The Join of the Varieties of R-trivial and L-trivial Monoids via Combinatorics on Words

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The join of two varieties is the smallest variety containing both. In finite semigroup theory, the varieties of \mathcal{R} -trivial and \mathcal{L} -trivial monoids are two of the most prominent classes of finite monoids. Their join is known to be decidable due to a result of Almeida and Azevedo. In this paper, we give a new proof for Almeida and Azevedo's effective characterization of the join of \mathcal{R} -trivial and \mathcal{L} -trivial monoids. This characterization is a single identity of ω -terms using three variables.

Keywords: finite semigroup theory, join of pseudovarieties, Green's relations, combinatorics on words

1 Introduction

Green's relations \mathcal{R} and \mathcal{L} are a standard tool in the study of semigroups [5]. In the context of finite monoids, among other results, they have been used to give effective characterizations of language classes such as star-free languages [3, 11] and piecewise testable languages [6, 12]. A deterministic extension of piecewise testable languages yields the class of languages corresponding to \mathcal{R} -trivial monoids, and a codeterministic extension corresponds to \mathcal{L} -trivial monoids [4, 9].

Almeida and Azevedo gave an effective characterization for the least variety of finite monoids containing all \mathcal{R} -trivial and all \mathcal{L} -trivial monoids [2], *i.e.*, for the *join* of the two varieties. Their proof is based on sophisticated algebraic techniques, on Reiterman's Theorem [10], and on a combinatorial result of König [7]. In this paper, we give a new proof of Almeida and Azevedo's Theorem. The current proof was inspired by another proof of the authors [8], which in turn uses ideas of Klíma [6]. The main ingredient is a system of congruences which relies on simple combinatorics on words.

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2 Preliminaries

Let A be a finite alphabet. The set of finite words over A is denoted by A^* . It is the free monoid over A. The *empty word* is 1. The *content* of a word $u = a_1 \cdots a_n$ with $a_i \in A$ is $\alpha(u) = \{a_1, \dots, a_n\}$, and its *length* is |u| = n. The length of the empty word is 0. A word u is a *prefix* (respectively *suffix*) of v if there exists $x \in A^*$ such that ux = v (respectively xu = v); if $x \ne 1$, then u is a *proper* prefix.

For more details concerning the algebraic concepts introduced in the remainder of this section, we refer the reader to textbooks such as [1, 4, 9]. *Green's relations* \mathcal{R} and \mathcal{L} are important tools in the study of finite monoids. Let M be a finite monoid. We set $u \mathcal{R} v$ for $u, v \in M$ if uM = vM, and the latter condition is equivalent to the existence of $x, y \in M$ with u = vx and v = uy. Symmetrically, $u \mathcal{L} v$ if Mu = Mv. The monoid M is \mathcal{R} -trivial (respectively \mathcal{L} -trivial) if \mathcal{R} (respectively \mathcal{L}) is the identity relation on M. We write $u <_{\mathcal{R}} v$ if $uM \subseteq vM$, and we write $u <_{\mathcal{L}} v$ if $uM \subseteq vM$.

A variety of finite monoids is a class of monoids closed under finite direct products, submonoids, and quotients. A variety of finite monoids is often called a *pseudovariety* in order to distinguish from varieties in Birkhoff's sense. Since we do not need this distinction in the current paper, whenever we use the term *variety* we mean a variety of finite monoids. The *join* $V_1 \vee V_2$ of two varieties V_1 and V_2 is the smallest variety containing $V_1 \cup V_2$. A monoid M is in $V_1 \vee V_2$ if and only if there exist $M_1 \in V_1$ and $M_2 \in V_2$ such that M is a quotient of a submonoid of $M_1 \times M_2$. If M is a finite monoid, then there exists an integer $\omega_M \geqslant 1$ such that, for all $u \in M$, the element u^{ω_M} is idempotent. Moreover, the element u^{ω_M} is the unique idempotent generated by u. Usually, the monoid M is clear from the context and thus, we simply write ω instead of ω_M . This leads to the following definition. An ω -term over a finite alphabet X is either a word in X^* , or of the form t^ω for some ω -term t, or the concatenation t_1t_2 of two ω -terms t_1, t_2 . A homomorphism $\varphi: X^* \to M$ to a finite monoid M uniquely extends to ω -terms over X by setting $\varphi(t^\omega) = \varphi(t)^{\omega_M}$. Let u, v be two ω -terms over X. A finite monoid M satisfies the identity u = v if $\varphi(u) = \varphi(v)$ for all homomorphisms $\varphi: X^* \to M$. The class of finite monoids satisfying the identity u = v is denoted by [u = v]. For all ω -terms u, v, the class [u = v] forms a variety. We need the following three varieties in this paper:

$$\mathbf{R} = [(xy)^{\omega}x = (xy)^{\omega}],$$

$$\mathbf{L} = [x(zx)^{\omega} = (zx)^{\omega}],$$

$$\mathbf{W} = [(xy)^{\omega}x(zx)^{\omega} = (xy)^{\omega}(zx)^{\omega}].$$

A monoid is in \mathbf{R} if and only if it is \mathscr{R} -trivial. Symmetrically, a monoid is in \mathbf{L} if and only if it is \mathscr{L} -trivial. The aim of this paper is to give a new proof of Almeida and Azevedo's result $\mathbf{R} \vee \mathbf{L} = \mathbf{W}$. The inclusion $\mathbf{R} \vee \mathbf{L} \subseteq \mathbf{W}$ is trivial since $\mathbf{R} \cup \mathbf{L} \subseteq \mathbf{W}$ and \mathbf{W} is a variety.

3 Congruences

In this section, we introduce the main combinatorial tool for our proof. It is a family of congruences \equiv_n on A^* for some finite alphabet A such that $A^*/\equiv_n \in \mathbf{R} \vee \mathbf{L}$ for all integers $n \geqslant 0$, see Lemma 2 below. As a first step towards the definition of \equiv_n we need to introduce an asymmetric, weaker congruence $\equiv_n^{\mathscr{R}}$. Let $u, v \in A^*$. We let $u \equiv_0^{\mathscr{R}} v$ if $\alpha(u) = \alpha(v)$. For $n \geqslant 0$, we let $u \equiv_{n+1}^{\mathscr{R}} v$ if the following conditions hold:

- 1. $\alpha(u) = \alpha(v)$,
- 2. for all factorizations $u = u_1 a u_2$ and $v = v_1 a v_2$ with $a \in A \setminus (\alpha(u_1) \cup \alpha(v_1))$ we have $u_1 \equiv_n^{\Re} v_1$ and $u_2 \equiv_n^{\Re} v_2$, and

3. for all factorizations $u = u_1 a u_2$ and $v = v_1 a v_2$ with $a \in A \setminus (\alpha(u_2) \cup \alpha(v_2))$ we have $u_1 \equiv_n^{\mathscr{R}} v_1$.

By a straightforward verification we see that $\equiv_n^{\mathscr{R}}$ is an equivalence relation. The factorization $u_1 a u_2$ with $a \in A \setminus \alpha(u_1)$ is unique. Therefore, induction on n shows that the index of $\equiv_n^{\mathscr{R}}$ is finite. If $u \equiv_{n+1}^{\mathscr{R}} v$, then $u \equiv_n^{\mathscr{R}} v$. Moreover, if $u \equiv_n^{\mathscr{R}} v$ and $u \in A$, then $u \equiv_n^{\mathscr{R}} u$ and $u \in A$. Therefore, the relation $u \in A$ is a finite index congruence on $u \in A$.

Lemma 1 For every finite alphabet A and every integer $n \ge 0$ we have $A^*/\equiv_n^{\mathscr{R}} \in \mathbf{R}$.

Proof: It suffices to show $(xy)^{n+1}x \equiv_n^{\mathscr{R}} (xy)^{n+1}$ for all words $x,y \in A^*$. We note that for y=1 this yields $x^{n+2} \equiv_n^{\mathscr{R}} x^{n+1}$. The proof is by induction on n. For n=0, the claim is true since $\alpha(xyx) = \alpha(xy)$. Let now n>0. As before, $\alpha((xy)^{n+1}x) = \alpha((xy)^{n+1})$. Suppose $(xy)^{n+1}x = u_1au_2$ and $(xy)^{n+1} = v_1av_2$ for $a \in A \setminus (\alpha(u_1) \cup \alpha(v_1))$. Then $u_1 = v_1$ and both are proper prefixes of xy. Thus $u_2 = p(xy)^n x$ and $v_2 = p(xy)^n$ for some $p \in A^*$. By induction $(xy)^n x \equiv_{n-1}^{\mathscr{R}} (xy)^n$ and hence, $u_2 \equiv_n^{\mathscr{R}} v_2$.

