{ and } n > 5000.$$

If $k = n$, then $\underline{k} \cdot \underline{n} = \{1, \ldots, n\} \cdot \{1, \ldots, n\}$. If $a \ne b$, then pairing ab with ba only the products of the form $a \cdot a$ are left, hence $\underline{n} * \underline{k} = \{1^2, 2^2, \ldots, n^2\}$. Thus
$$|\underline{n} * \underline{k}| = n.$$
Assume now that $k < n$. Then
$$|\underline{n} * \underline{k}| = |(\underline{k} * \underline{k})\Delta((\underline{n} \setminus \underline{k}) * \underline{k})| = |\underline{k} * \underline{k}| + |(\underline{n} \setminus \underline{k}) * \underline{k}| - 2|(\underline{k} * \underline{k}) \cap ((\underline{n} \setminus \underline{k}) * \underline{k})|. \tag{11}$$
For the first term on the right side of (11) we have
$$|\underline{k} * \underline{k}| = |\{1^2, 2^2, \ldots, k^2\}| = k. \tag{12}$$

Lemma 6 *For the second term of (11) we have*

$$|(\underline{n} \setminus \underline{k}) * \underline{k}| \geq 2k - n. \tag{13}$$

Proof: We use the following observation: If

$$i \leq \frac{k}{n-k} \quad \text{and} \quad k+1 \leq j \leq n,$$

then ij appears exactly once in $(\underline{n} \setminus \underline{k}) \cdot \underline{k}$, so $ij \in (\underline{n} \setminus \underline{k}) * \underline{k}$. Let us assume that $ij = i'j'$ such that $1 \leq i' \leq k$ and $k+1 \leq j' \leq n$. If $i = i'$, then $j = j'$. If $i' < i$, then $1 \leq i' \leq \frac{k}{n-k}$ and $k+1 \leq j' \leq n$. Now, changing the roles of (i,j) and (i',j') we may assume that $i < i'$. As $ij = i'j'$, we have $\frac{i}{i'} = \frac{j'}{j}$ and

$$\frac{i}{i'} \leq \frac{i}{i+1} \leq \frac{\frac{k}{n-k}}{\frac{k}{n-k}+1} = \frac{k}{n} < \frac{k+1}{n} \leq \frac{j'}{j},$$

which is a contradiction. For $(\underline{n} \setminus \underline{k}) * \underline{k}$ we obtain that

$$|(\underline{n} \setminus \underline{k}) * \underline{k}| \geq \left\lfloor \frac{k}{n-k} \right\rfloor (n-k) \geq \left(\frac{k}{n-k} - 1 \right)(n-k) = k - (n-k) = 2k - n. \tag{14}$$

□

Now, we focus on the third term of (11).

Lemma 7 *For the third second term of (11)*

$$|(\underline{k} * \underline{k}) \cap ((\underline{n} \setminus \underline{k}) * \underline{k})| \leq 0.431 \cdot k. \tag{15}$$

holds.

Proof: It is enough to show that among the numbers 1^2, 2^2, ..., k^2 at most $0.431k$ many has a divisor in the interval $[k+1, n]$. Let $k+1 \leq m \leq n$ and $m = a_m b_m^2$, where b_m^2 is the largest square divisor of m. Since a_m is squarefree, $m|i^2$ if and only if $a_m b_m | i$. Let S denote the following upper bound of the number of elements of the set $\{1^2, 2^2, \ldots, k^2\}$ which have a divisor in $[k+1, n]$:

$$S = \sum_{m=k+1}^{n} \left\lfloor \frac{k}{a_m b_m} \right\rfloor \leq \sum_{m=k+1}^{n} \frac{k}{a_m b_m} = k \sum_{m=k+1}^{n} \frac{b_m}{m}.$$

Recall that $m = a_m b_m^2$, where a_m is squarefree. Now, summing by $j = b_m \leq \sqrt{m}$:

$$S = k \sum_{j=1}^{\lfloor \sqrt{n} \rfloor} \sum_{\substack{j^2 | m, \\ k+1 \leq m \leq n, \\ |\mu(m/j^2)|=1}} \frac{j}{m} \leq k \sum_{j=1}^{\lfloor \sqrt{n} \rfloor} j \sum_{\substack{j^2 | m, \\ k+1 \leq m \leq n}} \frac{1}{m}.$$

Rewrite $S = k(S_1 + S_2)$, where

$$S_1 := \sum_{j=1}^{\lfloor\sqrt{n}/2\rfloor} j \sum_{\substack{j^2|m,\\ k+1\le m\le n}} \frac{1}{m} \quad \text{and} \quad S_2 := \sum_{j=\lfloor\sqrt{n}/2\rfloor+1}^{\lfloor\sqrt{n}\rfloor} j \sum_{\substack{j^2|m,\\ k+1\le m\le n}} \frac{1}{m}.$$

First, we give an upper bound for S_1.

