# Bipartite Powers of k-chordal Graphs

# L. Sunil Chandran<sup> $\dagger$ </sup> and Rogers Mathew<sup> $\ddagger$ </sup>

Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India.

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Let k be an integer and  $k \ge 3$ . A graph G is k-chordal if G does not have an induced cycle of length greater than k. From the definition it is clear that 3-chordal graphs are precisely the class of chordal graphs. Duchet proved that, for every positive integer m, if  $G^m$  is chordal then so is  $G^{m+2}$ . Brandstädt et al. in [Andreas Brandstädt, Van Bang Le, and Thomas Szymczak. Duchet-type theorems for powers of HHD-free graphs. Discrete Mathematics, 177(1-3):9-16, 1997.] showed that if  $G^m$  is k-chordal, then so is  $G^{m+2}$ .

Powering a bipartite graph does not preserve its bipartitedness. In order to preserve the bipartitedness of a bipartite graph while powering Chandran et al. introduced the notion of *bipartite powering*. This notion was introduced to aid their study of boxicity of chordal bipartite graphs. The *m*-th bipartite power  $G^{[m]}$  of a bipartite graph *G* is the bipartite graph obtained from *G* by adding edges (u, v) where  $d_G(u, v)$  is odd and less than or equal to *m*. Note that  $G^{[m]} = G^{[m+1]}$  for each odd *m*.

In this paper we show that, given a bipartite graph G, if G is k-chordal then so is  $G^{[m]}$ , where k, m are positive integers with  $k \ge 4$ .

Keywords: k-chordal graph, hole, chordality, graph power, bipartite power.

## 1 Introduction

A hole is a chordless (or an induced) cycle in a graph. The *chordality* of a graph G, denoted by C(G), is defined to be the size of a largest hole in G, if there exists a cycle in G. If G is acyclic, then its chordality is taken as 0. A graph G is k-chordal if  $C(G) \leq k$ . In other words, a graph is k-chordal if it has no holes with more than k vertices in it. Chordal graphs are exactly the class of 3-chordal graphs and chordal bipartite graphs are bipartite, 4-chordal graphs. k-chordal graphs have been studied in the literature in [2], [5], [6], [8], [9] and [16]. For example, Chandran and Ram [5] proved that the number of minimum cuts in a k-chordal graph is at most  $\frac{(k+1)n}{2} - k$ . Spinrad[16] showed that (k-1)-chordal graphs can be recognized in  $O(n^{k-3}M)$  time, where M is the time required to multiply two n by n matrices.

Powering and its effects on the chordality of a graph has been a topic of interest. The *m*-th power of a graph G, denoted by  $G^m$ , is a graph with vertex set  $V(G^m) = V(G)$  and edge set  $E(G^m) = \{(u, v) \mid u \neq v, d_G(u, v) \leq m\}$ , where  $d_G(u, v)$  represents the distance between u and v in G. Balakrishnan and

<sup>&</sup>lt;sup>†</sup>Email: sunil@csa.iisc.ernet.in

<sup>&</sup>lt;sup>‡</sup>Email: rogers@csa.iisc.ernet.in

<sup>1365-8050 © 2013</sup> Discrete Mathematics and Theoretical Computer Science (DMTCS), Nancy, France

Paulraja [1] proved that odd powers of chordal graphs are chordal. Chang and Nemhauser [7] showed that if G and  $G^2$  are chordal then so are all powers of G. Duchet [10] proved a stronger result which says that if  $G^m$  is chordal then so is  $G^{m+2}$ . Brandstädt et al. in [3] showed that if  $G^m$  is k-chordal then so is  $G^{m+2}$ , where  $k \ge 3$  is an integer. Studies on families of graphs that are closed under powering can also be seen in the literature. For instance, it is known that interval graphs, proper interval graphs [14], strongly chordal graphs [13], circular-arc graphs [15][12], cocomparability graphs [11] etc. are closed under taking powers.

