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Probe interval graphs and probe unit interval graphs on superclasses of cographs

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A graph is *probe (unit) interval* if its vertices can be partitioned into two sets: a set of probe vertices and a set of nonprobe vertices, so that the set of nonprobe vertices is a stable set and it is possible to obtain a (unit) interval graph by adding edges with both endpoints in the set of nonprobe vertices. Probe (unit) interval graphs form a superclass of (unit) interval graphs. Probe interval graphs were introduced by Zhang for an application concerning the physical mapping of DNA in the human genome project. The main results of this article are minimal forbidden induced subgraphs characterizations of probe interval and probe unit interval graphs within two superclasses of cographs: P_4-tidy graphs and tree-cographs. Furthermore, we introduce the concept of *graphs class with a companion* which allows to describe all the minimally non–(probe $\mathcal{G}$) graphs with disconnected complement for every graph class $\mathcal{G}$ with a companion.

Keywords: P_4-tidy graphs, forbidden induced subgraphs, probe interval graphs, probe unit interval graphs, tree-cographs

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1 Introduction

A graph G is an *interval graph* if it is the intersection graph of a family of open intervals in the real line; i.e., there is a one-to-one correspondence between the vertex set of G and a collection $\mathcal{S}$ of intervals in the real line such that two vertices of G are adjacent if their corresponding intervals intersect. If so, $\mathcal{S}$ is called an *interval model* of G. A *unit interval graph* is an interval graph admitting an interval model with all its intervals having the same length. Interval and unit interval graphs have well-known characterizations by minimal forbidden induced subgraphs [15, 18]. A graph is *probe (unit) interval* if its vertices can be partitioned into two sets: a set of probe vertices and a set of nonprobe vertices, so that the set of nonprobe vertices is a stable set and it is possible to obtain a (unit) interval graph by adding edges with both endpoints in the set of nonprobe vertices.

Probe interval graphs form a superclass of interval graphs and were introduced by Zhang [23] as a research tool in the frame of the human genome project. An open question from a combinatorial viewpoint is that of characterizing the classes of probe interval and probe unit interval graphs by forbidden induced subgraphs. Partial results in this direction were obtained in the following articles. Brown and Lundgren [2] prove that the class of bipartite probe interval graphs and the class of complements of 2-clique circular-arc graphs are each equivalent to the class of point interval bigraphs. In the same article the authors characterize those probe bipartite graphs such that the partition into probe and nonprobe vertices coincides with the bipartition (as bipartite graphs) but the problem of characterizing bipartite probe interval graphs remains open. However, they present a conjecture describing the complete family of forbidden induced subgraphs for the class of bipartite probe interval graphs. Sheng [19] characterizes probe interval graphs by forbidden induced subgraphs within the class of cycle-free graphs. Brown, Lundgren, and Sheng [3] do the same work for the class of probe unit interval graphs. Pržulj and Corneil [17] study the class of probe interval graphs within the class of 2-trees, finding a wide list of forbidden induced subgraphs which belong to this class.

The main results of this paper are characterizations of probe interval graph and probe unit interval graphs by a set of minimal forbidden induced subgraphs within two superclasses of cographs: tree-cographs and P_4-tidy graphs. Both superclasses of cographs are defined recursively from the disjoint union and the join operation. The study of minimal forbidden induced subgraphs of probe interval graphs and probe unit interval graphs within these classes in connection with the join operation gives rise to introduce the concept of companion of a graph class (cf. Section 5). The results on probe unit interval graphs presented in this article follow from the connection between the behavior of the join operation and the companion class of unit interval graphs. The characterizations of probe interval and probe unit interval graphs within the class of tree-cographs are also based on the characterizations presented in [19] and [3] respectively, as well as on Theorem 10, which characterizes probe interval graphs and probe unit interval within the class of co-bipartite graphs. The last result follows from results on domination in graphs presented in [4] leads to characterizations of probe interval graphs and probe unit interval graphs within the class of complements of trees.

The paper is organized as follows. In Section 2, we introduce some terminology and state some results used throughout the article. Section 3 is devoted to characterize by forbidden induced subgraphs probe interval graphs and probe unit interval graphs within co-bipartite graphs. In Section 4, we present characterizations by forbidden induced subgraphs within two superclasses of cographs, namely tree-cographs and P_4-tidy graphs. The results given in Section 4 can be proved using a theorem presented in Section 5. Nevertheless, we decided to show an alternative proof that implicitly uses this theorem, for the convenience of the reader. In Section 5 the concept of hereditary class with a companion is introduced. Theorem 22 is the main result of Section 5. Mostly, the characterizations of Section 7 are based on Theorem 22. In Section 6, characterizations for probe $\{3K_1, C_4, C_5\}$-free graphs are presented. The class of $\{3K_1, C_4, C_5\}$-free graphs is the companion of the class of unit interval graphs. Finally, Section 7 is devoted to forbidden induced subgraphs characterizations of probe unit interval graphs within tree-cographs and P_4-tidy graphs.

2 Preliminaries

For concepts and notation not defined here we refer the reader to [22]. All graphs in this article are simple (i.e., without loops or multiple edges). Let G be a graph. The vertex set of G is denoted by $V(G)$ and the edge set by $E(G)$. If $u, v \in V(G)$ and $uv \notin E(G)$, uv is called a *nonedge* of G. For each vertex v of G, $N_G(v)$ denotes the *neighborhood* of v in G and $N_G[v]$ denotes *closed neighborhood* $N_G(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $d_G(v)$, is the number of vertices in $N_G(v)$

Given a subset $A \subseteq V(G)$, $G[A]$ stands for the subgraph of G induced by A. Two vertices v and w of G are *false twins* (resp. *true twins*) if $N_G(v) = N_G(w)$ (resp. $N_G[v] = N_G[w]$). The *complement graph* of G is denoted $\overline{G}$. Let G_1 and G_2 two graphs. The *disjoint union of* G_1 *and* G_2 is denoted by $G_1 + G_2$. If $\mathcal{H}$ is a collection of graphs, a graph G is $\mathcal{H}$*-free* if G contains no induced subgraph isomorphic to some graph belonging to $\mathcal{H}$. Given a graph $H \in \mathcal{H}$, we say that G *contains no induced* H if G contains no induced subgraph isomorphic to H.

Let $\mathcal{G}$ be a family of graphs. A graph belonging to $\mathcal{G}$ is called a $\mathcal{G}$-graph. The family of graphs formed by the complements of the graphs belonging to $\mathcal{G}$ is denoted *co-*$\mathcal{G}$. If $G \in \mathcal{G}$ implies that every induced subgraph of G is a $\mathcal{G}$-graph, $\mathcal{G}$ is said to be *hereditary*. If $\mathcal{G}$ is a hereditary class, a graph H is a *minimal forbidden induced subgraph of* $\mathcal{G}$, or more briefly, *minimally non-*$\mathcal{G}$, if H does not belong to $\mathcal{G}$ but every proper induced subgraph of H is a $\mathcal{G}$-graph. Let (N, P) be a partition of $V(G)$ where N is a stable set. A *completion* of G is a graph G^* obtained from G by adding an edge set F whose endpoints belong to the stable set N. Let $\mathcal{G}$ be a hereditary class of graphs. G is defined to be *probe* $\mathcal{G}$ if there exists a partition (N, P) of $V(G)$ and a completion G^* of G belonging to $\mathcal{G}$. Under such conditions, (N, P) is called a *probe* $\mathcal{G}$ *partition* of G, and the vertices belonging to N (resp. P) are called *nonprobe vertices* (resp. *probe vertices*). $\mathcal{P}(\mathcal{G})$ denotes the class of probe $\mathcal{G}$ graphs. Probe classes of graphs have been studied for instance in [1, 4, 5, 6, 9, 14].

The induced path (resp. induced cycle) on n vertices is denoted P_n (resp. C_n). The *complete graph* on n vertices is denoted K_n and nK_1 stands for a stable set on n vertices. Particularly, C_3 is called *triangle*. A *paw* is a triangle with a pendant vertex. A *diamond* is a four vertex complete graph less an edge. A graph is *bipartite* if its vertex set can be partitioned into two stable sets. If, in addition, every vertex of one stable set is adjacent to every vertex of the other stable set the graph is defined as *complete bipartite*. $K_{n,m}$ stands for the complete bipartite graph partitioned into two stable sets with n and m vertices, respectively. For some graphs needed in what follows, see Figure 1.

