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# *A Stronger Recognizability Condition for Two-dimensional Languages*

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The paper presents a condition necessarily satisfied by (tiling system) recognizable two-dimensional languages. The new recognizability condition is compared with all the other ones known in the literature (namely three conditions), once they are put in a uniform setting: they are stated as bounds on the growth of some complexity functions defined for two-dimensional languages. The gaps between such functions are analyzed and examples are shown that asymptotically separate them. Finally the new recognizability condition is shown to be the strongest one, while the remaining ones are its particular cases. The problem of deciding whether a two-dimensional language is recognizable is here related to the one of estimating the minimal size of finite automata recognizing a sequence of (one-dimensional) string languages.

**Keywords:** Two-dimensional languages. Recognizability conditions.

## 1 Introduction

Picture languages are a generalization of string languages to two dimensions: a picture is a two-dimensional array of elements from a finite alphabet. The increasing interest for pattern recognition and image processing has motivated the research on two-dimensional languages, and especially tile-based models. The aim is to generalize or extend the well-founded theory of formal languages. Since the sixties, the problem has been approached from different points of view: finite automata, grammars, logics and regular expressions have been proposed. Among the various defined classes of languages, probably the most successful one, as far as theoretical aspects are concerned, is the class of *tiling system recognizable languages* (recognizable languages, for short), also known as REC (see [12, 13]). A two-dimensional language is said to be recognizable when it is the alphabetic projection of a local language defined in terms of a finite set of $2 \times 2$ pictures, the allowed tiles; the recognition is given by a so-called *tiling system*.

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Since its introduction, family REC has been intensively studied (see e.g. [13, 4, 9, 1]). The definition in terms of tiling systems turns out to be robust in comparison with the other models in the literature: REC has a characterization in terms of monadic second-order logic formulas [13, 15]; it has a counterpart as machine model in the two-dimensional on-line tessellation acceptor [19] and in other models of automata as proposed in [2, 7, 20]. Tiling systems can be also simulated by domino systems [19], Wang systems [10] and grammars [9].

Despite a wide literature on two-dimensional languages, the problem of establishing whether a language is recognizable or not still deserves to be investigated. There do not exist, in the literature, feasible characterizations of REC based on the structure of the language itself, that could be handily used for this goal. In formal language theory, the problem of deciding whether a given language is regular, can be solved considering some congruence classes and the Myhill-Nerode Theorem. Then, a very useful tool to disprove that a language is regular is given by the Pumping Lemma [17]. The Pumping Lemma gives a condition necessarily satisfied by any regular language. The problem seems much more complex in two dimensions. Recall that the runs of a Turing machine form a recognizable two-dimensional language (cf. [13]) so that one cannot expect simple periodicity results.

The paper continues the research of conditions necessarily satisfied by recognizable languages, principally as tools to disprove recognizability of languages. Recently, some steps in this direction were done in [3, 14, 5, 1]. The idea is to bring the problem from two dimensions to one dimension, and then rely on results of the more consolidate theory of string languages. More specifically, a first step was done in [21], where O. Matz isolated a technique for showing that a language is not recognizable. It consists of considering, for any recognizable picture language $L$ and integer $m$, the subset $L(m)$, of all pictures in $L$ of fixed height $m$, as a *string* language over the alphabet of the columns. Hence a picture language can be considered as the sequence of such string languages, when $m$ grows. Then if $L \in$ REC it is possible to associate to any tiling system recognizing $L$ a family $\{A_m\}$, where each $A_m$ is an automaton accepting $L(m)$ with $2^{O(m)}$ states. Using some known lower bound on the size of an automaton, he proved a condition satisfied by the languages in REC.

In [3] a further step forward was done: Matz's technique was for the first time used together with some bound on the Hankel matrices of the string languages $L(m)$. The combination of these two ideas (Matz's technique and Hankel matrices) has allowed to obtain necessary conditions for the belonging of a language to REC and to other meaningful sub-classes of REC [1]. In [14], some of these conditions have been put in a uniform framework, by introducing some complexity functions (the "row", "permutation" and "rank" complexity functions) of a picture language and studying their asymptotic growth. Note that other necessary conditions for recognizabilty have been proved in [11, 5] for a specific class of unary picture languages.

The paper states a new condition that is necessarily satisfied by any recognizable language. The condition is in fact a bound on the growth of a new complexity function here defined for two-dimensional languages. Following the approach in [14], Theorem 4.1 collects all necessary conditions for recognizability given in the past years (as far as we know) (referred to as conditions *1*, *2* and *3*), together with the new one (referred to as condition *4*), and puts them in a uniform setting useful to compare them. The properties are given as bounds on the complexity functions $R_L(m)$, $P_L(m)$, $F_L(m)$ and $C_L(m)$. Then the four conditions, namely the four complexity functions, are compared each other. Examples show the gaps existing on the growth of the complexity functions. Finally the results show that the condition introduced in this paper is the strongest one: all the other ones are particular cases. Moreover Section 5 shows examples in which it is possible to disprove recognizability by applying condition *4* and not by using the other ones. The computation of the three functions $P_L(m)$, $F_L(m)$ and $C_L(m)$ is algorithmically possible (cf. [16]), but, in general, not easy. So that it is important to take into account all the complexity functions, since the computation, in some cases, could be easier for one function rather than another.

Finding characterizations of two-dimensional recognizable languages seems to be difficult since it is related to hard questions concerning computational complexity. Unfortunately, none of these necessary conditions is also sufficient. Counter-examples for conditions *1-3* are given in [5]; Proposition 5.1 shows a counter-example for condition *4*.

The paper is organized as follows. After giving the basic definitions and results on two-dimensional languages in Section 2, in Section 3 the new necessary condition is presented. The comparison of the new condition with the other ones is given in Section 4 and some of its applications are given in Section 5. At last in Section 6 some conclusions and open problems are stated.

A preliminary version of this paper can be found in [6].

## 2 Preliminaries

In this section some definitions about two-dimensional recognizable languages are recalled. More details can be mainly found in [13].

A *two-dimensional string* (or a *picture*) over a finite alphabet $\Sigma$ is a two-dimensional rectangular array of elements of $\Sigma$. The set of all pictures over $\Sigma$ is denoted by $\Sigma^{**}$ and a *two-dimensional language* over $\Sigma$ is a subset of $\Sigma^{**}$.

Given a picture $p \in \Sigma^{**}$, let $p_{(i,j)}$ denote the symbol in $p$ with coordinates $(i, j)$, $|p|_r$ the number of rows and $|p|_c$ the number of columns of $p$; the pair $(|p|_r, |p|_c)$ is the *size* of $p$. Note that when a one-letter alphabet is concerned, a picture $p$ is totally defined by its size $(m, n)$, and it will be simply referred to as $p = (m, n)$. Remark that the set $\Sigma^{**}$ includes also all the empty pictures, i.e. all pictures of size $(m, 0)$ or $(0, n)$ for all $m, n \geq 0$, denoted by $\lambda_{m,0}$ and $\lambda_{0,n}$ respectively. The set of all pictures over $\Sigma$ of size $(m, n)$ is denoted by $\Sigma^{m,n}$, while $\Sigma^{m,*}$ ($\Sigma^{*,n}$, resp.) is the set of all pictures over $\Sigma$ with $m$ rows (with $n$ columns, resp.). In order to identify the symbols on the boundary of a given picture, for any picture $p$ of size $(m, n)$, we consider the *bordered picture* $\widehat{p}$ of size $(m + 2, n + 2)$ obtained by surrounding $p$ with a special *boundary symbol* $\# \notin \Sigma$.