Subclasses of bipartite graphs, like chordal bipartite graphs, are not closed under powering since the m-th power of a bipartite graph need not be even bipartite. Chandran et al. in [4] introduced the notion of bipartite powering to retain the bipartitedness of a bipartite graph while taking power. The m-th bipartite power  $G^{[m]}$  of a bipartite graph G is the bipartite graph obtained from G by adding edges (u, v) where  $d_G(u, v)$  is odd and less than or equal to m. Note that  $G^{[m]} = G^{[m+1]}$  for each odd m. It was shown in [4] that the m-th bipartite power of a tree is chordal bipartite. The intention there was to construct chordal bipartite graphs of high boxicity. The fact that the chordal bipartite graph under consideration was obtained as a bipartite graphs, a natural question that came up was the following: is it true that the m-th bipartite power of every chordal bipartite graph is chordal bipartite? In this paper we answer this question in the affirmative. In fact, we prove a more general result.

#### Our Result

Let m, k be positive integers with  $k \ge 4$ . Let G be a bipartite graph. If G is k-chordal, then so is  $G^{[m]}$ . Note that the special case when k = 4 gives us the following result: chordal bipartite graphs are closed under bipartite powering.

# 2 Graph Preliminaries

Throughout this paper we consider only finite, simple, undirected graphs. For a graph G, we use V(G) to denote the set of vertices of G. Let E(G) denote its edge set. For every  $x, y \in V(G)$ ,  $d_G(x, y)$  represents the distance between x and y in G. For every  $u \in V(G)$ ,  $N_G(u)$  denotes its open neighborhood in G, i.e.  $N_G(u) = \{v \mid (u, v) \in E(G)\}$ . A path P on the vertex set  $V(P) = \{v_1, v_2, \ldots, v_n\}$  (where  $n \geq 2$ ) has its edge set  $E(P) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\}$ . Such a path is denoted by  $v_1v_2 \ldots v_n$ . If  $v_i, v_j \in V(P)$ ,  $v_iPv_j$  is the path  $v_iv_{i+1} \ldots v_j$ . The length of a path P is the number of edges in it and is denoted by ||P||. A cycle C with vertex set  $V(C) = \{v_1, v_2, \ldots, v_n\}$ , and edge set  $E(C) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(v_n, v_1)\}$  is denoted as  $C = v_1v_2 \ldots v_nv_1$ . We use ||C|| to denote the length of cycle C.

# 3 Holes in Bipartite Powers

Let H be a bipartite graph. Let  $\mathcal{B}(H)$  be a family of graphs constructed from H in the following manner:  $H' \in \mathcal{B}(H)$  if corresponding to each vertex  $v \in V(H)$  there exists a nonempty bag of vertices, say  $B_v$ , in H' such that (a) for every  $x \in B_u$ ,  $y \in B_v$ ,  $(x, y) \in E(H')$  if and only if  $(u, v) \in E(H)$ , and (b) vertices within each bag in H' are pairwise non-adjacent. Below we list a few observations about H and every H' (, where  $H' \in \mathcal{B}(H)$ ):

**Observation 1.** H' is bipartite.

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**Observation 2.** H is an induced subgraph of H'.

**Observation 3.** Let k be an integer such that  $k \ge 4$ . If H is k-chordal, then so is H'.

**Proof:** Any hole of size greater than 4 in H' cannot have more than one vertex from the same bag, say  $B_v$ , as such vertices have the same neighborhood. Hence, the vertices of a hole (of size greater than 4) in H' belong to different bags and thus there is a corresponding hole of the same size in H.

**Theorem 4.** Let m, k be positive integers with  $k \ge 4$ . Let G be a bipartite graph. If G is k-chordal, then so is  $G^{[m]}$ .

**Proof:** We prove this by contradiction. Let p denote the size of a largest induced cycle, say  $C = u_0 u_1 \ldots u_{p-1} u_0$ , in  $G^{[m]}$ . Assume p > k. Then,  $p \ge 6$  (since  $k \ge 4$  and  $G^{[m]}$  is bipartite). Between each  $u_{i-1}$  and  $u_i$ , where  $i \in \{0, \ldots, p-1\}$ , there exists a shortest path of length not more than m in  $G^{(i)}$ . Let  $P_i$  be one such shortest path between  $u_{i-1}$  and  $u_i$  in G.