Let A be a vertex set that induces P_4 in G. A vertex v of G is said to be a *partner* of A if $G[A \cup \{v\}]$ contains at least two induced P_4's and G is called P_4*-tidy* if each vertex set A inducing a P_4 in G has at most one partner [8]. The class of P_4-tidy graphs is an extension of the class of cographs as well as several other classes defined by bounding the number of P_4's according to different criteria; e.g., P_4-sparse graphs [11], P_4-lite graphs [12], and P_4-extendible graphs [13]. A *spider* [11] is a graph whose vertex set can be partitioned into three sets S, C, and R, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set;

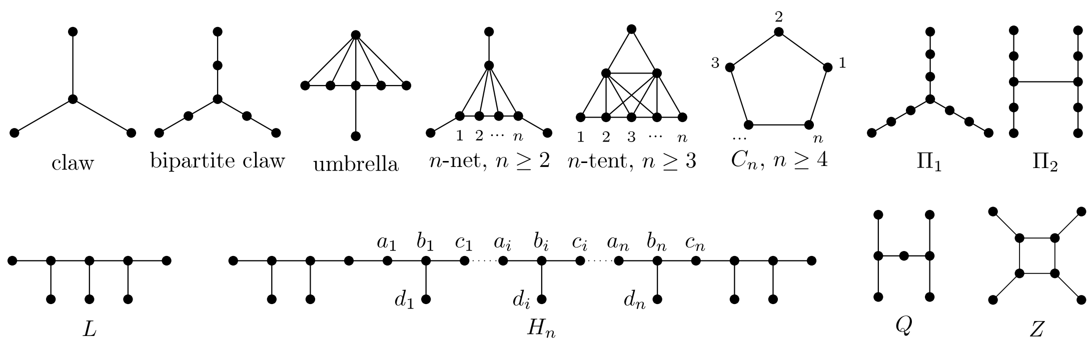

Fig. 1: List of graphs used throughout the paper.

$C = \{c_1, \ldots, c_k\}$ is a clique; s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and nonadjacent to all the vertices in S. The triple (S, C, R) is called the *spider partition*. By $\text{thin}_k(H)$ and $\text{thick}_k(H)$ we respectively denote the thin spider and the fat spider with $|C| = k$ and H the subgraph induced by R. If R is an empty set we denote the thin spider and the thick spider by thin_k and thick_k, respectively. Clearly, the complement of a thin spider is a thick spider, and vice versa. A *fat spider* is obtained from a spider by adding a true or false twin of a vertex $v \in S \cup C$. The following theorem characterizes P_4-tidy graphs.

Theorem 1 [8] *Let G be a P_4-tidy graph with at least two vertices. Then, exactly one of the following conditions holds:*

1. *G is disconnected.*
2. *$\overline{G}$ is disconnected.*
3. *G is isomorphic to P_5, $\overline{P_5}$, C_5, a spider, or a fat spider.*

Tree-cographs [20] are another generalization of cographs. They are defined recursively as follows: trees are tree-cographs; the disjoint union of tree-cographs is a tree-cograph; and the complement of a tree-cograph is also a tree-cograph.

The problem of characterizing interval graphs by forbidden induced subgraphs was solved by Boland and Lekkerkerker in [15].

Theorem 2 [15] *A graph is an interval graph if and only if it contains no induced bipartite-claw, umbrella, n-net for any $n \geq 2$, n-tent for any $n \geq 3$, or C_n for any $n \geq 4$.*

Roberts was able to prove that an interval graph is unit if and only if it is claw-free [18]. Combining this result and the above theorem, the characterization below by forbidden induced subgraphs for unit interval graphs follows.

Corollary 3 *A graph is a unit interval graph if and only if it contains no induced claw, 2-net, 2-tent, or C_n for any $n \geq 4$.*

For other characterizations of unit interval graphs, the reader is referred to the works of Roberts [18] and Wegner [21].

Probe interval and probe unit interval graphs have been already characterized by forbidden induced subgraphs within the trees. In this work, we present generalizations of these theorems, which characterize in terms of forbidden induced subgraphs probe interval graphs and probe unit interval graphs within tree-cographs (see Sections 4 and 7).

Theorem 4 [19] *Let G be a tree. Then G is a probe interval graph if and only if G contains no induced Π_1 or Π_2.*

Theorem 5 [3] *Let G be a tree. Then G is a probe unit interval graph if and only if G contains no induced bipartite-claw, L, or H_n for any $n \geq 0$.*

The *length* of an induced cycle is the number of its edges. The following lemma shows an infinite family of forbidden induced subgraphs for the class of probe interval graphs.

Lemma 6 [10] *No odd cycle of length at least five is probe interval.*

Notice that this result is also true if we replace probe interval by probe unit interval.

Let $\mathcal{K}$ be the class of graphs formed by the complete graphs. Let us study the class $\mathcal{P}(\mathcal{K})$, used throughout this article.

Lemma 7 *A graph G is probe $\mathcal{K}$ if and only if G does not contain any induced C_4 or $\overline{P}_3$.*

Proof: It is easy to see that neither C_4 nor $\overline{P}_3$ is a probe $\mathcal{K}$ graph. Conversely, if G is a $\{\overline{P}_3, C_4\}$-free graph, then $\overline{G}$ is a complete graph plus a disjoint union of isolated vertices. So, G can be partitioned into a stable set and a clique in such a way that every vertex of the stable set is adjacent to every vertex of the clique. Therefore, G is a probe complete graph. $\square$

We will also refer to a probe complete graph as a *split complete graph*.

3 Co-bipartite graphs and trees

In this section we present a forbidden induced subgraph characterization for probe interval graphs within the class of those graphs whose complement is bipartite. Furthermore, we show that, restricted to the class of co-bipartite graphs, probe interval graphs, probe $\{3K_1, C_4, C_5\}$-free graphs and probe unit interval graphs are the same classes. The class of probe $\{3K_1, C_4, C_5\}$-free graphs will play a very important role in the following sections. The main results of this section are Theorem 10 which implies Corollary 11 (a result used to characterize probe interval graphs and probe unit interval graphs within the class of complement of trees) and Theorem 12 that characterizes $\{3K_1, C_4, C_5\}$-free graphs within the class of complement of trees. Theorem 12 will play a central role in the characterization of probe unit interval graphs within the class of tree-cographs (cf. Section 7).

Given a graph G, a set $D \subseteq V(G)$ is called a *dominating set* if every vertex $v \in V(G)$ either belongs to D or is adjacent to a vertex in D.

Lemma 8 [16] *Let G be a triangle-free graph. Then G is $\{P_6, C_6\}$-free if and only if every induced connected subgraph of G has a dominating complete bipartite subgraph isomorphic to $K_{n,m}$ with $n, m \geq 1$.*

The following lemma follows from Lemma 8.

Lemma 9 *Let $G = (V, E)$ be a connected bipartite $\{2P_3, 3K_2, P_6, C_6, Z\}$-free graph. Then, either G has diameter at most 3, or there exists a pendant vertex $v \in V$ such that $H = G - v$ has diameter at most 3.*

Proof: Let G be a connected bipartite $\{2P_3, 3K_2, P_6, C_6, Z\}$-free graph. By Lemma 8, there exists a dominating complete bipartite graph $H = (V', E')$ such that V' can be partitioned into two stable sets A and B. Since G is bipartite, either $N_H(v) \cap A = \emptyset$ or $N_H(v) \cap B = \emptyset$ for every vertex $v \in V - (A \cup B)$. We will call A' the set of vertices of $V - V'$ whose neighbors belong to A and B' the set of vertices of $V - V'$ whose neighbors belong to B. Assume that there exist two vertices $u, v \in V$ such that $d(u, v) = 4$. Notice that either $u, v \in A'$ or $u, v \in B'$. Suppose without loss generality, that $u, v \in A'$, let $u' \in A$ and $v' \in A$ be a neighbor of u and v, respectively. Since G is Z-free, $\bigcap_{b' \in B'} N(b') \neq \emptyset$; i.e., all vertices in B' have a common neighbor. Consequently, given a vertex $b' \in B'$, $d(b', z) \leq 3$ for all vertices $z \in V$. On the other hand, since G is $3K_2$-free, every vertex $w \in A'$ is either adjacent to u' or adjacent to v'. Since G is $2P_3$-free, either u or v is a pendant vertex. If $A' = \{u, v\}$, u or v would satisfy the condition of the lemma. So, we can assume that $A' - \{u, v\} \neq \emptyset$. Suppose, without loss of generality that u is the pendant vertex. Since G is $2P_3$-free, if $u_1 \in A' - \{u\}$ is adjacent to u' and $v_1 \in A' - \{v\}$ is adjacent to v', then u_1 is adjacent to v' or v_1 is adjacent to u'. Consequently, u is a pendant vertex satisfying the conditions of the lemma. □

Lemma 9 implies the following characterization.