A *tile* is a picture of size $(2, 2)$ and $B_{2,2}(p)$ is the set of all sub-blocks of size $(2, 2)$ of a picture $p$. Given an alphabet $\Gamma$, a two-dimensional language $L \subseteq \Gamma^{**}$ is *local* if there exists a set $\Theta$ of tiles over $\Gamma \cup \{\#\}$ such that $L = L(\Theta)$, where $L(\Theta)$ is defined as $L(\Theta) = \{p \in \Gamma^{**} | B_{2,2}(\widehat{p}) \subseteq \Theta\}$.

A *tiling system* is a quadruple $(\Sigma, \Gamma, \Theta, \pi)$ where $\Sigma$ and $\Gamma$ are finite alphabets, $\Theta$ is a finite set of tiles over $\Gamma \cup \{\#\}$ and $\pi : \Gamma \to \Sigma$ is a projection. A two-dimensional language $L \subseteq \Sigma^{**}$ is *tiling recognizable* if there exists a tiling system $(\Sigma, \Gamma, \Theta, \pi)$ such that $L = \pi(L(\Theta))$ (extending $\pi$ in the usual way). For any $p \in L$, a local picture $p' \in L(\Theta)$, such that $p = \pi(p')$, is called a $pre\text{-}image$ of $p$. The family of all *tiling recognizable* picture languages is called *REC*.

Given two pictures $p$ and $q$, the *column concatenation* of $p$ and $q$ (denoted by $p \varobar q$) and the *row concatenation* of $p$ and $q$ (denoted by $p \ominus q$) are partial operations, defined only if $|p|_r = |q|_r$ and if $|p|_c = |q|_c$, respectively, and are given by:

$$p \obar q = \begin{array}{|c|c|}\hline \quad p \quad & \quad q \quad \\ \hline \end{array} \qquad\qquad p \ominus q = \begin{array}{|c|}\hline \quad p \quad \\ \hline \quad q \quad \\ \hline \end{array}$$

These definitions of picture concatenations can be extended to define two-dimensional language concatenations. Furthermore the *column closure* of $L$ (denoted by $L^{*\obar}$) and the *row closure* of $L$ (denoted by $L^{*\ominus}$) are defined by iterating the concatenation operations: $L^{*\obar} = \bigcup_i L^{i\obar}$ and $L^{*\ominus} = \bigcup_i L^{i\ominus}$ where $L^{0\obar} = \{\lambda_{m,0} \mid m \geq 0\}$, $L^{n\obar} = L^{(n-1)\obar} \obar L$ and $L^{0\ominus} = \{\lambda_{0,m} \mid m \geq 0\}$, $L^{n\ominus} = L^{(n-1)\ominus} \ominus L$. REC family is closed under row and column concatenation and their closures, under union, intersection and under rotation, but not under complementation (see [13] for all the proofs).

**Example 2.1** *Consider the language $L$ of square pictures over a one-letter alphabet, say $\Sigma = \{a\}$, that is non-empty pictures with same number of rows as columns. $L$ is not a local language, but it belongs to REC. Indeed it can be obtained as projection of the local language of squares over the alphabet $\{0,1\}$ in which all the symbols in the main diagonal are $1$, whereas the remaining positions carry symbol $0$. Below it is given a picture $p \in L$ together with its pre-image $p'$. The reader can infer the set of all tiles by taking all possible sub-pictures of size $(2,2)$ of the bordered pictures $\widehat{p'}$ and $\widehat{p''}$ where $p'' = \boxed{a}$.*

$$p = \begin{array}{|c|c|c|c|}\hline a & a & a & a \\ \hline a & a & a & a \\ \hline a & a & a & a \\ \hline a & a & a & a \\ \hline \end{array} \qquad\qquad p' = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array}$$

**Example 2.2** *Consider the language over a one-letter alphabet $L_{mult} = \{(m, km) \mid m \geq 0, k \geq 0\}$ and remark that $L_{mult} = L^{*\obar}$, where $L$ is the language of square pictures in Example 2.1. Since $L \in$ REC and REC is closed under the column closure, $L_{mult} \in$ REC. Indeed a tiling system $\mathcal{T}$ for $L_{mult}$ can be obtained following the construction of a tiling system for the column star of a language in [13]. Consider two disjoint copies of the local language for $L$ and force them to alternate: one copy is used to recognize squares in odd positions (thanks to the subscript "$o$" that stands for "odd") and the other copy for squares in even positions (thanks to the subscript "$e$" that stands for "even") .*

*For example, pictures $p_1 = (4,8)$ and $p_2 = (4,12)$ of $L_{mult}$ have pre-images, respectively, $p_1'$ and $p_2'$ given below. The reader can infer the set of all tiles in $\mathcal{T}$ by taking all possible sub-pictures of size $(2,2)$ of the bordered pictures $\widehat{p_1'}$ and $\widehat{p_2'}$.*

$$p_1' = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1_o & 0_o & 0_o & 0_o & 1_e & 0_e & 0_e & 0_e \\ \hline 0_o & 1_o & 0_o & 0_o & 0_e & 1_e & 0_e & 0_e \\ \hline 0_o & 0_o & 1_o & 0_o & 0_e & 0_e & 1_e & 0_e \\ \hline 0_o & 0_o & 0_o & 1_o & 0_e & 0_e & 0_e & 1_e \\ \hline \end{array}$$

$$p'_2 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1_o & 0_o & 0_o & 0_o & 1_e & 0_e & 0_e & 0_e & 1_o & 0_o & 0_o & 0_o \\ \hline 0_o & 1_o & 0_o & 0_o & 0_e & 1_e & 0_e & 0_e & 0_o & 1_o & 0_o & 0_o \\ \hline 0_o & 0_o & 1_o & 0_o & 0_e & 0_e & 1_e & 0_e & 0_o & 0_o & 1_o & 0_o \\ \hline 0_o & 0_o & 0_o & 1_o & 0_e & 0_e & 0_e & 1_e & 0_o & 0_o & 0_o & 1_o \\ \hline \end{array}$$

*Consider now the language $\overline{L_{mult}}$. Also $\overline{L_{mult}} \in REC$: a tiling system can be obtained redefining in $\mathcal{T}$ its tiles for the bottom-right and the top-right corners.*

**Example 2.3** *Let $\Sigma = \{a, b\}$ and consider the language $L_{2col} = \{p \in \Sigma^{*,2} \mid$ there exist $1 \leq i_1 \leq i_2 \leq |p|_r$ such that $p_{(i_1,1)} = p_{(i_2,2)} = b$, $p_{(i,1)} = a$ for every $i \neq i_1$ and $p_{(i,2)} = a$ for every $i \neq i_2\}$. Roughly, $L_{2col}$ is the language of pictures $p$ with two columns such that there is only one symbol $b$ in each column and the entry that carries the symbol $b$ in the second column of $p$ is not higher than the one in the first column (see below).*