Let H be the subgraph induced on the vertex set  $\bigcup_{i=0}^{p-1} V(P_i)$  in G. As mentioned in the beginning of this section, construct a graph H' from H, where  $H' \in \mathcal{B}(H)$ , in the following manner: for each  $v \in V(H)$ , let  $|B_v| = |\{P_i \mid 0 \le i \le p-1, v \in V(P_i)\}|$  i.e., let  $B_v$  have as many vertices as the number of paths in  $\{P_0 \ldots P_{p-1}\}$  that share vertex v in H. For each  $i \in \{0, \ldots, p-1\}$ , let  $Q'_i = u_{i-1}Q_i$  be a shortest path between  $u_{i-1}$  and  $u_i$  in H' such that no two paths  $Q_i$  and  $Q_j$  (where  $i \ne j$ ) share a vertex <sup>(i)</sup>. From our construction of H' from H it is easy to see that such paths exist. Let  $Q_i = v_{i,1}v_{i,2}\ldots v_{i,r_i}u_i$ , where  $r_i = ||Q_i|| \ge 0$ . Thus,  $Q'_i = u_{i-1}v_{i,1}v_{i,2}\ldots v_{i,r_i}u_i$ . Clearly,  $||Q'_i|| = ||P_i|| \le m$ . The reader may also note that the cycle  $C (= u_0u_1 \ldots u_{p-1}u_0)$  which is present in  $G^{[m]}$  will be present in  $H^{[m]}$  and thereby in  $H'^{[m]}$  too.

In order to prove the theorem, it is enough to show that there exists an induced cycle of size at least p in H'. Then by combining Observation 3 and the fact that H is an induced subgraph of G, we get  $k \ge C(G) \ge C(H) \ge C(H') \ge p$  contradicting our assumption that p > k. Hence, in the rest of the proof we show that  $C(H') \ge p$ .

Consider the following drawing of the graph H'. Arrange the vertices  $u_0, u_1, \ldots, u_{p-1}$  in that order on a circle in clockwise order. Between each  $u_{i-1}$  and  $u_i$  on the circle arrange the vertices  $v_{i,1}, v_{i,2}, \ldots, v_{i,r_i}$  in that order in clockwise order. Recall that these vertices are the internal vertices of path  $Q'_i$ .

*Claim* 4.1. In this circular arrangement of vertices of H', each vertex has an edge (in H') with both its left neighbor and right neighbor in the arrangement.

Let  $x_1, x_2 \in V(H')$ , where  $x_1 \in V(Q_i), x_2 \in V(Q_j)$ . We define the *clockwise distance from*  $x_1$  to  $x_2$ , denoted by  $clock\_dist(x_1, x_2)$ , as the minimum non-negative integer s such that j = i + s. Similarly, the *clockwise distance from*  $x_2$  to  $x_1$ , denoted by  $clock\_dist(x_2, x_1)$ , is the minimum non-negative integer s' such that i = j + s'. Let  $x, y, z \in V(H')$ . We say  $y <_x z$  if scanning the vertices of H' in clockwise direction along the circle starting from x, vertex y is encountered before z. Let  $x \in V(Q_i)$ . Vertex y is called the *farthest neighbor of* x before z if  $y \in N_{H'}(x), y \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2}), y <_x z$ , and for every other  $w \in N_{H'}(x)$  either  $z <_x w$  or  $w \notin V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$  or both.

Claim 4.2. There always exists a vertex which is the farthest neighbor of x before z, unless  $(x, z) \in E(H')$  and  $z \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$ .

<sup>&</sup>lt;sup>(i)</sup> throughout this proof expressions involving subscripts of u, P, Q, and Q' are to be taken modulo p. Every such expression should be evaluated to a value in  $\{0, \ldots, p-1\}$ . For example, consider a vertex  $u_i$ , where i < p Then, p + i = i.

Let  $\{A, B\}$  be the bipartition of the bipartite graph H'. We categorize the edges of H' as follows: an edge  $(x, y) \in E(H')$  is called an *l-edge*, if  $l = \min(clock\_dist(x, y), clock\_dist(y, x))$ .