Suppose now $(xy)^{n+1}x = u_1au_2$ and $(xy)^{n+1} = v_1av_2$ for $a \in A \setminus (\alpha(u_2) \cup \alpha(v_2))$. Then av_2 is a suffix of xy and au_2 is a suffix of yx. We can therefore write $v_1 = (xy)^n p'$ for some prefix p' of xy. Similarly, $u_1 = (xy)^k p$ for some $k \in \{n, n+1\}$ and some prefix p of xy, i.e., we have pq = xy for some $q \in A^*$. By induction, we have $(xy)^{n+1} \equiv_{n-1}^{\mathscr{R}} (xy)^n$ and thus $(xy)^{n+1} p \equiv_{n-1}^{\mathscr{R}} (xy)^n p$. We can therefore assume k = n. Without loss of generality, let $|p| \leq |p'|$, i.e., p' = ps for some $s \in A^*$. It follows

$$u_1 = (pq)^n p$$
 and $v_1 = (pq)^n ps$.

Since p' = ps is a prefix of xy = pq, the word s is a prefix of q. In particular, there exists $t \in A^*$ such that qp = st. This yields

$$u_1 = p(st)^n$$
 and $v_1 = p(st)^n s$.

By induction, $(st)^n \equiv_{n-1}^{\mathscr{R}} (st)^n s$ and thus $u_1 \equiv_{n-1}^{\mathscr{R}} v_1$. This shows $(xy)^{n+1} x \equiv_n^{\mathscr{R}} (xy)^{n+1}$ which concludes the proof.

There is a left-right symmetric congruence $\equiv_n^{\mathscr{L}}$ on A^* . It can be defined by setting $u \equiv_n^{\mathscr{L}} v$ if and only if $u^{\rho} \equiv_n^{\mathscr{R}} v^{\rho}$. Here, $u^{\rho} = a_n \cdots a_1$ is the *reversal* of the word $u = a_1 \cdots a_n$ with $a_i \in A$. It satisfies $A^*/\equiv_n^{\mathscr{L}} \in \mathbf{L}$ for every $n \geqslant 0$. We define $u \equiv_n v$ if and only if both $u \equiv_n^{\mathscr{R}} v$ and $u \equiv_n^{\mathscr{L}} v$. The following lemma puts together some properties of the finite index congruence \equiv_n .

Lemma 2 For every finite alphabet A and every integer $n \ge 0$ the following properties hold:

- 1. $A^*/\equiv_n \in \mathbf{R} \vee \mathbf{L}$.
- 2. If $u_1au_2 \equiv_{n+1} v_1av_2$ for $a \in A \setminus (\alpha(u_1) \cup \alpha(v_1))$, then $u_1 \equiv_n^{\mathscr{R}} v_1$ and $u_2 \equiv_n v_2$.
- 3. If $u_1 a u_2 \equiv_{n+1} v_1 a v_2$ for $a \in A \setminus (\alpha(u_2) \cup \alpha(v_2))$, then $u_1 \equiv_n v_1$ and $u_2 \equiv_n^{\mathscr{L}} v_2$.

Proof: "1": We have $A^*/\equiv_n \in \mathbf{R} \vee \mathbf{L}$ since it is a submonoid of $(A^*/\equiv_n^{\mathscr{R}}) \times (A^*/\equiv_n^{\mathscr{L}})$, and $A^*/\equiv_n^{\mathscr{R}} \in \mathbf{R}$ and $A^*/\equiv_n^{\mathscr{L}} \in \mathbf{L}$ by Lemma 1 and its left-right dual. The properties "2" and "3" trivially follow from the definition of \equiv_n .

4 An Equation for the Join

The goal of this section is to prove $\mathbf{W} \subseteq \mathbf{R} \vee \mathbf{L}$. By Lemma 2 it suffices to show that for every A-generated monoid $M \in \mathbf{W}$ there exists an integer $n \geqslant 0$ such that M is a quotient of A^*/\equiv_n . The outline of the proof is as follows. First, in Lemma 3, we give a substitution rule valid in \mathbf{W} . Then, in Lemma 5, we show that \equiv_n -equivalence allows a factorization satisfying the premise for applying this substitution rule; this relies on a property of \mathbf{W} shown in Lemma 4. Finally, in Theorem 6, all the ingredients are put together.