*Language $L_{2col} \in REC$. Indeed it can be obtained as the projection of the local language of two-columns pictures over the alphabet $\{U_1, U_2, D_1, D_2, B_1, B_2\}$ such that: in the first column (second column, resp.) there is only one occurrence of the symbol $B_1$ ($B_2$, resp.), whereas all the positions above it carry symbol $U_1$ ($U_2$, resp.) and all the positions below it carry symbol $D_1$ ($D_2$, resp.). The projection $\pi$ is defined by: $\pi(U_1) = \pi(U_2) = \pi(D_1) = \pi(D_2) = a$ and $\pi(B_1) = \pi(B_2) = b$. Below the example of a picture $p \in L_{2col}$ together with its pre-image $p'$ is given.*

$$p = \begin{array}{|c|c|} \hline a & a \\ \hline b & a \\ \hline a & a \\ \hline a & b \\ \hline a & a \\ \hline \end{array} \qquad p' = \begin{array}{|c|c|} \hline U_1 & U_2 \\ \hline B_1 & U_2 \\ \hline D_1 & U_2 \\ \hline D_1 & B_2 \\ \hline D_1 & D_2 \\ \hline \end{array}$$

# 3 A new recognizability condition

The present section shows a condition, namely a bound on the growth of a complexity function, that is necessarily satisfied by any recognizable two-dimensional language. The condition is a useful tool to disprove the recognizability of a two-dimensional language.

The problem of the existence of a tiling system recognizing a given two-dimensional language, is here afforded by investigating the sequence of its subsets of pictures of fixed height, that in turn, can be viewed as string languages on growing alphabets. More precisely, as in [21], let $L \subseteq \Sigma^{**}$ be a picture language and for any $m \geq 1$, consider the subset $L(m) \subseteq L$ containing all pictures in $L$ with exactly $m$ rows. The language $L(m)$ can be viewed as a string language over the alphabet of the columns of height $m$. Using this reduction from two dimensions to one dimension, O. Matz stated the following recognizability condition for picture languages.

**Lemma 3.1** *[21] If $L$ is in REC then it is possible to associate to any tiling system recognizing $L$ a family $\{\mathcal{A}_m\}$, where each $\mathcal{A}_m$ is an automaton accepting $L(m)$ with $2^{O(m)}$ states.*

This result is the starting point for the new necessary condition for the recognizability. The condition is formulated in terms of the Hankel matrix of a language. Let us recall some terminology. The infinite boolean Hankel matrix associated to any string language $S \subseteq \Sigma^*$, is $M_S = \|a_{\alpha\beta}\|_{\alpha\in\Sigma^*,\beta\in\Sigma^*}$ where $a_{\alpha\beta} = 1$ if and only if $\alpha\beta \in S$ (see [18]). Observe that, when $S \subseteq \Sigma^*$ is a regular language, the number of different rows of $M_S$ is finite and equal to the number of different columns. Moreover this number coincides with the number of classes of the Myhill-Nerode equivalence relation $\equiv_S$ for $S$: given $x, y \in \Sigma^*$, $x \equiv_S y$ if and only if (for any $z \in \Sigma^*$, $xz \in S$ if and only if $yz \in S$).

The sub-matrix of an Hankel matrix $M_S$ specified by a pair of languages $(U, V)$, with $U, V \subseteq \Sigma^*$, is the matrix obtained by intersecting all rows and all columns of $M_S$ that are indexed by the strings in $U$ and $V$, respectively. The size of a finite matrix with $m$ rows and $n$ columns is denoted $(m, n)$, or simply $m$, if it is a square matrix.

Remark that, if $L$ is a picture language in REC, then $L(m)$ is a regular language for any $m$ (see Lemma 3.1), and, therefore, the Hankel matrix $M_{L(m)}$ has a finite number of different rows and columns. In the following examples, for a Hankel matrix $M$ with a finite number of different rows and columns, the sub-matrix indexed by the distinct rows and the distinct columns (in some order) will be referred to as the "finite part" of $M$.

**Example 3.1** *Consider again the language $L_{mult} = \{(m, km) \mid m \geq 0, k \geq 0\}$ defined in Example 2.2 and, for any $m \geq 2$, the Hankel matrix $M = M_{L_{mult}(m)}$. $M$ can be obtained as a juxtaposition, both along the rows and along the columns, of the following block $M'$, where $\lambda = \lambda_{m,0}$ and $v = (m, 1)$. Then $M'$ is the finite part of $M$.*

| | $v$ | $v^2$ | $\cdots$ | $v^{m-2}$ | $v^{m-1}$ | $v^m$ |
|---|---|---|---|---|---|---|
| $\lambda$ | 0 | 0 | $\cdots$ | 0 | 0 | 1 |
| $v$ | 0 | 0 | $\cdots$ | 0 | 1 | 0 |
| $v^2$ | 0 | 0 | $\cdots$ | 1 | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $v^{m-2}$ | 0 | 1 | $\cdots$ | 0 | 0 | 0 |
| $v^{m-1}$ | 1 | 0 | $\cdots$ | 0 | 0 | 0 |

.

Let us now introduce a complexity function of picture languages, based on the notion of cover of a boolean matrix. This complexity function will be used to obtain a necessary condition for a picture language to be in REC. A boolean matrix is a *1-monochromatic matrix* if all its entries are 1.

**Definition 3.1** *Let $A = \|a_{ij}\|$ be a boolean matrix and $S = \{A_1, \ldots, A_k\}$ be a set of 1-monochromatic sub-matrices of $A$ where, for any $t = 1, \ldots, k$, $A_t$ is specified by the pair of languages $(U_t, V_t)$. $S$ is a* cover *for $A$ if, for any pair $(i, j)$ such that $a_{ij} = 1$, there exists an integer $t$, $1 \leq t \leq k$, such that $(i, j) \in U_t \times V_t$.*

**Definition 3.2** *Let L be a picture language. The* covering complexity function *$C_L(m)$, defined from $\mathbb{N}$ to $\mathbb{N} \cup \{\infty\}$, gives the cardinality of a minimal cover for $M_{L(m)}$.*

**Theorem 3.1** *Let $L \subseteq \Sigma^{**}$. If $L \in$ REC then $C_L(m)$ is $2^{O(m)}$.*

**Proof:** If $L$ is in REC then, from Lemma 3.1, it is possible to associate to any tiling system recognizing $L$ a family $\{\mathcal{A}_m\}$, where, for some constant $c$, each $\mathcal{A}_m$ is an automaton accepting $L(m)$ with a number of states that is at most $c^m$. For any $m$, consider the NFA $\mathcal{A}_m = (Q_m, q_m^0, F_m, \delta_m)$ over the alphabet $\Sigma^{m,1}$. For any $q \in Q_m$ consider the sets $X_q$ and $Y_q$ of strings over $\Sigma^{m,1}$, defined as follows: $X_q = \{x \in (\Sigma^{m,1})^* : q \in \delta_m^*(q_m^0, x)\}$ and $Y_q = \{y \in (\Sigma^{m,1})^* : \delta_m^*(q, y) \cap F_m \neq \emptyset\}$.