**Fig. 1:**  $x \in V(Q_i), y \in V(Q_{i+l})$  and let  $(x, y) \in E(H')$  be an *l*-edge, where l > 2. The dotted line between  $u_{i-1}$  and  $u_i$  indicate the path  $Q_i$ . Similarly, the dotted line between  $u_{i+l-1}$  and  $u_{i+l}$  indicate the path  $Q_{i+l}$ .

Claim 4.3. H' cannot have an *l*-edge, where l > 2.

**Proof:** Suppose H' has an *l*-edge, where l > 2, between  $x \in Q_i$  and  $y \in Q_{i+l}$  (see Fig. 1). Let  $a_1 = ||u_{i-1}Q'_ix||, b_1 = ||xQ'_iu_i||, a_2 = ||u_{i+l-1}Q'_{i+l}y||$  and  $b_2 = ||yQ'_{i+l}u_{i+l}||$ . We consider the following two cases:

#### Case 1: l is even

In this case  $u_{i-1}$  and  $u_{i+l-1}$  will be on the same side of the bipartite graph H'. Without loss of generality, let  $u_{i-1}, u_{i+l-1} \in A$ . Then,  $u_i, u_{i+l} \in B$ . We know that, for every  $w_1, w_2 \in V(H'^{[m]})$  with  $w_1 \in A$  and  $w_2 \in B$ , if  $(w_1, w_2) \notin E(H'^{[m]})$  then  $d_{H'}(w_1, w_2) \ge m + 2$  (recalling m and  $d_{H'}(w_1, w_2)$  are odd integers). Therefore, we have  $a_1 + 1 + b_2 \ge d_{H'}(u_{i-1}, u_{i+l}) \ge m + 2$ . Similarly,  $b_1 + 1 + a_2 \ge d_{H'}(u_i, u_{i+l-1}) \ge m + 2$ . Summing up the two inequalities we get,  $(a_1 + b_1) + (a_2 + b_2) \ge 2m + 2$ . This implies that either  $||Q'_i||$  or  $||Q'_{i+l}||$  is greater than m which is a contradiction. *Case* 2: l is odd

(proof is similar to the above case and hence omitted).

Hence we prove the claim.

We find a cycle  $C' = z_0 z_1 \dots z_q z_0$  in H' using Algorithm 3.1<sup>(i)</sup>. Please read the algorithm before proceeding further.

<sup>&</sup>lt;sup>(i)</sup> throughout this proof expressions involving subscripts of z are to be taken modulo q + 1. Every such expression should be evaluated to a value in  $\{0, \ldots, q\}$ . For example, consider a vertex  $z_a$ , where a < q + 1. Then, q + 1 + a = a.

Algorithm 3.1 Finding Cycle C' in H' such that  $||C'|| \ge ||C||$ 

1.  $l \leftarrow \max_{l'}(H' \text{has an } l' \text{-edge})$ . Without loss of generality assume that this l-edge is between a vertex in  $Q_0$  and a vertex in  $Q_l$ 2. Scan the vertices of  $Q_0$  in clockwise direction to find the first vertex  $z_0$ , where  $z_0 \in V(Q_0)$ , which has an l-edge to a vertex in  $Q_l$ . 3. Scan the vertices of  $Q_l$  in clockwise direction to find the last vertex in  $Q_l$  which is a neighbor of  $z_0$ in H'. Call it  $z_1$ . 4. Find the farthest neighbor of  $z_1$  before  $z_0$ . Call it  $z_2$ . /\* refer proof of Claim 4.4 for a proof of existence of such a  $z_2*/$ 5.  $s \leftarrow 2$ . while  $(z_s, z_0) \notin E(H')$  do 6. Find the farthest neighbor of  $z_s$  before  $z_0$ . Call it  $z_{s+1}$ . /\* such a neighbor exists by Claim 4.2\*/ 7.  $s \leftarrow s + 1$ . end while 8.  $q \leftarrow s$ . 9. Return cycle  $C' = z_0 z_1 \dots z_q z_0$ .

Claim 4.4. There always exists a farthest neighbor of  $z_1$  before  $z_0$ .

**Proof:** Note that  $z_0 \in Q_0$  and  $z_1 \in Q_l$ , where  $l \leq 2$  (by Claim 4.3). Recalling that  $||C|| = p \geq 6$ , we have  $z_0 \notin V(Q_l) \cup V(Q_{l+1}) \cup V(Q_{l+2})$ . Hence by Claim 4.2, the claim is true.