Theorem 10 *Let G be a co-bipartite graph. Then, the following statements are equivalent:*

1. *G is a probe interval graph;*
2. *G is a probe unit interval graph;*
3. *G is probe $\{3K_1, C_4, C_5\}$-free;*
4. *G is $\{\overline{2P_3}, \overline{3K_2}, \overline{P_6}, \overline{C_6}, \overline{Z}\}$-free.*

Proof: It is easy to see that $\overline{2P_3}, \overline{3K_2}, \overline{P_6}, \overline{C_6}$ and $\overline{Z}$ are neither probe interval, nor probe unit interval, nor probe $\{3K_1, C_4, C_5\}$-free.

Conversely, let $G = (V, E)$ be a $\{\overline{2P_3}, \overline{3K_2}, \overline{P_6}, \overline{C_6}, \overline{Z}\}$-free co-bipartite graph. Consider the complement graph of G, $\overline{G}$. By Theorem 2, if $\overline{G}$ had diameter at most 3; i.e., $\overline{G}$ were $2K_2$-free, then G would be an interval graph. Therefore, we can assume that $\overline{G}$ has diameter 4. By Lemma 9, there exists a pendant vertex $v \in V$, whose neighbor we will call v', such that $H = \overline{G}[V - v]$ has diameter at most 3. Consequently, the completion $G^*(N \cup P, E \cup F)$, where $N = \{v, v'\}$, $P = V - N$ and $F = \{vv'\}$, is an interval graph. Finally, since G^* is also co-bipartite and interval, G is $\{C_4, C_5\}$-free, $3K_1$-free (consequently, claw-free) and thus $\{3K_1, C_4, C_5\}$-free and unit interval. Therefore, G is probe $\{3K_1, C_4, C_5\}$-free and probe unit interval. □

As a consequence of Lemma 9, we obtain the following corollary.

Corollary 11 *Let G be the complement of a tree. Then the following assertions are equivalent:*

1. *G is a probe $\{3K_1, C_4, C_5\}$-free graph.*
2. *G is a probe unit interval graph.*
3. *G is a probe interval graph.*
4. *G is $\{\overline{3K_2}, \overline{2P_3}, \overline{P_6}\}$-free.*
5. *G is $\{$co-bipartite-claw$, \overline{Q}, \overline{P_6}\}$-free.*

(Here, item 4. is a minimal forbidden induced subgraph characterization, while item 5. is a minimal forbidden connected induced subgraph characterization.)

Proof: The equivalence between the first four statements follows from Theorem 10.

Let us see the equivalence between 4. and 5.. Since $\overline{G}$ is $\{3K_2, 2P_3, P_6\}$-free, then $\overline{G}$ is $\{$bipartite-claw, H, $P_6\}$-free. Conversely, let $\overline{G}$ be a P_6-free tree. If $\overline{G}$ contains an induced $2P_3$ or an induced $3K_2$, then $\overline{G}$ contains either an induced H or an induced bipartite-claw. □

The following theorem gives a forbidden induced subgraph characterization of probe $\{3K_1, C_4, C_5\}$-free graphs among trees. The class of probe $\{3K_1, C_4, C_5\}$-free graphs will be useful in the following sections when dealing with the class of probe unit interval graphs. The graph E, mentioned in the following statement, is the graph obtained from the bipartite-claw by deleting a vertex of degree one.

Theorem 12 *Let G be a tree. Then the following assertions are equivalent:*

1. *G is a probe $\{3K_1, C_4, C_5\}$-free graph.*
2. *G contains no induced $2K_2 + K_1$ or $P_4 + K_1$.*
3. *G contains no induced E or P_6.*

(Here, item 2. is a minimal forbidden induced subgraph characterization, while item 3. is a minimal forbidden connected induced subgraph characterization.)

Proof: We first prove the equivalence of 1. and 3.. It is straightforward to verify that neither E nor P_6 is a probe $\{3K_1, C_4, C_5\}$-free graphs. Conversely, suppose that G is $\{E, P_6\}$-free. Let $P = v_1v_2v_3 \cdots v_n$ be a chordless path of maximum length of G. Since G is a tree and P is of maximum length, v_1 and v_n are pendant vertices of G. Since G contains no induced P_6, $n \leq 5$. Since G is an E-free tree and P is of maximum length, for each $i \in \{2, \ldots, n-1\}$, the neighbors of v_i in G different from v_{i-1} and v_{i+1} are pendant vertices of G. If $n \leq 3$, G is probe complete and, in particular, it is a probe $\{3K_1, C_4, C_5\}$-free graph. So, assume that $4 \leq n \leq 5$. If $n = 5$, then $d_G(v_3) = 2$ because G contains no induced E. Let $N_1 = N_G(v_2) - \{v_3\}$ and let $N_2 = N_G(v_{n-1}) - \{v_2\}$ (so, if $n = 5$, $N_2 = N_G(v_{n-1})$). Hence, we split $V(G)$ into $N = N_1 \cup N_2$ which is clearly a stable set of G and $P = V(G) - N$. The graph G^* that arises from G by adding all the edges joining two vertices of N_1 and all the edges joining two vertices of N_2 is $3K_1$-free and chordal. Thus, G^* is a $\{3K_1, C_4, C_5\}$-free completion of G.

We now prove the equivalence of 2. and 3.. It can be easily seen that if G is $\{2K_2 + K_1, P_4 + K_1\}$-free, then G does not contain any induced E and P_6. Conversely, if G is P_6-free and contains either an induced $2K_2 + K_1$ or an induced $P_4 + K_1$ and G is a tree and thus connected, then it contains an induced E. Consequently, if G is a $\{E, P_6\}$-free tree, then G contains no induced $2K_2 + K_1$ or $P_4 + K_1$.

□

4 Probe interval graphs

Given two graphs G and H, the *join* of G and H, denoted $G \vee H$, is the graph obtained from the disjoint union of G and H by adding the edges $\{uv : u \in V(G), v \in V(H)\}$.

Lemma 13 *Let G_1 and G_2 be two graphs. Then $G_1 \vee G_2$ is an interval graph if and only if one of G_1 and G_2 is interval and the other one is a complete graph.*

Proof: Since interval graphs are a hereditary class, if $G_1 \vee G_2$ is an interval graph then G_1 and G_2 are interval graphs. Suppose that none of G_1 and G_2 is a complete graph. Then, there exist two nonadjacent vertices $v_1^i, v_2^i \in V(G_i)$ for $i = 1, 2$. Consequently, $\{v_1^1, v_2^1, v_1^2, v_2^2\}$ induces C_4 in $G_1 \vee G_2$ and thus $G_1 \vee G_2$ is not interval. Conversely, suppose that G_1 or G_2 is interval and the other one a complete graph, say G_1 is interval and G_2 is a complete graph. So, we can construct an interval model for $G_1 \vee G_2$ from the interval model $\mathcal{S}$ of G_1 by adding as many intervals as the number of vertices of G_2, covering the whole interval model $\mathcal{S}$. □

Lemma 14 *Let G_1 and G_2 be two graphs. Then $G_1 \vee G_2$ is a probe interval graph if and only if one of the following assertions holds:*

1. *One of G_1 and G_2 is a complete graph and the other one is probe interval.*
2. *One of G_1 and G_2 is probe complete and the other one is interval.*

Proof: Let G_1 and G_2 be two graphs and let $H = G_1 \vee G_2$ be probe interval. Therefore, there exists a probe interval completion $H^* = (N \cup P, E \cup F)$ of H such that H^* is an interval graph. Since $H = G_1 \vee G_2$, either $N \subseteq V(G_1)$ or $N \subseteq V(G_2)$. Assume, without loss of generality, that $N \subseteq V(G_1)$; i.e., $H^* = G_1^* \vee G_2$ with $G_1^* = (V(G_1), E(G_1) \cup F)$. By Lemma 13, since H^* is an interval graph, one of G_1^* and G_2 is a complete graph and the other one is interval. So, either G_1 is probe complete and G_2 is an interval graph or G_1 is a probe interval and G_2 is a complete graph. □

Notice the following immediate class inclusion:

$$\text{complete} \subseteq \text{probe complete} \subseteq \text{interval} \subseteq \text{probe interval}.$$

The following theorem characterizes all minimal non-(probe interval) graphs whose complement is disconnected.