Let $M_{L(m)}$ be the Hankel matrix of the language $L(m)$ and denote $M_q$ its sub-matrix specified by the pair of languages $(X_q, Y_q)$. $M_q$ is a 1-monochromatic sub-matrix of $M_{L(m)}$ since $xy \in L(m)$ for all $x \in X_q$ and $y \in Y_q$. Moreover $S_m = \{M_q : q \in Q_m\}$ is a cover of $M_{L(m)}$. Indeed, if an entry of $M_{L(m)}$, indexed by a pair $(\overline{x}, \overline{y})$, carries a symbol 1, then $\overline{x}\,\overline{y} \in L(m)$ and then there exists at least a state $\overline{q} \in Q_m$ such that $\overline{q} \in \delta_m^*(q_m^0, \overline{x})$ and $\delta_m^*(\overline{q}, \overline{y}) \cap F_m \neq \emptyset$. Therefore $\overline{x} \in X_{\overline{q}}$ and $\overline{y} \in Y_{\overline{q}}$ and the entry indexed by $(\overline{x}, \overline{y})$ belongs to $M_{\overline{q}}$.

Since $C_L(m)$ is the cardinality of a minimal cover for $M_{L(m)}$, we have $C_L(m) \leq |S_m| \leq |Q_m|$. But $|Q_m| \leq c^m$ and the result follows. $\square$

**Remark:** Theorem 3.1 (as well as next Prop. 4.1) with its proof can be viewed as the picture counterpart of a result given in [16] in the graph setting. Indeed, one can prove that, for any $L \subseteq \Sigma^{**}$, there exists a strict relation between the Hankel matrix of $L(m)$ and the bipartite graph associated to the string language $L(m)$ as defined in [16]. In particular, one can prove that $C_L(m)$ is equal to the bipartite dimension of the graph of $L(m)$. $\square$

**Example 3.2** *Consider the language $L = L_{mult} \in$ REC as defined in Example 2.2, and, for any $m$, the Hankel matrix $M = M_{L(m)}$. Starting from a cover for its finite part $M'$ (as shown in Example 3.1), it is possible to obtain a cover for $M$. More exactly, a cover for $M'$ is given by the set $S'$ of all the $m$ matrices of size 1 corresponding to the counterdiagonal positions of $M'$. Formally, $S' = \{A'_1, \ldots, A'_m\}$ where, for $i = 1, \ldots, m$, $A'_i$ is specified by the pair of picture languages $(X'_i, Y'_i)$ with $X'_i = \{(m, i-1)\}$ and $Y'_i = \{(m, m-i+1)\}$. Remark that the sets $X'_i$ and $Y'_i$ both contain just one element. Then a cover for $M$ can be obtained by considering the set $S = \{A_1, \ldots, A_m\}$ where, for $i = 1, \ldots, m$, $A_i$ is specified by the pair of picture languages $(X_i, Y_i)$ with $X_i = \{(m, i-1+km) \mid k \geq 0\}$ and $Y_i = \{(m, m-i+1+km) \mid k \geq 0\}$. Remark that the sets $X_i$ and $Y_i$ both contain an infinite number of elements.*

*Moreover, it is easy to show that $S$ is a minimal cover for $M$: indeed, the occurrences of 1 in two different rows in $M$ cannot be "covered" by a same monochromatic matrix in $S$. Hence $C_L(m)$ is linear in $m$, and then $C_L(m) = 2^{O(m)}$, in line with Theorem 3.1.*

**Example 3.3** *Consider, for any $m \geq 0$, the function $f(m) = lcm(2^m + 1, \ldots, 2^{m+1})$ and the language $L_M$ over the unary alphabet $\Sigma = \{a\}$, $L_M = \{(m, n) \mid n$ is not a multiple of $f(m)\}$. It was shown that $L_M \in$ REC (see [22, 23]). Consider now the language $\overline{L}_M$, the complement of $L_M$, and, for any $m > 1$, languages $\overline{L}_M(m)$ and their Hankel matrices $M_{\overline{L}_M(m)}$. The finite part of $M_{\overline{L}_M(m)}$ can be arranged in a square matrix of size $f(m)$ with 1 in all counter-diagonal positions and 0 elsewhere. More exactly, if $\lambda = \lambda_{m,0}$ and $v = (m, 1)$, then the finite part of $M_{\overline{L}_M(m)}$ can be arranged as follows.*

| | $\lambda$ | $v$ | $v^2$ | $\cdots$ | $v^{f(m)-2}$ | $v^{f(m)-1}$ |
|---|---|---|---|---|---|---|
| $v$ | 0 | 0 | 0 | $\cdots$ | 0 | 1 |
| $v^2$ | 0 | 0 | 0 | $\cdots$ | 1 | 0 |
| $v^3$ | 0 | 0 | 0 | $\cdots$ | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $v^{f(m)-2}$ | 0 | 0 | 1 | $\cdots$ | 0 | 0 |
| $v^{f(m)-1}$ | 0 | 1 | 0 | $\cdots$ | 0 | 0 |
| $v^{f(m)}$ | 1 | 0 | 0 | $\cdots$ | 0 | 0 |

.

*Reasoning as in Example 3.2, it can be shown that a minimal cover for $M_{\overline{L}_M(m)}$ has cardinality $f(m)$ and, therefore, $C_{\overline{L}_M}(m) = f(m)$. Applying Theorem 3.1, since $f(m) = 2^{\Theta(2^m)}$ (see [21, 22]), it follows that $\overline{L}_M \notin$ REC.*

## 4 Comparison with other recognizability conditions

The investigation of recognizable two-dimensional languages has provided, in the last two decades, several properties necessarily satisfied by recognizable languages. The section collects all of them (as far as we know), together with the condition presented in previous section, and put them in a uniform setting. The properties are given as bounds on some complexity functions defined for two-dimensional languages. Then the four conditions, namely the four complexity functions, are compared to each other. The results show that the condition introduced in Section 3 is the strongest one: all the other ones are particular cases. Examples show the gaps existing on the growth of these complexity functions.

Let us fix some definitions.

A $permutation\ matrix$ is a boolean square matrix that has exactly one occurrence of 1 in each row and in each column. A boolean square matrix $A$ is a *fooling matrix* if there exists a permutation of its rows such that, in the resulting matrix $B = \|b_{ij}\|$, for any $i$, $b_{ii} = 1$ and, for any $i, j$ with $i \neq j$, if $b_{ij} = 1$ then $b_{ji} = 0$ (cf. [5]). Then, given a picture language $L$, one can define the following functions from $\mathbb{N}$ to $\mathbb{N} \cup \{\infty\}$: the *row complexity function* $R_L(m)$ gives the number of distinct rows of $M_{L(m)}$, the *permutation complexity function* $P_L(m)$ gives the size of the maximal permutation matrix that is a sub-matrix of $M_{L(m)}$ (cf. [14]), while the *fooling complexity function* $F_L(m)$ gives the size of the maximal fooling matrix that is a sub-matrix of $M_{L(m)}$ (cf. [5]).