Claim 4.5. The while loop in Algorithm 3.1 terminates after a finite number of iterations.

**Proof:** From Claim 4.1, we know that each vertex has an edge (in H') with both its left neighbor and right neighbor in the circular arrangement. Each time when Step 6 of Algorithm 3.1 is executed, a vertex  $z_{s+1}$  is chosen such that  $z_{s+1}$  is the farthest neighbor of  $z_s$  before  $z_0$ . Since H' is a finite graph, there will be a point of time in the execution of the algorithm when in Step 6 it picks a  $z_{s+1}$  such that  $(z_{s+1}, z_0) \in E(H')$ .

From Claim 4.5, we can infer that C' is a cycle.

Claim 4.6. C' is an induced cycle in H'.

**Proof:** Suppose C' is not an induced cycle. Then there exists a chord  $(z_a, z_b)$  in C'. Since  $(z_a, z_b)$  is a chord, we have  $b \neq a - 1$  or  $b \neq a + 1$ . Let  $l = \max_{l'}(H' \text{ has an } l' \text{-edge})$ . Let  $z_a \in V(Q_i)$ ,  $z_b \in V(Q_j)$ . We know that  $\min(clock\_dist(z_a, z_b), clock\_dist(z_b, z_a)) \leq l$ . Without loss of generality, assume  $clock\_dist(z_a, z_b) \leq l \leq 2$  (from Claim 4.3). That is,  $j - i \leq l \leq 2$  and  $(z_a, z_b)$  is a (j - i)-edge. If  $z_a = z_0$ , then  $z_b \neq z_1$  and the algorithm exits from the while loop, when q = b, thus returning a cycle  $z_0 \dots z_b z_0$ . But in such a cycle  $(z_b, z_0)$  is not a chord. Therefore,  $z_a \neq z_0$ . Similarly,  $z_b \neq z_0$ . We know that  $z_{a+1} \neq z_b, z_{a+1} < z_a z_b$ , and  $z_{a+1} \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$ . Since  $j - i \leq 2$ ,  $z_b \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$ . If  $z_b <_{z_a} z_0$ , then it contradicts the fact that  $z_{a+1}$  is the farthest neighbor of  $z_a$  before  $z_0$ . Therefore,  $z_0 <_{z_a} z_b$ . Then, either  $z_b = z_1$  or  $z_1 <_{z_a} z_b$ . Recall that  $l = \max_{l'}(H' \text{has an } l' \text{-edge})$ , and  $(z_0, z_1)$  is an l-edge with  $z_0 \in V(Q_0)$  and  $z_1 \in V(Q_l)$ . Since (i)  $(z_a, z_b)$  is a (j - i)-edge, where  $j - i \leq l$ , (ii)  $z_0 <_{z_a} z_b$ , and (iii)  $z_b = z_1$  or  $z_1 <_{z_a} z_b$ , we have  $l \geq j - i = clock\_dist(z_a, z_b) \geq clock\_dist(z_0, z_b) \geq clock\_dist(z_0, z_1) = l$ . Hence, j - i = l and



**Fig. 2:** Figure illustrates the case when path P defined in Claim 4.8 is a trivial path. The dotted lines between each  $u_{i-1}$  and  $u_i$  indicate the path  $Q'_i$ . Each continuous arc corresponds to an edge in the cycle  $C' = z_0 \dots z_q z_0$ .

 $(z_a, z_b)$  is an *l*-edge. We know that  $(z_0, z_1)$  is also an *l*-edge with  $z_0 \in V(Q_0)$  and  $z_1 \in V(Q_l)$ . Since  $z_0 <_{z_a} z_b$  and  $z_b = z_1$  or  $z_1 <_{z_0} z_b$ , we get  $z_a \in V(Q_0)$  and  $z_b \in V(Q_l)$ . From Step 2 of the algorithm we know that  $z_0$  is the first vertex (in a clockwise scan) in  $Q_0$  which has an *l*-edge to a vertex in  $Q_l$ . This implies that, since  $z_0 <_{z_a} z_b, z_a = z_0$  which is a contradiction. Hence we prove the claim.