Theorem 15 *The only minimally non-(probe interval) graphs whose complement is disconnected are bipartite-claw $\vee\, 2K_1$, umbrella $\vee\, 2K_1$, n-net $\vee\, 2K_1$ where $n \geq 2$, n-tent $\vee\, 2K_1$ for any $n \geq 3$, $\overline{3K_2}$, and $\overline{2P_3}$.*

Proof: Let G be a minimally non-(probe interval) graph whose complement is disconnected. Since $\overline{G}$ is disconnected, there exist two graphs G_1 and G_2 such that G is the join between them; i.e., $G = G_1 \vee G_2$. By minimality, G_1 and G_2 are probe interval. Since $G = G_1 \vee G_2$ is non-(probe interval), by Lemma 13, none of G_1 and G_2 is a complete graph.

Suppose that one of G_1 and G_2 is probe complete, say G_2. Then, G_1 is not interval because otherwise $G_1 \vee G_2$ would be probe interval. Since, for each $v_1 \in V(G_1)$, $(G_1 - v_1) \vee G_2$ is probe interval and G_2 is

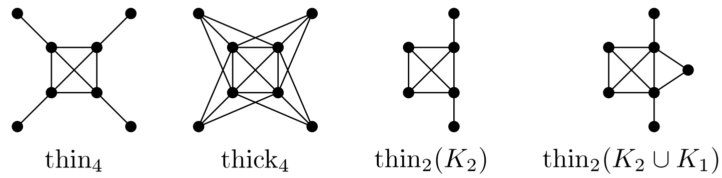

Fig. 2: Some spiders.

not a complete graph, $G_1 - v_1$ is an interval graph. Thus, G_1 is a minimally non interval graph. Since, for each $v_2 \in V(G_2)$, $G_1 \vee (G_2 - v_2)$ is probe interval and G_1 is not interval, $G_2 - v_2$ is a complete graph. Since G_2 is not a complete graph, $G_2 = 2K_1$. Since G_1 is probe interval, G_1 is not a cycle of length at least 5 (see Lemma 6). We conclude that G equals bipartite-claw $\vee\, 2K_1$, umbrella $\vee\, 2K_1$, n-net $\vee\, 2K_1$ for some $n \geq 2$, n-tent $\vee\, 2K_1$ for some $n \geq 3$, or $C_4 \vee 2K_1 = \overline{3K_2}$.

We can now assume that G_1 and G_2 are non-(probe complete). Since $(G_1 - v_1) \vee G_2$ is probe interval, $G_1 - v_1$ is probe complete, for each $v_1 \in V(G_1)$. So G_1 is a minimally non-(probe complete), i.e., $\overline{P_3}$ or C_4 (see Lemma 7). Symmetrically, G_2 is $\overline{P_3}$ or C_4. If $G_1 = C_4$ or $G_2 = C_4$, then G contains a proper induced $C_4 \vee 2K_1$, a contradiction. Therefore $G = \overline{2P_3}$. □

The following theorem characterizes those probe interval graphs among tree-cographs.

Theorem 16 *Let G be a tree-cograph. Then, G is a probe interval graph if and only if G contains no induced Π_1, Π_2, bipartite-claw $\vee\, 2K_1$, $\overline{3K_2}$, $\overline{2P_3}$, or $\overline{P_6}$.*

Proof: It suffices to prove that if G is a tree-cograph non-(probe interval) graph, then G contains an induced Π_1, Π_2, bipartite-claw $\vee\, 2K_1$, $\overline{3K_2}$, $\overline{2P_3}$, or $\overline{P_6}$.

So, assume that G is a tree-cograph that is not probe interval. Therefore, G contains an induced subgraph H that is a minimally non-(probe interval) graph. Since G is a tree-cograph, H is also a tree-cograph. Consequently, H is disconnected, or the complement of H is disconnected, or H is a tree, or H is the complement of a tree. By minimality of H, H is not disconnected because the disjoint union of probe interval graphs is also a probe interval graph. If the complement of H is disconnected, then (by Theorem 15) H equals bipartite-claw $\vee\, 2K_1$, $\overline{3K_2}$ or $\overline{2P_3}$ (notice that umbrella $\vee\, 2K_1$, n-net $\vee\, 2K_1$ for any $n \geq 2$, and n-tent $\vee\, 2K_1$ for any $n \geq 3$ are not tree-cographs). If H is a tree, Theorem 4 implies that H equals Π_1 or Π_2. Finally, consider the case when H is the complement of a tree. By Theorem 11, H equals $\overline{3K_2}$, $\overline{2P_3}$, or $\overline{P_6}$. □

In order to characterize those probe interval graphs within P_4-tidy graphs, we need the following lemma that characterizes those spiders that are probe interval.

Lemma 17 *Let H be a spider with spider partition (C, S, R). Then, H is probe interval if and only if one of the following conditions holds:*

1. *$|C| = 3$ and $H[R]$ is interval.*

2. *$|C| = 2$ and $H[R]$ is probe interval.*

Moreover, if H is probe interval, then a fat spider H' that arises from H is also probe interval except when $|C| = 2$, H' arises by making a false twin of a vertex of C, and $H[R]$ is not interval.

Proof: Let $H = (V, E)$ be a thick (thin) spider with partition (C, S, R) that is probe interval with a completion $H^* = (V, E \cup F)$ and probe interval partition (N, P). Suppose that $|C| \geq 4$ and let c_1, c_2, c_3, c_4 be different vertices in C. Notice that if a tent (net) is an induced subgraph of H, then exactly a vertex of degree four (three) belongs to N. Let s_1, s_2, s_3, s_4 be different vertices of S such that s_i adjacent to any vertex in C but c_i (s_i is only adjacent to c_i) for all $1 \leq i \leq |C|$. So, $\{c_1, c_2, c_3, s_1, s_2, s_3\}$ induces a tent (net) and thus one of c_1, c_2, c_3 belongs to N, say c_1. Analogously, $\{c_2, c_3, c_4, s_2, s_3, s_4\}$ also induces a tent (net) and one of c_2, c_3, c_4 belongs to N, but all all graphs of the lemma are adjacent to c_1, a contradiction. Consequently, $2 \leq |C| \leq 3$. Assume that $|C| = 3$. Since H is probe interval, $H[C + R] = H[C] \vee H[R]$ is also probe interval. On the one hand, since $H[C]$ is a complete graph, by Lemma 14, $H[R]$ is probe interval. On the other hand, since $\{s_1, s_2, s_3, c_1, c_2, c_3\}$ induces a tent (net), one of the vertices in C is nonprobe and thus any vertex in R is probe. So, $H[R]$ is interval. Now, assume that $|C| = 2$. Since C is a complete graph and H is probe interval, $H[C] \vee H[R]$ is probe interval. Thus, by Lemma 14, $H[R]$ is probe interval. Conversely, it is straightforward to construct a probe interval model of a thick (thin) spider that satisfies condition 1. or 2..

Let H' be a fat spider that arises from H. If H' arises by making a twin of a vertex $s \in S$, then H' is also probe interval. Indeed, if $H^* = (V(H), E(H) \cup F)$ is a probe interval completion of H with a probe interval partition (N, P) chosen (by symmetry) in such a way that $s \in N$ if s' is a false twin and $s \in P$ if s' is a true twin, then (N, P) can be extended to a probe interval partition $(N'P')$ of H' by taking also the twin s' of s as a nonprobe vertex ($N' = N \cup \{s\}$) if it is a false twin and as a probe vertex ($P' = P \cup \{s\}$) if it is a true twin. Therefore, $H'^* = (N' \cup P, E(H') \cup F')$, with $F' = F \cup \{ss'\} \cup \{vs' : vs \in F\}$ if s' is a false twin and $F' = F$ if s' is a true twin, is interval because the graph obtained by adding true twins to an interval graph is also interval. Suppose now that H' arises from a vertex $c \in C$ by making a true twin. Then H' is probe interval. In fact, any partition (N, P) of H where $c \in P$ can be extended to a partition of H' where the new vertex is also a probe vertex. Finally, consider the case where H arises by making a false twin of a vertex $c \in C$. If $H[R]$ were not interval and thus $|C| = 2$, then H would contain $H[R] \vee 2K_1$, where $H[R]$ is a forbidden induced subgraph for the class of interval graphs. Therefore, H' would not be probe interval. Notice that if $H[R]$ is interval, then clearly H' is an interval, simply look for a partition where c and the false twin of it are both nonprobe, and the vertices of R are all probe. $\square$

Let $\mathcal{H}$ be the set formed by all the minimally non interval graphs except for the induced cycles with at least five vertices. Graphs belonging to $\mathcal{H}$ are probe interval. In addition, it can be proved that every probe interval partition of a graph belonging to $\mathcal{H}$ contains at least two nonadjacent probe interval vertices. Therefore, graphs arising from a graph belonging to $\mathcal{H}$ by adding two nonadjacent universal vertices is a minimally non-(probe interval).