**Example 4.1** *Consider the language $L = L_{mult}$ as defined in Example 2.2. For any $m$, the finite part of the Hankel matrix $M = M_{L(m)}$ (see Example 3.1) is both a maximal permutation matrix and a maximal fooling matrix. Its size is $m$ and, hence, $P_L(m) = F_L(m) = m$. Moreover, we have $C_L(m) = m$ (see Example 3.2).*

*Consider now the language $L_1 = \{(m, km) \mid m \geq 1, k \geq 1\}$ that is a slight variation of $L$. $L_1 \in$ REC. For any $m$, the finite part of the Hankel matrix $M_1 = M_{L_1(m)}$ can be obtained by adding the column indexed by picture $(m, 0)$ and the row indexed by the picture $(m, m)$ to the finite part of $M_{L(m)}$. Then the finite part of $M_1$ can be arranged as follows, where $\lambda = \lambda_{m,0}$ and $v = (m, 1)$.*

| | $\lambda$ | $v$ | $v^2$ | $\cdots$ | $v^{m-2}$ | $v^{m-1}$ | $v^m$ |
|---|---|---|---|---|---|---|---|
| $\lambda$ | 0 | 0 | 0 | $\cdots$ | 0 | 0 | 1 |
| $v$ | 0 | 0 | 0 | $\cdots$ | 0 | 1 | 0 |
| $v^2$ | 0 | 0 | 0 | $\cdots$ | 1 | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $v^{m-2}$ | 0 | 0 | 1 | $\cdots$ | 0 | 0 | 0 |
| $v^{m-1}$ | 0 | 1 | 0 | $\cdots$ | 0 | 0 | 0 |
| $v^m$ | 1 | 0 | 0 | $\cdots$ | 0 | 0 | 1 |

.

*It is easy to see that the sub-matrix that contains the first $m$ rows and the last $m$ columns of $M_1$ is a maximal permutation matrix, therefore $P_{L_1}(m) = m$. Moreover, $M_1$ is a fooling matrix and it is a maximal one. Hence $F_{L_1}(m) = m + 1$. At last, it is possible to obtain a cover of $M_1$ with $m + 1$ elements and, it can be easily proved that $C_{L_1}(m) = m + 1$. Hence, $C_{L_1}(m)$ is linear in $m$, and then $C_{L_1}(m) = 2^{O(m)}$ according to Theorem 3.1.*

The following theorem collects all the known necessary conditions for the recognizability of picture languages and states them using the complexity functions just defined, following the approach in [14]. In particular, calling *condition* $i$ the one stated in item $i$, we have that conditions *1*, *2*, and *3* restate recognizability conditions proved in [8], [14] and [5], respectively, whereas condition *4* is the condition proved in Theorem 3.1.

**Theorem 4.1** *Let $L \subseteq \Sigma^{**}$.*

1. *If $L \in$ REC $\cup$ co-REC then $R_L(m)$ is $2^{2^{o(m)}}$.*
2. *If $L \in$ REC then $P_L(m)$ is $2^{O(m)}$.*
3. *If $L \in$ REC then $F_L(m)$ is $2^{O(m)}$.*
4. *If $L \in$ REC then $C_L(m)$ is $2^{O(m)}$.*

In the sequel we compare the four conditions and find that the new one extends the other ones. The comparison of condition *4* with condition *1* is in Proposition 4.2, while the comparison with conditions *2* and *3*, is in the following proposition.

**Proposition 4.1** *Let $L \subseteq \Sigma^{**}$. Then, for all $m \geq 1$, $P_L(m) \leq F_L(m) \leq C_L(m) \leq R_L(m)$.*

**Proof:** For any $m$, consider the Hankel matrix $M_{L(m)}$.

$P_L(m) \leq F_L(m)$ since every permutation matrix is a fooling matrix.

Moreover, $F_L(m) \leq C_L(m)$. Indeed, for any $m$, let $S = \{M_1, M_2, \ldots, M_n\}$ be a cover for $M_{L(m)}$ and let $A = \|a_{ij}\|$, $1 \leq i, j \leq k$, be a sub-matrix of $M_{L(m)}$ that is a fooling matrix. Suppose $A$ is specified by the pair of picture languages $(X, Y)$ with $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$, and that w.l.o.g., for any $1 \leq i \leq k$, $a_{x_i y_i} = 1$ and, for any $1 \leq i, j \leq k$ with $i \neq j$, if $a_{x_i y_j} = 1$ then $a_{x_j y_i} = 0$. Then consider two entries in $A$ indexed by the pairs $(x_i, y_i)$ and $(x_j, y_j)$, with $i \neq j$: since they carry symbol 1 and $S$ is a cover of $M_{L(m)}$, these entries belong to some matrices in $S$. But they cannot belong to the same matrix: $S$ contains only 1-monochromatic matrices and we cannot have, at the same time, $a_{x_iy_j} = a_{x_jy_i} = 1$. Therefore, for any $1 \le i \le k$, the entries in $A$ indexed by pair $(x_i, y_i)$ belong to different elements of $S$, i.e. the size of $A$ is less than or equal to the cardinality of $S$.

The last inequality $C_L(m) \le R_L(m)$ follows from the observation that it is always possible to obtain a cover for the Hankel matrix $M_{L(m)}$ with as many elements as the number of rows. Indeed, suppose that the finite part of $M_{L(m)}$ is $\|a_{\alpha_i\beta_j}\|$ with $i, j = 1, \dots, r$ and $r = R_L(m)$. Then the set $S = \{M_1, M_2, \dots, M_r\}$, where each $M_i$, for $i = 1, \dots, r$, is specified by the pair $U_i = \{\alpha_i\}$ and $V_i = \{\beta_j \mid a_{\alpha_i\beta_j} = 1\}$, is a cover for $M_{L(m)}$. $\square$

**Remark:** There exist languages such that, for any $m$, $P_L(m) = F_L(m) = C_L(m) = R_L(m)$. Consider for example language $L = L_{mult}$: then $P_L(m) = F_L(m) = C_L(m) = R_L(m) = m$ (see Example 4.1). $\square$

**Proposition 4.2** *Let $L \subseteq \Sigma^{**}$. Then, for all $m \ge 1$, $R_L(m) \le 2^{C_L(m)}$.*

**Proof:** For any $m$, consider the Hankel matrix $M_{L(m)}$, let $C_L(m) = n$ and let $S = \{M_1, M_2, \dots, M_n\}$ be a minimal cover for $M_{L(m)}$. Suppose that $M_i$ is indexed by the pair of languages $(X_i, Y_i)$, $i = 1, \dots, n$. For any $x \in \Sigma^{m,*}$, let $B(x) = \{M_i \in S \mid x \in X_i\}$, i.e. the set of all matrices in $S$ that contain the row indexed by $x$.

We claim that for all $x, x' \in \Sigma^{m,*}$, $B(x) = B(x')$ implies $x \equiv_{L(m)} x'$, where $\equiv_{L(m)}$ is the Myhill-Nerode equivalence relation for $L(m)$. Indeed consider $y \in \Sigma^{m,*}$. If there exists $\bar{\imath} \in \{1, \dots, n\}$ such that the entry $(x, y)$ belongs to $M_{\bar{\imath}}$, then $M_{\bar{\imath}} \in B(x)$ and in this case $x \in X_{\bar{\imath}}$ and $y \in Y_{\bar{\imath}}$. Since $B(x) = B(x')$, $M_{\bar{\imath}} \in B(x')$ also (and, therefore, $x' \in X_{\bar{\imath}}$). Recalling that $M_{\bar{\imath}}$ is a 1-monochromatic matrix, we have both $xy \in L(m)$ and $x'y \in L(m)$. Otherwise, if there is no $\bar{\imath} \in \{1, \dots, n\}$ such that the entry $(x, y)$ belongs to $M_{\bar{\imath}}$, then $B(x) = B(x')$ implies both $xy \notin L(m)$ and $x'y \notin L(m)$. We can conclude that, if $B(x) = B(x')$ then $xy \in L(m)$ if and only if $x'y \in L(m)$, i.e. $x \equiv_{L(m)} x'$.