What is left now is to show that  $q + 1 \ge p$ , i.e.,  $||C'|| \ge ||C||$ , where  $C' = z_0 \dots z_q z_0$  and  $C = u_0 \dots u_{p-1} u_0$ . In order to show this, we state and prove the following claims. Claim 4.7. For every  $j \in \{0, \dots, p-1\}$ ,  $(V(Q_j) \cup V(Q_{j+1})) \cap V(C') \ne \emptyset$ .

**Proof:** Suppose the claim is not true. Find the minimum j that violates the claim. Clearly,  $j \neq 0$  as  $z_0 \in V(Q_0)$ . We claim that  $z_q \in V(Q_{j-1})$ . Suppose  $z_q \notin V(Q_{j-1})$ . Let  $a = \max\{i \mid z_i \in V(Q_{j-1})\}$  (note that, since  $j \neq 0$ , by the minimality of j,  $(V(Q_{j-1}) \cup V(Q_j)) \cap V(C') \neq \emptyset$  and therefore  $V(Q_{j-1}) \cap V(C') \neq \emptyset$ ). Since  $z_a \neq z_q$ , by the maximality of a, we have  $z_{a+1} \notin V(Q_{j-1})$ . From our assumption,  $(V(Q_j) \cup V(Q_{j+1})) \cap V(C') = \emptyset$  and therefore  $z_{a+1} \notin V(Q_{j-1}) \cup V(Q_j) \cup V(Q_{j+1})$ . Thus  $z_a \neq z_q$  and  $z_{a+1}$  is not the farthest neighbor of  $z_a$  before  $z_0$ . This is a contradiction to the way  $z_{a+1}$  is chosen by Algorithm 3.1. Hence,  $z_q \in V(Q_{j-1})$ . We know that  $(z_q, z_0) \in E(H')$  with  $z_q \in V(Q_{j-1})$  and  $z_0 \in V(Q_0)$ . Since  $l = \max_{l'}(H'$  has an l'-edge), we have  $\min(clock\_dist(z_q, z_0), clock\_dist(z_0, z_q)) \leq l$ . That is,  $j \geq p+1-l$  or  $j \leq 1+l$ . As  $l \leq 2$  (by Claim 4.3), we have j = p-1 or  $j \leq 1+l$ . Since  $z_0 \in V(Q_0)$ ,  $(V(Q_{p-1}) \cup V(Q_0)) \cap V(C') \neq \emptyset$  and hence  $j \neq p-1$ . Therefore,  $j \leq 1+l$ . Since  $z_0 \in V(Q_0)$  and  $z_1 \in V(Q_l)$  (recall  $l \leq 2$ ), we get j = 1+l. We know that, for every  $z_a, z_b \in V(C')$ , if a < b then  $z_a <_{z_0} z_b$ . Therefore,  $z_1 <_{z_0} z_q$ . But this contradicts the fact that  $z_1$  is the last vertex in  $Q_l$  encountered in a clockwise scan that has  $z_0$  as its neighbor.

Claim 4.8. Let  $(z_a, z_{a+1}), (z_b, z_{b+1}) \in E(C')$  be two 2-edges, where a < b. Let P, P' denote the clockwise  $z_{a+1} - z_b, z_{b+1} - z_a$  paths respectively in C'. Both P and P' contain at least one 0-edge.



**Fig. 3:** Figure illustrates the case when path P defined in Claim 4.8 is  $P = z_{a+1}z_{a+2} \dots z_{a+1+s}$ , where  $s \ge 1$  and  $z_{a+1+s} = z_b$ . The dotted lines between each  $u_{i-1}$  and  $u_i$  indicate the path  $Q'_i$ . Each continuous arc corresponds to an edge in the cycle  $C' = z_0 \dots z_q z_0$ .