Lemma 18 *The minimally non-(probe interval) graphs that are spiders or fat spiders are the graphs: thick$_3(H)$, thin$_3(H)$, for $H \in \mathcal{H}$, thin$_4$ and thick$_4$.*

Proof: It is straightforward to check that all the graphs of the lemma are minimally non-(probe interval). Let G be a thick (thin) spider with partition (C, S, R) that is minimally non-(probe interval). If $|C|$ were of size at least 4, then G would contain thick$_4$ (thin$_4$) as induced subgraph and by minimality $G =$ thick$_4$ ($G =$ thin$_4$). Now, we may assume that $|C| \leq 3$. Indeed, by Lemma 17, since G is non-(probe interval), it suffices to consider $|C| = 3$. By minimality, $G[R]$ is probe interval and by Lemma 17 it cannot be an interval graph. By Theorem 2, the only minimal not interval graphs that are probe interval are those graphs belonging to $\mathcal{H}$. Therefore, G contains thick$_3(H)$ (thin$_3(H)$) for some $H \in \mathcal{H}$ as induced subgraph. Furthermore, by minimality, G is exactly that graph.

Let G be a fat spider that is a minimally non-(probe interval). By Lemma 17, G arises from a spider with $|C| = 2$ and $H[R]$ is not an interval graph by making a false twin of a vertex in C. Assume that $C = \{c_1, c_2\}$, $S = \{s_1, s_2\}$ and v is a false twin of c_1. By minimality $G - v$ is probe interval. By Lemma 17 and minimality, $H[R]$ is probe interval but it is not an interval graph. Consequently, $H[R]$ contains an induced minimally non interval graph W. By minimality, $W \in \mathcal{H}$. Since v is not adjacent to c_1 and they are complete to W, G contains an induced $U = W \vee 2K_2$ with $W \in \mathcal{H}$. By Lemma 14 it follows that U is minimally non-(probe interval). By minimality, $G = U$, this leads to a contradiction because U is not a spider. $\square$

Theorem 19 *Let G be a P_4-tidy graph. Then G is a probe interval graph if and only if G contains no induced net $\vee 2K_1$, tent $\vee 2K_1$, $\overline{3K_2}$, $\overline{2P_3}$, C_5, thin$_3$(net), thick$_3$(net), thin$_3$(tent), thick$_3$(tent), thin$_4$, or thick$_4$.*

Proof: Let G be a minimally non-(probe interval) graph that is a P_4-tidy graph. By minimality, G is connected. If $\overline{G}$ were disconnected then, by Theorem 15, G would be isomorphic to either net $\vee 2K_1$, or tent $\vee 2K_1$, or $\overline{3K_2}$, or $\overline{2P_3}$. So, we may assume that $\overline{G}$ is connected. By Theorem 1, G is C_5, P_5, $\overline{P_5}$, a spider or a fat spider. Notice that P_5 and $\overline{P_5}$ are probe interval graphs. So, G is isomorphic to C_5 or, by Corollary 18, G is isomorphic to either thin$_3(H)$, or thick$_3(H)$, for $H \in \{\text{net}, \text{tent}\}$, or thin$_4$, or thick$_4$.$\square$

5 Graphs classes with a companion

Let $\mathcal{G}$ be a hereditary class. We say that a class $\mathcal{H}$ is the *companion* of $\mathcal{G}$ if and only if, given any two graphs G_1 and G_2, the following holds: $G_1 \vee G_2 \in \mathcal{G}$ if and only if one of G_1 and G_2 is a complete graph and the other one belongs to $\mathcal{H}$.

For example, by Lemma 13, the class of interval graphs is its own companion. Using the Robert's characterizations for unit interval graphs [18] it follows that the companion of the class of unit interval graphs is the class of $\{3K_1, C_4, C_5\}$-free graphs. Notice also that the companion of the class $\{3K_1, C_4, C_5\}$-free is itself.

The main result of this section is Theorem 22. This theorem allows to compute all the minimally non-(probe $\mathcal{G}$) graphs whose complement graph is disconnected, being $\mathcal{G}$ a hereditary graph class with a companion.

Lemma 20 *Let $\mathcal{G}$ be a hereditary graph class and $\mathcal{H}$ be its companion. Then, the following assertions hold:*

1. *$\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{G}$.*
2. *$\mathcal{H}$ is a hereditary class.*
3. *$\mathcal{H}$ is its own companion.*
4. *$C_4 \notin \mathcal{G}$.*
5. *If $\mathcal{H} \neq \mathcal{K}$, then C_4 is a minimally non-$\mathcal{G}$ graph and a probe $\mathcal{H}$ graph.*

Proof: Let $H \in \mathcal{H}$. Since $\mathcal{H}$ is the companion of $\mathcal{G}$, $K_n \vee H \in \mathcal{G}$ for every positive integer n. Since G is hereditary, $\mathcal{K} \subseteq \mathcal{G}$ and $\mathcal{H} \subseteq \mathcal{G}$. Since $\mathcal{K} \subseteq \mathcal{G}$, $K_n \vee K_n \in \mathcal{G}$ for every positive integer n, and therefore $\mathcal{K} \subseteq \mathcal{H}$. We conclude that $\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{G}$. On the other hand, since $\mathcal{G}$ is an hereditary class $H' \vee K_1 \in \mathcal{G}$ for any H' subgraph of H. Therefore, $H' \in \mathcal{K} \subseteq \mathcal{H}$ or $H' \in \mathcal{H}$. Consequently, $\mathcal{H}$ is a hereditary class. 3. is immediate from 1. and the definition of hereditary class with a companion. C_4 does not belong to $\mathcal{G}$ because $C_4 = 2K_1 \vee 2K_1$ and $2K_1 \notin \mathcal{K}$.

Assume now that $\mathcal{H} \neq \mathcal{K}$. Since $\mathcal{H}$ is hereditary, $2K_1 \in \mathcal{H}$ and consequently $P_3 = K_1 \vee 2K_1 \in \mathcal{G}$. Moreover, since $2K_1 \in \mathcal{H}$, $K_1 \vee \text{diamond} = K_3 \vee 2K_1 \in \mathcal{G}$, which implies that diamond $\in \mathcal{H}$. Therefore, C_4 is a minimally non-$\mathcal{G}$ graph but it is a probe $\mathcal{H}$ graph. □

The following lemma shows the behavior of the join operator with respect to a hereditary class with a companion.

Lemma 21 *Let $\mathcal{G}$ be a hereditary graph class with companion $\mathcal{H}$ and let G_1 and G_2 be two nonempty graphs. Then, $G_1 \vee G_2$ is a probe $\mathcal{G}$ graph if and only if at least one of the following conditions holds:*

1. *One of G_1 and G_2 is a complete graph and the other one is a probe $\mathcal{H}$-graph.*
2. *One of G_1 and G_2 is probe complete and the other one is an $\mathcal{H}$-graph.*

Proof: The *"if"* part is straightforward. So, we are going to prove the *"only if"* part. Let $G = G_1 \vee G_2$ be a probe $\mathcal{G}$ graph, with $\mathcal{G}$ a hereditary class with a companion $\mathcal{H}$. Therefore, there exists a completion $G^* = (V(G), E(G) \cup F)$ with a probe interval partition (N, P) such that $G^* \in \mathcal{G}$. Since N is an independent set, $N \subseteq V(G_1)$ or $N \subseteq V(G_2)$. Assume, without loss of generality, that $N \subseteq V(G_1)$ and we call G_1^* to the graph whose vertex set is $V(G_1)$ and whose edge set is $E(G_1) \cup F$. Consequently, since $\mathcal{H}$ is the companion of $\mathcal{G}$, either G_1^* is a complete graph (G_1 is probe complete) or $G_1^* \in \mathcal{H}$ (G_1 is probe $\mathcal{H}$) and $G_2 \in \mathcal{H}$ or G_2 is a complete graph, respectively. □

The following theorem gives a tool to calculate the minimally non-probe $\mathcal{G}$ graphs for a hereditary class $\mathcal{G}$ with a companion $\mathcal{H} \neq \mathcal{K}$.