Then, define $x \sim x'$ with $x, x' \in \Sigma^{m,*}$ if and only if $B(x) = B(x')$. This equivalence relation induces $2^{|S|}$ equivalence classes and it is a refinement of the Myhill-Nerode equivalence relation. Therefore, $R_L(m)$, that is equal to the number of classes of the $\equiv_{L(m)}$, by Myhill-Nerode Theorem, is less than or equal to the number of classes of the $\sim$ relation, that is $2^{C_L(m)}$. $\square$

Note that Propositions 4.1, 4.2 and Theorem 3.1 together, provide an alternative proof of conditions *1*, *2*, *3*.

**Corollary 4.1** *Let $L \subseteq \Sigma^{**}$. If $L \in$ REC then $R_L(m)$ is $2^{2^{o(m)}}$, $P_L(m)$ and $F_L(m)$ are $2^{O(m)}$.*

Propositions 4.1 and 4.2 show that condition *4* extends conditions *1*, *2*, and *3*. A typical use of such conditions is as a tool to disprove the recognizability of a language, by showing that some of its complexity functions exceeds the corresponding bound. For example, one can prove that a given language is not in REC, by showing that its cover complexity function grows more than exponentially, or prove that neither a language nor its complement are in REC showing that its row complexity has a more than doubly exponential growth. Propositions 4.1 and 4.2 ensure that if we succeed using condition *1*, *2* or *3*, then we can also succeed by condition *4*. Nevertheless it is convenient to take into account all of them: the computation of the values of one complexity function could be simpler than for another one, and at the meantime, sufficient to disprove recognizability. Regarding condition *1*, note that it concerns the recognizability of a language and its complement. Indeed, since the Hankel matrix of the complement of a language can be obtained by exchanging 0 with 1, then $R_L(m) = R_{\overline{L}}(m)$. Hence if $R_L(m) = R_{\overline{L}}(m)$ exceeds any doubly exponential function then both $C_L(m)$ and $C_{\overline{L}}(m)$ exceed any exponential function, since $C_L(m) \geq \log_2 R_L(m)$ and $C_{\overline{L}}(m) \geq \log_2 R_{\overline{L}}(m)$ (see Proposition 4.2).

From Propositions 4.1 and 4.2, we know that $P_L(m) \leq F_L(m) \leq C_L(m) \leq R_L(m) \leq 2^{C_L(m)}$. Now we want to understand how much bigger can be $F_L(m)$ with respect to $P_L(m)$, $C_L(m)$ with respect to $F_L(m)$, and $R_L(m)$ with respect to $C_L(m)$. Remark that the complexity functions $P_L(m)$, $F_L(m)$ and $C_L(m)$ are related to some known lower bounds on the state complexity of regular string languages (see [6]). More exactly the bounds on $P_L(m)$, $F_L(m)$ and $C_L(m)$ follow from the application to the Hankel matrices of a two-dimensional language, of the fooling set, the extended fooling set and the non-deterministic message complexity techniques, respectively. Different lower bounds techniques give raise to different necessary conditions. In [16] it is shown that the gap between the best bounds in the three techniques can be arbitrarily large, for some given string languages. Next Propositions 4.3 and 4.4 show the analogous result for picture languages, by constructing appropriate two-dimensional counterexamples, partially inspired from the examples of string languages in [16].

**Proposition 4.3** *There exists $L \in$REC such that $P_L(m) = 3$ and $F_L(m) = m + 2$.*

**Proof:** Consider the language $L = L_{2col}$ in Example 2.3, containing pictures $p$ over $\{a, b\}$, with two columns such that there is only one symbol $b$ in each column and the entry that carries symbol $b$ in the second column of $p$ is not higher than the one in the first column. For any $m \geq 1$, consider the language $L(m)$ and the corresponding Hankel matrix $M_{L(m)}$. Then for any $1 \leq i \leq m$, denote by $p_i$ the one-column picture, with symbol $b$ in its $i$-th row, and symbol $a$ elsewhere, by $p$ the picture $p_1 ⦶ p_1$, and by $\lambda$ the picture $\lambda_{m,0}$. The Hankel matrix $M_{L(m)}$ is given by the following matrix, where $|L(m)| - 1$ copies of the row indexed by $p$ and of the column indexed by $p$ (one for any other picture of $L(m)$ with two columns) are omitted, as well as an infinite number of 0-rows and 0-columns (for any picture not in $L(m)$).

| | $p_1$ | $p_2$ | $p_3$ | $\cdots$ | $p_m$ | $\lambda$ | $p$ |
|---|---|---|---|---|---|---|---|
| $p_1$ | 1 | 1 | 1 | $\cdots$ | 1 | 0 | 0 |
| $p_2$ | 0 | 1 | 1 | $\cdots$ | 1 | 0 | 0 |
| $p_3$ | 0 | 0 | 1 | $\cdots$ | 1 | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $p_m$ | 0 | 0 | 0 | $\cdots$ | 1 | 0 | 0 |
| $p$ | 0 | 0 | 0 | $\cdots$ | 0 | 1 | 0 |
| $\lambda$ | 0 | 0 | 0 | $\cdots$ | 0 | 0 | 1 |

.

Consider now a permutation matrix $A = \|a_{ij}\|$, $1 \leq i, j \leq k$, that is a sub-matrix of $M_{L(m)}$. Let $A$ be specified by the pair of languages $(X, Y)$ with $X = \{x_1, x_2, \ldots x_k\}$ and $Y = \{y_1, y_2, \ldots y_k\}$, and suppose w.l.o.g. that, for any $1 \leq i \leq k$, $a_{x_i y_i} = 1$ and, for any $1 \leq i, j \leq k$ with $i \neq j$, $a_{x_i y_j} = 0$. Then for any $1 \leq i \leq k$, $x_i y_i \in L(m)$ and, for any $1 \leq i, j \leq k$ with $i \neq j$, $x_i y_j \notin L(m)$.

Remark that, for any $1 \leq i \leq k$, $x_i, y_i \in \Sigma^{m,0} \cup \Sigma^{m,1} \cup \Sigma^{m,2}$. Since a permutation matrix cannot have two equal rows or columns, in $X$ there is only one index $i$ at most such that $x_i = \lambda$. Similarly, we can have only one index $i$ such that $x_i \in \Sigma^{m,2}$. At last, in $A$, there is only one index $i$ such that $x_i \in \Sigma^{m,1}$: otherwise, if there were two different indexes, say $i_1$ and $i_2$, such that $x_{i_1}, x_{i_2} \in \Sigma^{m,1}$ then, since $x_{i_1}y_{i_1}, x_{i_2}y_{i_2} \in L(m)$, necessarily $x_{i_1} = p_{j_1}$, $x_{i_2} = p_{j_2}$ and $y_{i_2} = p_{j'_2}$ with $j_2 \leq j'_2$. Suppose w.l.o.g. $j_1 \leq j_2$. This implies $x_{i_1}y_{j_2} = p_{j_1}p_{j'_2} \in L(m)$ and, hence, the row indexed by $x_{i_1} = p_{j_1}$ would carry two occurrences of symbol 1, that is a contradiction. Therefore any permutation matrix that is a sub-matrix of $M_{L(m)}$, has size at most 3. The bound is tight: it suffices to consider in the figure above the sub-matrix specified by the pair of picture languages $(X, X)$ with $X = \{p_m, p, \lambda\}$.