**Proof:** Consider the path P (proof is similar in the case of path P'). Path P is a non-trivial path only if  $z_{a+1} \neq z_b$ . Suppose  $z_{a+1} = z_b$  (see Fig. 2). Let  $z_a \in V(Q_f)$ . For the sake of ease of notation, assume f = 1 (the same proof works for any value of f). Let  $a_1 = ||u_0Q'_1z_a||$ ,  $b_1 = ||z_aQ'_1u_1||$ ,  $a_2 = ||u_2Q'_3z_b||$ ,  $b_2 = ||z_bQ'_3u_3||$ ,  $a_3 = ||u_4Q'_5z_{b+1}||$ , and  $b_3 = ||z_{b+1}Q'_5u_5||$ . We know that, for every  $w_1, w_2 \in V(H'^{[m]})$  with  $w_1 \in A$  and  $w_2 \in B$ , if  $(w_1, w_2) \notin E(H'^{[m]})$  then  $d_{H'}(w_1, w_2) \geq m + 2$ . Since  $(u_0, u_3) \notin E(H'^{[m]})$ ,  $(u_1, u_4) \notin E(H'^{[m]})$  and  $(u_2, u_5) \notin E(H'^{[m]})$ , we have  $a_1 + b_2 \geq m + 1$ ,  $b_1 + a_3 \geq m$ , and  $a_2 + b_3 \geq m + 1$ . Adding the three inequalities and by applying an easy averaging argument we can infer that either  $a_1 + b_1 = ||Q_1|| > m$ ,  $a_2 + b_2 = ||Q_3|| > m$ , or  $a_3 + b_3 = ||Q_5|| > m$  which is a contradiction. Therefore P is a non-trivial path i.e.,  $z_{a+1} \neq z_b$ . Assume P does not contain any 0-edge. Let  $P = z_{a+1}z_{a+2} \dots z_{a+1+s}$ , where  $s \geq 1$ , a + 1 + s = b, and  $(z_{a+1}, z_{a+2}) \dots (z_{a+s}, z_{a+1+s})$  are 1-edges (see Fig. 3). Since  $(u_0, u_3) \notin E(H'^{[m]})$ ,  $(u_1, u_4) \notin E(H'^{[m]})$ , we have  $c_a + d_{a+1} \geq m + 1$  and  $d_a + d_{a+2} \geq m$  (please refer Fig. 3 for knowing what  $c_a, d_a, \dots, c_{b+1}, d_{b+1}$  are). Summing up the two inequalities, we get  $d_{a+1} + d_{a+2} \geq 2m + 1 - (c_a + d_a)$ . We know that, for each  $i \in \{0, \dots, p-1\}$ ,  $||Q'_i|| \leq m$ . Therefore, we have  $c_a + d_a \leq m$ . Hence,  $d_{a+1} + d_{a+2} \geq m + 1$ . Since  $(c_{a+1} + d_{a+1}) + (c_{a+2} + d_{a+2}) \leq 2m$ , we get

$$c_{a+1} + c_{a+2} \leq m - 1 \tag{1}$$

Since  $(u_{s+2}, u_{s+5}) \notin E(H'^{[m]}), (u_{s+1}, u_{s+4}) \notin E(H'^{[m]})$ , we have,

$$c_b + d_{b+1} \ge m+1$$
  
$$c_{a+s} + c_{b+1} \ge m$$

Summing up the two inequalities, we get

$$c_b + c_{a+s} \geq 2m + 1 - (c_{b+1} + d_{b+1})$$

Since b = a + s + 1 and  $c_{b+1} + d_{b+1} \le m$ , we get

$$c_{a+s+1} + c_{a+s} \ge m+1 \tag{2}$$

Substituting for s = 1 in Inequality 2, we get  $c_{a+2} + c_{a+1} \ge m + 1$ . But this contradicts Inequality 1. Hence s > 1. Suppose s = 2. Since  $(u_2, u_5) \notin E(H'^{[m]})$ , we have  $c_{a+1} + d_{a+3} \ge m$ . Adding this with Inequality 2, we get  $c_{a+1} + c_{a+2} \ge (2m + 1) - (c_{a+3} + d_{a+3}) \ge m + 1$ . But this contradicts Inequality 1. Hence s > 2. Since  $(u_s, u_{s+3}) \notin E(H'^{[m]})$ ,  $\ldots$ ,  $(u_2, u_5) \notin E(H'^{[m]})$ , we have the following inequalities:-