Theorem 22 *Let $\mathcal{G}$ be a hereditary graph class and $\mathcal{H}$ be its companion. If $\mathcal{H} \neq \mathcal{K}$ then the minimally non-(probe $\mathcal{G}$) graphs with disconnected complements are:*

1. *the graphs $W \vee K_1$ for each $W \in \mathcal{P}(G)$ that is minimally non-(probe $\mathcal{H}$);*
2. *the graphs $W \vee 2K_1$ for each $W \in \mathcal{P}(G)$ that is minimally non-$\mathcal{X}$, where $\mathcal{X} = \mathcal{P}(\mathcal{K}) \cup \mathcal{H}$;*
3. *the graphs $W_1 \vee W_2$ for each $W_1, W_2 \in \mathcal{P}(\mathcal{K})$ that are minimally non-$\mathcal{H}$;*
4. *the graph $\overline{2P_3}$.*

Proof: Let G be a minimally non-(probe $\mathcal{G}$) graph with disconnected complement. Then, $G = G_1 \vee G_2$ where G_1 and G_2 are nonempty graphs.

Suppose that G_2 is a complete graph. Since $G_1 \vee G_2$ is not a probe $\mathcal{G}$ graph, G_1 is not a probe $\mathcal{H}$ graph, see Lemma 21. By minimality and Lemma 21, G_2 is isomorphic to K_1 and thus $G_1 \vee K_1$ is non-(probe $\mathcal{G}$). Since G_1 is not a probe $\mathcal{H}$ graph, in particular, G_1 is not a complete graph. Since $(G_1 - v_1) \vee K_1$ is probe $\mathcal{G}$, $G_1 - v_1$ is a probe $\mathcal{H}$ graph for each $v_1 \in V(G_1)$. So, G is isomorphic to $W \vee K_1$ where W is minimally non-(probe $\mathcal{H}$) and a probe $\mathcal{G}$ graph. In what follows, we can assume that G_1 and G_2 are not complete graphs.

Suppose that G_2 contains an induced C_4. Since $(G_1 - v_1) \vee C_4$ is probe $\mathcal{G}$ and C_4 is neither an $\mathcal{H}$ graph nor probe complete, then $G_1 - v_1$ is a complete graph for each $v_1 \in V(G_1)$. So, by Lemma 21, since G_1 is not a complete graph, G_1 is isomorphic to $2K_1$. Since $C_4 \vee 2K_1$ is not a probe $\mathcal{G}$ graph, by minimality $G = C_4 \vee 2K_1$. Notice that C_4 is minimally non-$\mathcal{X}$ and probe $\mathcal{G}$. In what follows, we can assume that G_1 and G_2 contain no induced C_4.

Suppose that G_2 is probe complete and an $\mathcal{H}$ graph. Since $G_1 \vee G_2$ is not a probe $\mathcal{G}$ graph, G_1 is not a $\mathcal{X}$ graph. Since $(G_1 - v_1) \vee G_2$ is probe a $\mathcal{G}$ graph, $G_1 - v_1$ is a $\mathcal{X}$ graph. So, G_1 is minimally non-$\mathcal{X}$ graph. Since $G_1 \vee (G_2 - v_2)$ is a probe $\mathcal{G}$ graph, $G_2 - v_2$ is a complete graph for each $v_2 \in V(G_2)$. Since G_2 is not a complete graph, $G_2 = 2K_1$. So, $G = W \vee 2K_1$ where W is a minimally non-$\mathcal{X}$ graph that is probe $\mathcal{G}$.

Suppose that G_2 is probe complete but it is not an $\mathcal{H}$ graph. Since $G_1 \vee G_2$ is not a probe $\mathcal{G}$ graph, G_1 is not an $\mathcal{H}$ graph. Since $(G_1 - v_1) \vee G_2$ is a probe $\mathcal{G}$ graph, $G_1 - v_1$ is an $\mathcal{H}$ graph. So, G_1 is minimally non-$\mathcal{H}$. Suppose, by way of contradiction, that G_1 is non-(probe complete). Since $G_1 \vee (G_2 - v_2)$ is a probe $\mathcal{G}$ graph, $G_2 - v_2$ is a complete graph for each $v_2 \in V(G_2)$. Since G_2 is not a complete graph, G_2 is isomorphic to $2K_1$. Since G_2 is not an $\mathcal{H}$ graph, then $\mathcal{H} \subseteq \mathcal{K}$, a contradiction. The contradiction arose by assuming that G_1 is non-(probe complete). Therefore, G_1 is probe complete. Since G_1 is not an $\mathcal{H}$ graph, by symmetry, G_2 is a minimally non-$\mathcal{H}$ graph. We conclude that $G = W_1 \vee W_2$ where W_1 and W_2 are minimally non-$\mathcal{H}$ graphs and probe complete.

Finally, we can assume that G_1 and G_2 are non-(probe complete). Since G_1 and G_2 contain no induced C_4, by Lemma 7, G_1 and G_2 contain an induced $\overline{P_3}$ each. By minimality, G is isomorphic to $\overline{2P_3}$. □

Notice that Theorem 15 follows easily from the above theorem. Indeed, by Lemma 13, the class $\mathcal{I}$ of interval graphs is the companion of itself. Since $\mathcal{P}(\mathcal{K}) \subseteq \mathcal{I}$, $\mathcal{P}(\mathcal{K}) \cup \mathcal{I} = \mathcal{I}$ and none of the minimally non-$\mathcal{I}$ graphs is probe complete, the only minimally non-$\mathcal{I}$ graphs that are non-(probe $\mathcal{I}$) graphs are the cycles C_n for each $n \geq 5$.

Remark 1 *If $\mathcal{H} = \mathcal{K}$, the graphs belonging to $\mathcal{G}$ are P_3-free (i.e., are the disjoint union of complete graphs) and the minimally non-(probe $\mathcal{G}$) graphs with disconnected complement are C_4 and $\overline{P_3}$, if $\overline{P_3} \notin \mathcal{G}$; or C_4 and paw, otherwise.*

6 Partial characterization of probe $\{3K_1, C_4, C_5\}$-free graphs

The main results of this section are Theorem 25 that characterizes probe $\{3K_1, C_4, C_5\}$-free graphs within the class of tree-cographs and Theorem 28 that characterizes probe $\{3K_1, C_4, C_5\}$-free graphs within the class of P_4-tidy graphs. They both will be used in the next section with Theorem 22 to characterize probe unit interval graphs among tree-cographs and P_4-tidy graphs.

Threshold graphs, introduced by Chvátal and Hammer in 1975 [7], can be defined as $\{2K_2, P_4, C_4\}$-free graphs. Threshold graphs are a subclass of split graphs.

In what follows, $\mathcal{T}$ denotes the class of $\{3K_1, C_4, C_5\}$-free graphs and $\mathcal{L}$ denotes the class $\mathcal{P}(\mathcal{K}) \cup \mathcal{T}$.

Lemma 23 *The minimally non-$\mathcal{P}(\mathcal{T})$ disconnected graphs are $2K_2 + K_1$, $P_4 + K_1$, and $C_4 + K_1$.*

Proof: It is straightforward to check that $2K_2 + K_1$, $P_4 + K_1$, and $C_4 + K_1$ are minimally non-$\mathcal{P}(\mathcal{T})$ graphs.

Conversely, let H be a disconnected minimally non-$\mathcal{P}(\mathcal{T})$ graph. Suppose, by way of contradiction, that H does not contain $2K_2 + K_1$, $P_4 + K_1$, and $C_4 + K_1$ as induced subgraph. Consequently, H is either a threshold graph or the union of two threshold graphs H_1, H_2 with no induced $K_2 + K_1$ (i.e., split complete graphs).