To compute the fooling complexity of $L$ consider the sub-matrix $F = \|f_{ij}\|$ of $M_{L(m)}$ above depicted. $F$ is a fooling matrix (on the diagonal $f_{\lambda p} = f_{p\lambda} = 1$ and, for any $1 \leq i \leq m$, $f_{p_i p_i} = 1$; for any $1 \leq i \leq m$, $f_{\lambda p_i} = f_{p_i \lambda} = 0$; elsewhere, for any $1 \leq i, j \leq m$, $i \neq j$, $f_{p_i p_j} = 1$ if and only if $i < j$, and in this case $f_{p_j p_i} = 0$). Since, from Proposition 4.1, the size of a fooling sub-matrix of $M_{L(m)}$ must be less than or equal to $R_L(m) = m + 2$, it follows that $F$, that has size $m + 2$, is a maximal fooling sub-matrix of $M_{L(m)}$ and, hence, $F_L(m) = m + 2$. □

**Remark:** Consider language $L = L_{2col}$. In the proof of Proposition 4.3, we have shown that $P_L(m) = 3$ and $F_L(m) = m + 2$; moreover $R_L(m) = m + 2$. From Proposition 4.1, $F_L(m) \leq C_L(m) \leq R_L(m) = m + 2$. Hence, it easily follows $C_L(m) = m + 2$. □

In order to compare the asymptotic growth of functions $F_L(m)$, $C_L(m)$ and $R_L(m)$, let us state a technical lemma.

**Lemma 4.1** *Let $A_m$ denote the boolean square matrix of size $m$ with symbol $0$ in all the counterdiagonal positions and symbol $1$ elsewhere. The cardinality of a (minimal) covering for matrix $A_m$ is less than or equal to $2 \log_2 m$, for any $m$.*

**Proof:** The statement is proved by exhibiting, for any $m$, a covering for the matrix $A_m$ of cardinality $2 \log_2 m$. More exactly, the proof shows how to obtain a covering of cardinality $\gamma_m = \gamma_{\lceil m/2 \rceil} + 2$ for $A_m$, starting from a covering of cardinality $\gamma_{\lceil m/2 \rceil}$ for $A_{\lceil m/2 \rceil}$. Remarking that there is a covering of cardinality 2 for matrix $A_2$, it follows $\gamma_m = 2 \log_2 m$.

Indeed, let us first consider the case $m$ is even (in this case we have $\lceil m/2 \rceil = m/2$), and split $A_m$ in four square sub-blocks of size $m/2$: $A_m = \begin{array}{|c|c|} \hline B_1 & B_2 \\ \hline B_3 & B_4 \\ \hline \end{array}$. Then $B_2 = B_3 = A_{m/2}$, while $B_1$ and $B_4$ are 1-monochromatic matrices. Suppose now that $\{M'_1, \ldots, M'_{\gamma_{m/2}}\}$ is a covering for $B_2$ and that any $M'_i$ is specified by the pair of sets of indexes $(X'_i, Y'_i)$, $1 \leq i \leq \gamma_{m/2}$. Consider the set $\mathcal{B}$ of the $\gamma_{m/2}$ sub-matrices of $A_m$ specified by the pairs $(X'_i \cup X''_i, Y'_i \cup Y''_i)$ where $X''_i = \{k + m/2 \mid k \in X'_i\}$ and $Y''_i = \{k - m/2 \mid k \in Y'_i\}$. Then easily the set $\mathcal{B} \cup \{B_1, B_4\}$ is a cover for $A_m$ and the claim follows.

In the case $m$ is odd, consider the matrix $A_{m+1}$: since $m + 1$ is even, starting from a covering of cardinality $\gamma_{(m+1)/2}$ for the matrix $A_{(m+1)/2}$ and using the previous construction, we can obtain a cover $S$ for $A_{m+1}$ of cardinality $\gamma_{m+1} = \gamma_{(m+1)/2} + 2 = \gamma_{\lceil m/2 \rceil} + 2$. By erasing the first column and the last row from all the matrices in $S$, one has a cover for $A_m$, preserving cardinality $\gamma_{\lceil m/2 \rceil} + 2$ (in $S$ there is not a matrix indexed by the first column and the last row of $A_{m+1}$ only). □

**Proposition 4.4** *There exists $L \in$REC such that $F_L(m) = 3$, $C_L(m) = \Theta(\log m)$ and $R_L(m) = m$.*

**Proof:** Consider the language $L = \overline{L_{mult}}$, where $L_{mult}$ is the language defined in the Example 2.2. $L \in$ REC. For any $m$, consider the Hankel matrix $M_{L(m)}$: it can be obtained by exchanging entries $0$ with entries $1$ in the Hankel matrix for $L_{mult}$. It easy to see that $R_L(m) = m$.

From Proposition 4.2 we have $R_L(m) \leq 2^{C_L(m)}$, i.e. $C_L(m) \geq \log(R_L(m)) = \log m$. Moreover, remarking that, for any $m$, the finite part of $M_{L(m)}$ is equal to $A_m$ as defined in Lemma 4.1, and applying the same lemma, we have $C_L(m) \leq 2\log m$. Therefore $C_L(m) = \Theta(\log m)$.

Now, we will show that $F_L(m) = 3$. Let $F$ be a fooling sub-matrix of $M_{L(m)}$ of size $k$, for some integer $k$. Let $b_0$ be the number of symbols 0 that occur in $F$. Remark that $b_0 \leq k$, since any row of $F$ has one symbol 0 at most. On the other hand, the number of symbols 1 that occur in $F$, apart from the $k$ on the diagonal positions, is $k^2 - k - b_0$. Since $F$ is fooling, it must be $k^2 - k - b_0 \leq b_0$, i.e. recalling $b_0 \leq k$, $k^2 - 3k \leq 0$ that is $k \leq 3$. Hence $F_L(m) \leq 3$. But it is easy to find a fooling sub-matrix of $M_{L(m)}$ of size 3 (consider for example the sub-matrix specified by the pair $(X, Y)$ with $X = \{(m,1),(m,2),(m,3)\}$ and $Y = \{(m,m-1),(m,m-2),(m,m-3)\}$) and hence the equality $F_L(m) = 3$ follows. □

# 5 Application of the new condition

Conditions *1-4* in Theorem 4.1 state properties necessarily satisfied by recognizable two-dimensional languages. A typical application is as a tool to disprove that a given language is recognizable. For example the language $\overline{L}_M$ in Example 3.3 is not recognizable because the function $C_{\overline{L}_M}(m)$ is not exponentially bounded. Results in previous section show that condition *4* can be used with this aim for a wider class of two-dimensional languages than conditions *1-3*. This section shows examples in which it is possible to disprove recognizability by applying condition *4* and not by using the other ones. Unfortunately, none of these necessary conditions is also sufficient; finding characterizations of two-dimensional recognizable languages, based on their definitions, seems to be a hard task since it is related to hard questions in computational complexity. Counter-examples to conditions *1-3* are given in [5]; Proposition 5.1 shows a counter-example to condition *4*.