$$c_{a+s-1} + d_{a+s+1} \ge m$$

$$\vdots \vdots \vdots$$

$$c_{a+1} + d_{a+3} \ge m$$

Adding the above set of inequalities and applying the fact that  $c_i + d_i \leq m, \forall i \in \{0, \dots, q\}$ , we get  $c_{a+1} + c_{a+2} + d_{a+s} + d_{a+s+1} \geq 2m$ . Adding this with Inequality 2, we get  $c_{a+1} + c_{a+2} \geq (3m+1) - (c_{a+s+1} + d_{a+s+1}) - (c_{a+s} + d_{a+s}) \geq m + 1$ . But this contradicts Inequality 1. Hence we prove the claim.

Claim 4.9. For every  $j, j' \in \{0, \ldots, p-1\}$ , where j < j' and  $(V(Q_j) \cup V(Q_{j'})) \cap V(C') = \emptyset$ , there exist  $i, i' \in \{0, \ldots, p-1\}$ , where only i satisfies j < i < j', such that  $|V(Q_i) \cap V(C')| \ge 2$  and  $|V(Q_{i'}) \cap V(C')| \ge 2$ .

**Proof:** By Claim 4.7, (i)  $j' \neq j + 1$  or  $j' \neq j - 1$ , and (ii) there exist  $r, r' \in \{0, \ldots, q\}$  such that  $(z_r, z_{r+1})$  is a 2-edge with its endpoints on  $Q_{j-1}$  and  $Q_{j+1}$  and  $(z_{r'}, z_{r'+1})$  is a 2-edge with its endpoints on  $Q_{j'-1}$  and  $Q_{j'+1}$ . By Claim 4.8, we know that if P, P' denote the clockwise  $z_{r+1} - z_{r'}, z_{r'+1} - z_r$  paths respectively in C', then both P and P' contains at least one 0-edge. This proves the claim.

In order to show that the size of cycle C' (=  $z_0 \dots z_q z_0$ ) is at least p, we consider the following three cases:-

Case  $|\{Q_j \in \{Q_0 \dots Q_{p-1}\} | V(Q_j) \cap V(C') = \emptyset\}| = 0$ : In this case, for every  $j \in \{0, \dots, p-1\}, Q_j$  contributes to V(C') and therefore  $||C'|| \ge p = ||C||$ .

Case  $|\{Q_j \in \{Q_0 \dots Q_{p-1}\} | V(Q_j) \cap V(C') = \emptyset\}| = 1$ : Let  $Q_j$  be that only path (among  $Q_0 \dots Q_{p-1}$ ) that does not contribute to V(C'). Then we claim that there exists a  $Q_{j'}$ , where  $j' \neq j$ , such that  $V(C') \cap V(Q_{j'}) \geq 2$ . Suppose the claim is not true then it is easy to see that ||C'|| = p - 1 which is an odd number thus contradicting the bipartitedness of H'. Hence the claim is true. Now, by applying the claim it is easy to see that  $||C'|| = \sum_{j} |V(C') \cap V(Q_j)| \geq p = ||C||$ .

Case  $|\{Q_j \in \{Q_0 \dots Q_{p-1}\} | V(Q_j) \cap V(C') = \emptyset\}| > 1$ : Scan vertices of H' starting from any vertex in clockwise direction. Claim 4.9 ensures that between every  $Q_j$  and  $Q_{j'}$ , which do not contribute to V(C'), encountered there exists a  $Q_i$  which compensates by contributing at least two vertices to V(C'). Therefore,  $||C'|| \ge p = ||C||$ .

### 4 Discussion

An interesting open question that naturally follows from our result is the following: given a graph G and positive integers k, m where  $k \ge 4$ , if  $G^{[m]}$  is k-chordal, then is  $G^{[m+2]}$  also k-chordal? As mentioned earlier, Brandstädt et al. in [3] showed a similar result in the context of *ordinary* graph powering. They showed that, for every graph G, if  $G^m$  is k-chordal, then so is  $G^{m+2}$ , where k, m are positive integers with  $k \ge 3$ . A straightforward extension of their proof technique doesn't seem to work in our context due to the bipartite nature of the powering that we consider.

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