Lemma 3 Let $M \in \mathbf{W}$ and let $u, v, x \in M$. If $u \mathcal{R}$ ux and $v \mathcal{L}$ xv, then uxv = uv.

Proof: Since $u \mathcal{R} ux$ and $v \mathcal{L} xv$, there exist $y, z \in M$ with u = uxy and v = zxv. In particular, we have $u = u(xy)^{\omega}$ and $v = (zx)^{\omega}v$. By $M \in \mathbf{W}$ we conclude $uxv = u(xy)^{\omega}x(zx)^{\omega}v = u(xy)^{\omega}(zx)^{\omega}v = uv$.

We will apply the previous lemma as follows. Let $M \in \mathbf{W}$ and $u,v,s,t \in M$ such that $u \mathcal{R}$ us \mathcal{R} ut and $v \mathcal{L}$ sv \mathcal{L} tv. Then usv = utv since usv = uv and utv = uv by Lemma 3. The \mathcal{R} -equivalences and \mathcal{L} -equivalences for being able to apply this substitution rule are established in Lemma 5. Before, we give a simple property of \mathbf{W} . It is the link between Green's relations and the congruence \equiv_n .

Lemma 4 Let $M \in \mathbf{W}$ and let $u, v, a \in M$. If $u \mathcal{R} v \mathcal{R} va$, then $u \mathcal{R} ua$. If $u \mathcal{L} v \mathcal{L} av$, then $u \mathcal{L} au$.

Proof: Since $u \mathcal{R} v$ and $u \mathcal{R} va$, there exist $x, y \in M$ with v = ux and u = vay. Now, $u = uxay = u(xay)^{2\omega+1} = u(xay)^{\omega}x(ayx)^{\omega}ay = u(xay)^{\omega}(ayx)^{\omega}ay = u(ayx)^{\omega}ay \in uaM$ where the fourth equality uses $M \in \mathbf{W}$. This shows $u \in \mathbf{W}$ and thus $u \mathcal{R} ua$. The second implication is left-right symmetric.

The intuitive interpretation of the algebraic statement in Lemma 4 is the following: For $M \in \mathbf{W}$ it only depends on the element a and the \mathcal{R} -class of u whether u \mathcal{R} ua or not (but not on the element u itself). The statement for \mathcal{L} -classes is analogous.

Lemma 5 Let $M \in \mathbf{W}$ and let $\varphi : A^* \to M$ be a homomorphism. If $u \equiv_n v$ for $n \geqslant 2|M|$, then there exist factorizations $u = a_1s_1 \cdots a_{\ell-1}s_{\ell-1}a_{\ell}$ and $v = a_1t_1 \cdots a_{\ell-1}t_{\ell-1}a_{\ell}$ with $a_i \in A$ and $s_i, t_i \in A^*$ and with $\ell \leqslant 2|M|$ such that for all $i \in \{1, \dots, \ell-1\}$ we have:

$$\varphi(a_1s_1\cdots a_{i-1}s_{i-1}a_i) \mathcal{R} \varphi(a_1s_1\cdots a_is_i) \mathcal{R} \varphi(a_1s_1\cdots a_{i-1}s_{i-1}a_it_i),$$

$$\varphi(a_{i+1}t_{i+1}\cdots a_{\ell-1}t_{\ell-1}a_{\ell}) \mathcal{L} \varphi(t_ia_{i+1}\cdots t_{\ell-1}a_{\ell}) \mathcal{L} \varphi(s_ia_{i+1}t_{i+1}\cdots a_{\ell-1}t_{\ell-1}a_{\ell}).$$

Proof: To simplify notation, for some relation \mathscr{G} on M we write $u \mathscr{G} v$ for words $u, v \in A^*$ if $\varphi(u) \mathscr{G} \varphi(v)$. Consider the \mathscr{R} -factorization of u, *i.e.*, let $u = b_1 u_1 \cdots b_k u_k$ with $b_i \in A$ such that

$$b_1u_1\cdots b_i \mathcal{R} \ b_1u_1\cdots b_iu_i$$
 for all $i\in\{1,\ldots,k\}$,
 $b_1u_1\cdots b_iu_i>_{\mathcal{R}} b_1u_1\cdots b_iu_ib_{i+1}$ for all $i\in\{1,\ldots,k-1\}$.