Let us consider a one-letter alphabet. Recall that, given a function $f$, *the picture language defined by* $f$, is the set $L_f = \{(m, f(m)) \mid m \geq 0\}$ (cf. [11], see also [13]). The sequel will concern languages $L_f$, together with their closure $L_f^{*\mathbb{D}}$, that will be denoted $L_{f^*}$ for short, and their complements $\overline{L_f}$ and $\overline{L_{f^*}}$. In [11], it is proved that, if $f(m)$ is a super-exponential function, then $L_f \notin$ REC. Moreover, under the same hypothesis, $\overline{L_f} \notin$ REC (the result is in [5] and could be inferred by some properties in [15, 24]).

**Proposition 5.1** *Let* $L \subseteq \Sigma^{**}$.

*The condition that* $C_L(m)$ *is* $2^{O(m)}$ *does not imply that* $L \in REC$.

**Proof:** Examples of languages not in REC with covering complexity less than exponential are all languages $\overline{L_f}$ where $f(m)$ is a super-exponential function and $f(m) = 2^{O(c^m)}$, for some constant $m$ (an example is $f(m) = m!$). Indeed, if $f(m)$ is a super-exponential function, then $\overline{L_f} \notin$ REC [5]. For any $m$, the Hankel matrix associated to the language $\overline{L_f}(m)$ is given by the following matrix "surrounded" by an infinite number of columns and rows of 1's, where $\lambda = \lambda_{m,0}$ and $v = (m, 1)$.

| | $\lambda$ | $v$ | $\cdots$ | $v^{f(m)-2}$ | $v^{f(m)-1}$ | $v^{f(m)}$ |
|---|---|---|---|---|---|---|
| $\lambda$ | 1 | 1 | $\cdots$ | 1 | 1 | 0 |
| $v$ | 1 | 1 | $\cdots$ | 1 | 0 | 1 |
| $v^2$ | 1 | 1 | $\cdots$ | 0 | 1 | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $v^{f(m)-1}$ | 1 | 0 | $\cdots$ | 1 | 1 | 1 |
| $v^{f(m)}$ | 0 | 1 | $\cdots$ | 1 | 1 | 1 |

.

From Lemma 4.1, $C_L(m)$ is logarithmic in $f(m)$ and, hence, $C_L(m)$ is $O(c^m)$. □

Let us give examples of languages for which it is possible to disprove recognizability by applying condition *4* and not by other ones. More precisely, Proposition 5.2 shows an example where condition *4* outperforms conditions *2* and *3*, while Proposition 5.3 shows an example where condition *4* outperforms condition *1*. Unfortunately, we don't have a unique language such that the new condition is useful whereas all other conditions fail.

**Proposition 5.2** *There exists $L \subseteq \Sigma^{**}$ such that $P_L(m)$ and $F_L(m)$ are $2^{O(m)}$, whereas $C_L(m)$ is not $2^{O(m)}$.*

**Proof:** Consider the language $L = \overline{L_{f^*}}$ with $f(m) = 2^{2^{2^m}}$. For any $m$, the Hankel matrix for $L(m)$ can be obtained by gluing an infinite number of copies of the following matrix, where $\lambda = \lambda_{m,0}$ and $v = (m, 1)$.

| | $v$ | $v^2$ | $v^3$ | $\cdots$ | $v^{f(m)-2}$ | $v^{f(m)-1}$ | $v^{f(m)}$ |
|---|---|---|---|---|---|---|---|
| $\lambda$ | 1 | 1 | 1 | $\cdots$ | 1 | 1 | 0 |
| $v$ | 1 | 1 | 1 | $\cdots$ | 1 | 0 | 1 |
| $v^2$ | 1 | 1 | 1 | $\cdots$ | 0 | 1 | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $v^{f(m)-3}$ | 1 | 1 | 0 | $\cdots$ | 1 | 1 | 1 |
| $v^{f(m)-2}$ | 1 | 0 | 1 | $\cdots$ | 1 | 1 | 1 |
| $v^{f(m)-1}$ | 0 | 1 | 1 | $\cdots$ | 1 | 1 | 1 |

.

Now, $R_L(m) = f(m) = 2^{2^{2^m}}$ and therefore $C_L(m) \geq \log(R_L(m)) = 2^{2^m}$ applying Proposition 4.2. Moreover, it is easy to see that $P_L(m) = 2$ and $F_L(m) = 3$, using a technique similar to the one used in Proposition 4.4. □

**Proposition 5.3** *There exists $L \subseteq \Sigma^{**}$ such that $R_L(m)$ and $R_{\overline{L}}(m)$ are $2^{2^{O(m)}}$, whereas $C_L(m)$ is not $2^{O(m)}$.*

**Proof:** Consider the language $L = L_f$ with $f(m) = 2^{2^{O(m)}}$. For any $m$, the Hankel matrix for $L(m)$ is given by the following matrix "surrounded" by an infinite number of columns and rows of 0's. Here $\lambda = \lambda_{m,0}$ and $v = (m, 1)$.

| | $v$ | $v^2$ | $v^3$ | $\cdots$ | $v^{f(m)-2}$ | $v^{f(m)-1}$ | $v^{f(m)}$ |
|---|---|---|---|---|---|---|---|
| $\lambda$ | 0 | 0 | 0 | $\cdots$ | 0 | 0 | 1 |
| $v$ | 0 | 0 | 0 | $\cdots$ | 0 | 1 | 0 |
| $v^2$ | 0 | 0 | 0 | $\cdots$ | 1 | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $v^{f(m)-3}$ | 0 | 0 | 1 | $\cdots$ | 0 | 0 | 0 |
| $v^{f(m)-2}$ | 0 | 1 | 0 | $\cdots$ | 0 | 0 | 0 |
| $v^{f(m)-1}$ | 1 | 0 | 0 | $\cdots$ | 0 | 0 | 0 |

.

Consider now the language $\overline{L}$, complement of $L$. Then, for any $m$, the Hankel matrix for $\overline{L}(m)$ can be obtained by exchanging entries 0 with entries 1 in the Hankel matrix for $L(m)$ and therefore, $R_L(m) = R_{\overline{L}}(m) = f(m) = 2^{2^{O(m)}}$. Finally it is easy to see that $C_L(m)$ is $f(m) = 2^{2^{O(m)}}$ (cf. Example 3.2). $\square$

# 6 Conclusions and open problems

The characterization of tiling recognizable languages appears as a difficult problem. The paper presents a new tool to prove that a language is not recognizable. The comparison with other similar results highlighted the possibility to gain necessary conditions for the recognizability of two-dimensional languages, from lower bound techniques for regular string languages. Indeed to estimate the number of states of a minimal NFA for a regular language is still an open problem, while computing such an NFA is PSPACE-complete. This field is therefore an active research area, where problems are tackled by different methods (communication complexity as well as graph theory, for instance). Hopefully, further results in the area could provide new insights also on two-dimensional languages.

In particular, consider the unary language given in the proof of Proposition 5.1 as an example of a language whose non-recognizability cannot be proved using the function $C_L$. Its non-recognizability follows from some result on automata over a one-letter alphabet. It should be interesting to see whether it is possible to obtain a stronger non-recognizability condition for unary languages, by reformulating in terms of Hankel matrices some known result of the unary automata theory.

Finally note that all results given in this paper are based on the investigation of the sequence of languages $L(m)$, the languages of pictures with fixed number of rows $m$. They can be straightaway translated to get further recognizability conditions concerning the languages of pictures with fixed number of columns. The combination of both bounds (on fixed number of rows and columns) could be a possible way to strengthen the conditions in order to better exploit the two-dimensional nature of picture languages.

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