In the first case, let N be the stable set in the split partition of H. The graph H^* that arises from H by adding all the edges uv with $u, v \in N$ is co-bipartite. So, H^* is a completion of H with partition (N, P) ($P = V(H) - N$) that is $\{3K_1, C_5\}$-free. Next, we will prove that H^* is also C_4-free. Let $A = \{u, v, x, y\}$ be a set of vertices of H^* such that $H^*[A]$ is isomorphic to C_4. Notice that, by construction, we can assume that $u, v \in N$, $x, y \in P$, u is adjacent to x and v is adjacent to y. But then A induces a P_4 in H, a contradiction.

In the second case, let N be the union of the stable sets in the split partitions of H_1 and H_2. Let H^* be the graph that arises from H by adding all the edges uv with $u, v \in N$. Then H^* can be obtained from P_4 by adding true twin vertices. It is easy to see then that it is a $\mathcal{T}$-graph. □

As a consequence of Theorem 22 we can calculate all minimally non-$\mathcal{P}(\mathcal{T})$ graphs whose complement is disconnected.

Lemma 24 *The minimally non-$\mathcal{P}(\mathcal{T})$ graphs with disconnected complement are* $(K_2 + 2K_1) \vee 2K_1$, $(P_3 + K_1) \vee 2K_1$, $\overline{3K_2}$, $K_{3,3}$, *and* $\overline{2P_3}$.

Proof: Recall that the class $\mathcal{T}$ is its own companion. Let W be a minimally non-$\mathcal{L}$ graph. We claim that W isomorphic to either $K_2 + 2K_1$, or $P_3 + K_1$, or C_4, or C_5. Indeed, since W is not a $\mathcal{T}$ graph, W contains an induced $3K_1$, C_4, or C_5. If W contained an induced C_4 or C_5, then, by minimality, W would be either isomorphic to C_4, or isomorphic to C_5. So, we may assume that W contains an induced $3K_1$ and no induced C_4 or C_5. Let S be a set that induces a $3K_1$ in W. Since W is non-(probe complete) and W contains no induced C_4, W contains an induced $\overline{P_3}$ (see Lemma 7). Let X be a set that induces a $\overline{P_3}$ in W and $e = uv$ be the only edge joining two vertices of X in W. If e has one endpoint either in S or adjacent to a vertex in S (say u), then W contains an induced $K_2 + 2K_1$ or $P_3 + K_1$, or $S \cup \{v\}$ induces a claw. If W contains an induced $K_2 + 2K_1$ or $P_3 + K_1$, then, by minimality, W is either isomorphic to $K_2 + 2K_1$ or isomorphic to $P_3 + K_1$. Suppose, by way of contradiction, that W contains neither an induced $K_2 + 2K_1$ nor an induced $P_3 + K_1$. Then, $S \cup \{v\}$ induces a claw. Let w be such that $X = \{u, v, w\}$. Since W contains no induced $P_3 + K_1$, w is adjacent to both vertices of $S - \{u\}$ and consequently W contains an induced C_4, a contradiction. Notice that $K_2 + 2K_1$, $P_3 + K_1$, C_4 are probe $\mathcal{T}$ graphs, but C_5 is not a probe $\mathcal{T}$ graph. Finally, the only minimally non-$\mathcal{T}$ graph that is probe complete is $3K_1$. The results follows now from Theorem 22. □

As a consequence of Lemmas 23 and 24, we will present characterizations for probe $\mathcal{T}$ graphs within the classes of tree-cographs and P_4-tidy graphs.

Theorem 25 *Let G be a tree-cograph. Then, G is a probe $\mathcal{T}$ graph if and only if G contains no induced* $2K_2 + K_1$, $P_4 + K_1$, $C_4 + K_1$, $(K_2 + 2K_1) \vee 2K_1$, $(P_3 + K_1) \vee 2K_1$, $\overline{3K_2}$, $K_{3,3}$, $\overline{2P_3}$, *or* $\overline{P_6}$.

Proof: Let H be a minimally non-$\mathcal{P}(\mathcal{T})$ graph that is a tree-cograph. If H is disconnected, then H is $2K_2 + K_1$, $P_4 + K_1$, or $C_4 + K_1$. If $\overline{H}$ is disconnected, then H is $(K_2 + 2K_1) \vee 2K_1$, $(P_3 + K_1) \vee 2K_1$, $\overline{3K_2}$, $K_{3,3}$, or $\overline{2P_3}$. By Theorem 12, there are no minimal probe non-$\mathcal{P}(\mathcal{T})$ graphs that are trees. If $\overline{H}$ is the complement of a tree then, by Theorem 11, H is $\overline{P_6}$, or $\overline{2P_3}$, or $\overline{3K_2}$. □

Lemma 26 *Let H be a spider with spider partition (S, C, R). Then, H is a probe $\mathcal{T}$ graph if and only if $H = thin_2(tK_1)$ for some $t \geq 0$. Moreover, if H is a probe $\mathcal{T}$ graph and H' is a fat spider that arises from H, then H' is also a probe $\mathcal{T}$ graph except when $t \geq 1$ and H' arises from H by making a false twin of a vertex of C.*

Proof: Let H be a spider with partition (S, C, R). Since H is $(P_4 + K_1)$-free and tent-free, $|C| = 2$. Notice that R is an independent set because $\text{thin}_2(K_2)$ is a non-$\mathcal{P}(\mathcal{T})$ graph. We conclude that $H = \text{thin}_2(tK_1)$ for some $t \geq 0$. Clearly, $\text{thin}_2(tK_1)$ is a probe $\mathcal{T}$ graph. By setting all vertices of $S \cup C$ as probe vertices (P) and the vertices of R as nonprobe vertices (N) and adding all the edges whose endpoints belong to N (F), we obtain the completion $H^* = (N \cup P, E(H) \cup F)$ of H that is a $\mathcal{T}$-graph. Therefore, H is a $\mathcal{P}(\mathcal{T})$-graph.

Suppose that H is a $\mathcal{P}(\mathcal{T})$-graph and let H' be a fat spider arising from H. Let v a false twin of a vertex $s \in S$. Consider the following probe partition (N, P): $N = R \cup \{v, s\}$ and $P = V(H) - N$ and denote by F the edges whose endpoints belong to N. Consequently, the completion $H^* = (N \cup P, E(H) \cup F)$ is $\mathcal{T}$-graph. Now, let v be a true twin of a vertex $s \in S$. Consider the following (N, P) probe partition: $N = R \cup (S - \{s, v\})$ and $P = V(H) - N$. So, the completion $H^* = (V(H), E(H) \cup F)$ with probe partition (N, P), where F are the edges whose endpoints belong to N, is a $\mathcal{T}$-graph. Therefore, if H' arises by making a twin of a vertex $s \in S$, then H' is a $\mathcal{P}(\mathcal{T})$-graph. We have already seen a probe interval partition of H where each $c \in C$ is a probe vertex having a probe $\mathcal{T}$ completion. Therefore, if H' arises by making a true twin of a vertex of C, H' is also a $\mathcal{P}(\mathcal{T})$-graph. Finally, assume that H' arises by making a false twin of a vertex $c \in C$. If $t = 0$, then H' is clearly a $\mathcal{P}(\mathcal{T})$-graph. If $t \geq 1$, then H' is not a probe $\mathcal{P}(\mathcal{T})$-graph because it contains an induced $C_4 + K_1$. □

Lemma 27 *The minimally non-$\mathcal{P}(\mathcal{T})$ graphs that are spiders or fat spiders are tent and $thin_2(K_2)$.*

Proof: By minimality, $|C| \leq 3$ (otherwise contains $P_4 + K_1$ or tent as proper induced subgraphs). If $|C| = 3$, then H is a thick spider (otherwise it contains $P_4 + K_1$ as proper induced subgraph). If $|C| = 3$ and H is thick, then H contains an induced tent and, by minimality, $H = \text{tent}$. Therefore, we can assume that $|C| = 2$. If $H[R]$ were a stable set then, by the above lemma, H is a spider, a contradiction. Therefore, R is not a stable set. So, H contains an induced $\text{thin}_2(K_2)$ and, by minimality, $H = \text{thin}_2(K_2)$.

Suppose, by way of contradiction, that there is a fat spider H' that is a minimally non-$\mathcal{P}(\mathcal{T})$ graph. By minimality, H' arises from a spider H that is a $\mathcal{P}(\mathcal{T})$-free graph. So, by the above lemma, $H = \text{thin}_2(tK_1)$ for some $t \geq 1$ and H arises by making a false twin of a vertex of C. Then, H' contains an induced $C_4 + K_1$. By minimality, $H' = C_4 + K_1$. This leads to a contradiction because H' is a fat spider. This contradiction proves that there are no minimally non-$\mathcal{P}(\mathcal{T})$ graphs that are fat spiders. □

By combining Lemmas 23, 24 and 27, the following characterization is obtained.