Similarly, let $v = v_1 c_1 \cdots v_{k'} c_{k'}$ be the \mathcal{L} -factorization of v, *i.e.*, we have $c_i \in A$ and

$$c_{i} \cdots v_{k'} c_{k'} \mathcal{L} v_{i} c_{i} \cdots v_{k'} c_{k'} \qquad \text{for all } i \in \{1, \dots, k'\},$$

$$v_{i} c_{i} \cdots v_{k'} c_{k'} >_{\mathcal{L}} c_{i-1} v_{i} c_{i} \cdots v_{k'} c_{k'} \qquad \text{for all } i \in \{2, \dots, k'\}.$$

We have $k, k' \leq |M|$ because neither the number of \mathscr{R} -classes nor the number of \mathscr{L} -classes can exceed |M|. By Lemma 4, we have $b_i \notin \alpha(u_{i-1})$ for all $i \in \{2, \dots, k\}$ and $c_i \notin \alpha(v_{i+1})$ for all $i \in \{1, \dots, k'-1\}$. We use these properties to convert the \mathscr{R} -factorization of u to v and to convert the \mathscr{L} -factorization of v to u: Let $v = b_1 v_1' \cdots b_k v_k'$ such that $b_i \notin \alpha(v_{i-1}')$, and let $u = u_1' c_1 \cdots u_{k'}' c_{k'}$ with $c_i \notin \alpha(u_{i+1}')$. These factorizations exist because $u \equiv_n v$; in particular, by Lemma 2,

$$u_{i}b_{i+1}u_{i+1}\cdots b_{k}u_{k} \equiv_{n-i} v'_{i}b_{i+1}v'_{i+1}\cdots b_{k}v'_{k}$$

$$v_{1}c_{1}\cdots v_{i-1}c_{i-1}v_{i} \equiv_{n-k'-1+i} u'_{1}c_{1}\cdots u'_{i-1}c_{i-1}u'_{i}$$

for all $i \in \{1, ..., k\}$ and $j \in \{1, ..., k'\}$. Moreover, we see that $\alpha(u_i) = \alpha(v_i')$ and $\alpha(v_i) = \alpha(u_i')$.

We now show that the relative positions of the b_i 's and c_j 's in the above factorizations are the same in u and v. Let p be the position of b_i in the \mathcal{R} -factorization of u and let q be the position of c_j in the above factorization of u. Similarly, let p' be the position of b_i in v and let q' be the position of c_j in v. First, suppose p < q. Let

$$u = b_1 u_1 \cdots b_{i-1} u_{i-1} b_i u' c_j u'_{j+1} c_{j+1} \cdots u'_{k'} c_{k'}.$$

By an *i*-fold application of property "2" in Lemma 2 with $a \in \{b_1, \ldots, b_i\}$ (which is possible for u) we obtain $v = b_1 v'_1 \cdots b_{i-1} v'_{i-1} b_i z$ with $z \equiv_{n-i} u' c_j u'_{j+1} c_{j+1} \cdots u'_{k'} c_{k'}$. By a (k'+1-j)-fold application of property "3" in Lemma 2 with $a \in \{c_{k'}, \ldots, c_j\}$ (which is possible for the word $u' c_j u'_{j+1} c_{j+1} \cdots u'_{k'} c_{k'}$) we obtain $z = v' c_j v_{j+1} c_{j+1} \cdots v_{k'} c_{k'}$. Thus

$$v = b_1 v'_1 \cdots b_{i-1} v'_{i-1} b_i v' c_i v_{i+1} c_{i+1} \cdots v_{k'} c_{k'}$$

showing that p' < q'. Symmetrically, one shows that p' < q' implies p < q. We conclude p < q if and only if p' < q'. Similarly, we have p = q if and only if p' = q'. It follows that the relative order of the b_i 's and c_j 's in u and v is the same. By factoring u and v at all b_i 's and c_j 's, we obtain $u = a_1s_1 \cdots a_{\ell-1}s_{\ell-1}a_\ell$ and $v = a_1t_1 \cdots a_{\ell-1}t_{\ell-1}a_\ell$ with $a_i \in A$ and $\ell \le k + k' \le 2|M|$.