Lemma 8

$$S_1 \le \left(\frac{\log n}{2} + 0.31\right)(\log n - \log k) + \frac{n+2\sqrt{n}}{8k}. \tag{16}$$

Proof: Let $r_j = \left\lceil \frac{k+1}{j^2} \right\rceil$ and $s_j = \left\lfloor \frac{n}{j^2} \right\rfloor$. Then

$$S_1 = \sum_{j=1}^{\lfloor\sqrt{n}/2\rfloor} j \sum_{l=r_j}^{s_j} \frac{1}{lj^2} = \sum_{j=1}^{\lfloor\sqrt{n}/2\rfloor} \frac{1}{j} \sum_{l=r_j}^{s_j} \frac{1}{l}. \tag{17}$$

The function $\frac{1}{x}$ is a nonnegative decreasing function on $(0,\infty)$, hence we can estimate the inside sum by

$$\sum_{l=r_j}^{s_j} \frac{1}{l} \le \int_{r_j}^{s_j} 1/x + \frac{1}{r_j} = \log s_j - \log r_j + \frac{1}{r_j}.$$

As $\frac{k}{j^2} \le r_j$ and $s_j \le \frac{n}{j^2}$ we have

$$\log s_j - \log r_j = \log \frac{s_j}{r_j} \le \log \frac{n/j^2}{k/j^2} = \log n - \log k.$$

Substituting into (17) we obtain

$$S_1 \le \sum_{j=1}^{\lfloor\sqrt{n}/2\rfloor} \frac{1}{j}\left(\log s_j - \log r_j + \frac{1}{r_j}\right) \le \sum_{j=1}^{\lfloor\sqrt{n}/2\rfloor} \frac{1}{j}\left(\log n - \log k + \frac{j^2}{k}\right). \tag{18}$$

Since

$$\sum_{j=1}^{\lfloor\sqrt{n}/2\rfloor} \frac{1}{j} \le \log\lfloor\sqrt{n}/2\rfloor + 1 \le \frac{\log n}{2} - \log 2 + 1 \le \frac{\log n}{2} + 0.31. \tag{19}$$

and

$$\sum_{j=1}^{\lfloor\sqrt{n}/2\rfloor} j = \frac{\lfloor\sqrt{n}/2\rfloor \cdot (\lfloor\sqrt{n}/2\rfloor + 1)}{2} \le \frac{n+2\sqrt{n}}{8}, \tag{20}$$

from the inequalities (18), (19), (20) we get (16). □

Now we give an upper bound for S_2.

Lemma 9

$$S_2 \le \left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)\cdot\frac{n-k}{2\sqrt{k}}\cdot\frac{\sqrt{n}}{k}+\frac{3\sqrt{n}}{k} < 1.15\cdot\frac{(n-k)\sqrt{n}}{k^{3/2}}+\frac{3\sqrt{n}}{k}. \tag{21}$$

Proof:

$$S_2 = \sum_{j=\lfloor\sqrt{n}/2\rfloor+1}^{\lfloor\sqrt{n}\rfloor}\ \sum_{\substack{j^2|m,\\ k+1\le m\le n}}\frac{j}{m} \tag{22}$$

In (22) for every j we have

$$n \ge j^2 \ge (\lfloor\sqrt{n}/2\rfloor+1)^2 > \frac{n}{4}.$$

Hence $m=j^2$ or $2j^2$ or $3j^2$. As $k<m\le n$, for $m=ij^2$ ($i=1,2,3$) we get

$$\sqrt{\frac{k}{i}} < j \le \sqrt{\frac{n}{i}} \quad \text{and} \quad \frac{j}{m}\le\frac{\sqrt{n}}{k}.$$

For fixed i, the number of j such that $m=ij^2$ is at most:

$$\left\lceil\frac{\sqrt{n}-\sqrt{k}}{\sqrt{i}}\right\rceil = \left\lceil\frac{1}{\sqrt{i}}\cdot\frac{n-k}{\sqrt{n}+\sqrt{k}}\right\rceil \le \frac{1}{\sqrt{i}}\cdot\frac{n-k}{2\sqrt{k}}+1,$$

thus

$$S_2 \le \left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)\cdot\frac{n-k}{2\sqrt{k}}\cdot\frac{\sqrt{n}}{k}+\frac{3\sqrt{n}}{k} < 1.15\cdot\frac{(n-k)\sqrt{n}}{k^{3/2}}+\frac{3\sqrt{n}}{k},$$

and this is what we wanted to show. □

Summarizing the results, from (16) and (21) we obtain:

$$\begin{aligned} S = k(S_1+S_2) \le& \\ \le k\Bigg\{&\left(\frac{\log n}{2}+0.31\right)(\log n-\log k)+\frac{n+2\sqrt{n}}{8k}+1.15\cdot\frac{(n-k)\sqrt{n}}{k^{3/2}}+\frac{3\sqrt{n}}{k}\Bigg\}. \end{aligned} \tag{23}$$

We assumed that $n-\frac{0.4\cdot n}{\log n+1.02}\le k$ and $n\ge 5000$. By using the inequality $e^{-x}<\frac{1}{1+x}$ we obtain that $ne^{-\frac{0.2}{\frac{\log n}{2}+0.31}} < n\cdot\frac{1}{1+\frac{0.2}{\frac{\log n}{2}+0.31}} = n-\frac{0.4\cdot n}{\log n+1.02}\le k$. As $n\ge 5000$, we have that $\frac{k}{n}>0.958$.

By easy calculation from these inequalities the following ones can be deduced:

$$\left(\frac{\log n}{2} + 0.31\right)(\log n - \log k) < 0.2, \tag{24}$$

$$\frac{n + 2\sqrt{n}}{8k} < 0.135, \tag{25}$$

$$1.15 \cdot \frac{(n-k)\sqrt{n}}{k^{3/2}} + \frac{3\sqrt{n}}{k} < 0.096. \tag{26}$$

Adding (24), (25) and (26) using (23) we arrive at:

$$S \leq k\,(0.2 + 0.135 + 0.096) = 0.431 \cdot k. \tag{27}$$

Then from inequalities (12), (13) and (15) in case $k/n > 0.958$ we get

$$|\underline{k} * \underline{n}| \geq k + 2k - n - 2S \geq 2.138 \cdot k - n > n,$$

thus we proved the statement in Case 3 as well. □

We proved the statement for all pairs n, k where $n \geq 5000$. Cases $k \leq n \leq 5000$ can be checked by brute force.

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