Theorem 28 *Let G be a P_4-tidy graph. Then, G is a $\mathcal{P}(\mathcal{T})$-graph if and only if G contains no induced $2K_2 + K_1$, $P_4 + K_1$, $C_4 + K_1$, $(K_2 + 2K_1) \vee 2K_1$, $(P_3 + K_1) \vee 2K_1$, $\overline{3K_2}$, $K_{3,3}$, $\overline{2P_3}$, C_5, tent, or $thin_2(K_2)$.*

Proof: Let H be a minimally non-$\mathcal{P}(\mathcal{T})$ graph that is a P_4-tidy graph. If H is disconnected, then H is isomorphic to either $2K_2 + K_1$, or $P_4 + K_1$, or $C_4 + K_1$. If $\overline{H}$ is disconnected, then H is isomorphic to either $(K_2 + 2K_1) \vee 2K_1$, or $(P_3 + K_1) \vee 2K_1$, or $\overline{3K_2}$, or $K_{3,3}$, or $\overline{2P_3}$. If H were C_5, P_5, or $\overline{P_5}$, then H would be necessarily isomorphic to C_5. Finally, if H is a spider or a fat spider, then H is tent or thin$_2(K_2)$. □

7 Partial characterizations of probe unit interval graphs

To the best of our knowledge, the problem of finding all minimally non-(probe unit interval) graphs whose complement is disconnected remains open. Nevertheless, we solve this problem for the classes of tree-cographs and P_4-tidy graphs.

Lemma 29 *The minimally non-(probe unit interval) graphs that are tree-cographs or P_4-tidy and whose complement is disconnected are $(2K_2 + K_1) \vee K_1$, $(P_4 + K_1) \vee K_1$, $(C_4 + K_1) \vee K_1$, thin$_2(K_2) \vee K_1$, $(K_2 + 2K_1) \vee 2K_1$, $(P_3 + K_1) \vee 2K_1$, $\overline{3K_2}$, $K_{3,3}$, and $\overline{2P_3}$.*

Proof: The result follows from Theorem 22. Indeed, the companion of the class of unit interval graphs is the class $\mathcal{T}$, and: (i) the minimally non-$\mathcal{P}(\mathcal{T})$ graphs that are tree-cographs or P_4-tidy and are probe unit interval graphs are $2K_2 + K_1$, $P_4 + K_1$, $C_4 + K_1$, and thin$_2(K_2)$ (see Theorem 25 and Theorem 28); (ii) by the proof of Lemma 24, the minimal forbidden induced subgraphs of $\mathcal{P}(\mathcal{K}) \cup \mathcal{T}$ are $K_2 + 2K_1$, $P_3 + K_1$, C_4, and C_5 (all of which are probe unit interval except for C_5); and (iii) the only minimally non-$\mathcal{T}$ graph that is probe complete is $3K_1$. □

Theorem 30 *Let G be a tree-cograph. Then, G is a probe unit interval graph if and only if G contains no induced bipartite-claw, L, H_n for any $n \geq 1$, $\overline{P_6}$, $(2K_2 + K_1) \vee K_1$, $(P_4 + K_1) \vee K_1$, $(C_4 + K_1) \vee K_1$, $(K_2 + 2K_1) \vee 2K_1$, $(P_3 + K_1) \vee 2K_1$, $\overline{3K_2}$, $K_{3,3}$, or $\overline{2P_3}$.*

Proof: Let H be a minimally non-(probe unit interval) graph that is a tree-cograph. By minimality, H is connected. If H is a tree, then, by Theorem 5, H is bipartite-claw, L, or H_n for some $n \geq 1$. If H is the complement of a tree, then, by Theorem 11, H is $\overline{P_6}$. If $\overline{H}$ is disconnected, by Lemma 29, H is $(2K_2+K_1)\vee K_1$, $(P_4+K_1)\vee K_1$, $(C_4+K_1)\vee K_1$, $(K_2+2K_1)\vee 2K_1$, $(P_3+K_1)\vee 2K_1$, $\overline{3K_2}$, $K_{3,3}$, $\overline{2P_3}$ (because thin$_2(K_2) \vee K_1$ is not a tree-cograph). □

Lemma 31 *Let H be a spider with spider partition (S, C, R). Then, H is a probe unit interval if and only if $|C| = 2$ and $H[R]$ is probe complete. Moreover, if H is probe unit interval and H' is a fat spider that arises from H, then H' is also probe unit interval except when $H[R]$ is not a complete graph and H' arises by making a false twin of a vertex of C.*

Proof: Let H be a probe unit interval spider with spider partition (S, C, R). Notice that $|C| = 2$ because otherwise H would contain either an induced net or an induced tent. In addition, $H[R]$ is $\{\overline{P_3}, C_4\}$-free (otherwise, H contains an induced thin$_2(\overline{P_3})$ which is not a probe unit interval graph or $(C_4+K_1)\vee K_1$). So, $H[R]$ is probe complete. Conversely, if $H[R]$ is probe complete and $|C| = 2$, clearly H is a probe unit interval graph.

Suppose now that H is probe unit interval. The vertices of S in any probe interval partition of H can be probe or nonprobe, so if H' arises by making a twin of a vertex of S, then H' is also a probe unit interval graph. The vertices of C in any probe interval partition of H can be set as probe, so if H' arises by making a true twin of C, H' is also probe unit interval. Finally, suppose that H' arises from H by making a false twin of C. If $H[R]$ is a complete graph, then H' is clearly probe interval, but if $H[R]$ is not a complete graph, H' contains an induced $(P_3 + K_1) \vee 2K_1$. □

Lemma 32 *The minimally non-(probe unit interval) graphs that are spiders or fat spiders are net, tent, and thin*$_2(\overline{P_3})$.

Proof: Let H be a spider with spider partition (S, C, R) that is a minimally non-(probe unit interval) graph. If $|C| \geq 3$, by minimality, H is net or tent. So we may assume that $|C| = 2$. Since H is not a probe unit interval graph, $H[R]$ is non-(probe complete). So, $H[R]$ contains an induced $\overline{P_3}$ or C_4. If $H[R]$ contained an induced C_4, H would contain an induced $(C_4 + K_1) \vee K_1$ and, by minimality, $H = (C_4 + K_1) \vee K_1$, contradicting the fact that H is a spider. So, necessarily $H[R]$ contains an induced $\overline{P_3}$. Therefore, H contains an induced thin$_2(\overline{P_3})$ and, by minimality, $H =$ thin$_2(\overline{P_3})$.

Suppose, by way of contradiction, that there is a fat spider H' that is a minimally non-(probe unit interval). By the minimality and the above lemma, H' arises from a spider H with $|C| = 2$ and $H[R]$ probe complete by making a false twin of a vertex of C. But then, H' contains an induced $(C_4 + K_1) \vee K_1$ and, by minimality, $H' = (C_4 + K_1) \vee K_1$, contradicting the fact that H' is a fat spider. □

Theorem 33 *Let G be a P_4-tidy graph. Then, G is a probe unit interval graph if and only if G contains no induced* $(2K_2 + K_1) \vee K_1$, $(P_4 + K_1) \vee K_1$, $(C_4 + K_1) \vee K_1$, *thin*$_2(K_2) \vee K_1$, $(K_2 + 2K_1) \vee 2K_1$, $(P_3 + K_1) \vee 2K_1$, $\overline{3K_2}$, $K_{3,3}$, $\overline{2P_3}$, C_5, *net, tent, or thin*$_2(\overline{P_3})$.

Proof: Let H be a minimally non-(probe unit interval) that is P_4-tidy. By minimality, H is connected. If $\overline{H}$ is disconnected, by Lemma 29, H is $(2K_2 + K_1) \vee K_1$, $(P_4 + K_1) \vee K_1$, $(C_4 + K_1) \vee K_1$, thin$_2(K_2) \vee K_1$, $(K_2 + 2K_1) \vee 2K_1$, $(P_3 + K_1) \vee 2K_1$, $\overline{3K_2}$, $K_{3,3}$, or $\overline{2P_3}$. If H were C_5, P_5, or $\overline{P_5}$, necessarily $H = C_5$. If H is a spider or a fat spider, then H is net, tent, or thin$_2(\overline{P_3})$. □

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