We have $a_1s_1 \cdots a_{i-1}s_{i-1}a_i \mathcal{R}$ $a_1s_1 \cdots a_{i-1}s_{i-1}a_is_i$ since the factorization $u = a_1s_1 \cdots a_{\ell-1}s_{\ell-1}a_{\ell}$ is a refinement of the \mathcal{R} -factorization. Note that we cannot assume $\alpha(s_i) = \alpha(t_i)$. But each t_i is a factor of some v_j , and at the same time s_i is a factor of u_j . More precisely, there exists $m \leq i$ such that

$$b_1 v_1' \cdots b_{j-1} v_{j-1}' b_j = a_1 t_1 \cdots a_{m-1} t_{m-1} a_m$$
 and $t_m a_{m+1} \cdots t_{i-1} a_i t_i$ is a prefix of v_j' .

Furthermore, $s_m a_{m+1} \cdots s_{i-1} a_i s_i$ is a prefix of u_j . Now, $\alpha(t_i) \subseteq \alpha(v_j') = \alpha(u_j)$ and, by Lemma 4, for all words z with $\alpha(z) \subseteq \alpha(u_j)$ we have $a_1 s_1 \cdots a_{i-1} s_{i-1} a_i \mathscr{R}$ $a_1 s_1 \cdots a_{i-1} s_{i-1} a_i z$. Symmetrically we see $a_{i+1} t_{i+1} \cdots a_{\ell-1} t_{\ell-1} a_{\ell} \mathscr{L}$ $s_i a_{i+1} t_{i+1} \cdots a_{\ell-1} t_{\ell-1} a_{\ell}$.

Theorem 6 (Almeida/Azevedo, 1989 [2])

$$\mathbf{R} \vee \mathbf{L} = [(xy)^{\omega} x (zx)^{\omega} = (xy)^{\omega} (zx)^{\omega}]$$

Proof: The inclusion $\mathbb{R} \vee \mathbb{L} \subseteq \mathbb{W}$ is trivial since $\mathbb{R} \cup \mathbb{L} \subseteq \mathbb{W}$ and \mathbb{W} is a variety of finite monoids. Let $M \in \mathbb{W}$ be generated by A, and let $\varphi : A^* \to M$ be the homomorphism induced by $A \subseteq M$. Let n = 2|M| and

suppose $u \equiv_n v$. Let $u = a_1 s_1 \cdots a_{\ell-1} s_{\ell-1} a_{\ell}$ and $v = a_1 t_1 \cdots a_{\ell-1} t_{\ell-1} a_{\ell}$ be the factorizations from Lemma 5. Applying Lemma 3 repeatedly, we get

$$\varphi(v) = \varphi(a_1t_1a_2t_2 \cdots a_{\ell-2}t_{\ell-2}a_{\ell-1}t_{\ell-1}a_{\ell})
= \varphi(a_1s_1a_2t_2 \cdots a_{\ell-2}t_{\ell-2}a_{\ell-1}t_{\ell-1}a_{\ell})
= \varphi(a_1s_1a_2s_2 \cdots a_{\ell-2}t_{\ell-2}a_{\ell-1}t_{\ell-1}a_{\ell})
\vdots
= \varphi(a_1s_1a_2s_2 \cdots a_{\ell-2}s_{\ell-2}a_{\ell-1}t_{\ell-1}a_{\ell})
= \varphi(a_1s_1a_2s_2 \cdots a_{\ell-2}s_{\ell-2}a_{\ell-1}t_{\ell-1}a_{\ell}) = \varphi(a_1s_1a_2s_2 \cdots a_{\ell-2}s_{\ell-2}a_{\ell-1}s_{\ell-1}a_{\ell}) = \varphi(u).$$

Note that the substitution rules $t_i \to s_i$ are φ -invariant only when applied from left to right. This shows that M is a quotient of A^*/\equiv_n , and the latter is in $\mathbb{R} \vee \mathbb{L}$ by Lemma 2. Thus $M \in \mathbb{R} \vee \mathbb{L}$.

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