Discrete Mathematics and Theoretical Computer Science DMTCS vol. **15:2**, 2013, 49–58

Bipartite Powers of k-chordal Graphs

L. Sunil Chandran† and Rogers Mathew‡

Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India.

received 4th May 2012, revised 17th April 2013, accepted 14th April 2013.

Let k be an integer and $k \geq 3$. A graph G is *k-chordal* if G does not have an induced cycle of length greater than k. From the definition it is clear that 3-chordal graphs are precisely the class of chordal graphs. Duchet proved that, for every positive integer m, if G^m is chordal then so is G^{m+2}. Brandstädt et al. in [Andreas Brandstädt, Van Bang Le, and Thomas Szymczak. Duchet-type theorems for powers of HHD-free graphs. *Discrete Mathematics*, 177(1-3):9-16, 1997.] showed that if G^m is k-chordal, then so is G^{m+2}.

Powering a bipartite graph does not preserve its bipartitedness. In order to preserve the bipartitedness of a bipartite graph while powering Chandran et al. introduced the notion of *bipartite powering*. This notion was introduced to aid their study of boxicity of chordal bipartite graphs. The m-th bipartite power $G^{[m]}$ of a bipartite graph G is the bipartite graph obtained from G by adding edges (u, v) where $d_G(u, v)$ is odd and less than or equal to m. Note that $G^{[m]} = G^{[m+1]}$ for each odd m.

In this paper we show that, given a bipartite graph G, if G is k-chordal then so is $G^{[m]}$, where k, m are positive integers with $k \geq 4$.

Keywords: k-chordal graph, hole, chordality, graph power, bipartite power.

1 Introduction

A *hole* is a chordless (or an induced) cycle in a graph. The *chordality* of a graph G, denoted by $\mathcal{C}(G)$, is defined to be the size of a largest hole in G, if there exists a cycle in G. If G is acyclic, then its chordality is taken as 0. A graph G is *k-chordal* if $\mathcal{C}(G) \leq k$. In other words, a graph is k-chordal if it has no holes with more than k vertices in it. Chordal graphs are exactly the class of 3-chordal graphs and chordal bipartite graphs are bipartite, 4-chordal graphs. k-chordal graphs have been studied in the literature in [2], [5], [6], [8], [9] and [16]. For example, Chandran and Ram [5] proved that the number of minimum cuts in a k-chordal graph is at most $\frac{(k+1)n}{2} - k$. Spinrad[16] showed that $(k-1)$-chordal graphs can be recognized in $O(n^{k-3}M)$ time, where M is the time required to multiply two n by n matrices.

Powering and its effects on the chordality of a graph has been a topic of interest. The m-th power of a graph G, denoted by G^m, is a graph with vertex set $V(G^m) = V(G)$ and edge set $E(G^m) = \{(u, v) \mid u \neq v, d_G(u, v) \leq m\}$, where $d_G(u, v)$ represents the distance between u and v in G. Balakrishnan and

†Email: `sunil@csa.iisc.ernet.in`
‡Email: `rogers@csa.iisc.ernet.in`

Paulraja [1] proved that odd powers of chordal graphs are chordal. Chang and Nemhauser [7] showed that if G and G^2 are chordal then so are all powers of G. Duchet [10] proved a stronger result which says that if G^m is chordal then so is G^{m+2}. Brandstädt et al. in [3] showed that if G^m is k-chordal then so is G^{m+2}, where $k \geq 3$ is an integer. Studies on families of graphs that are closed under powering can also be seen in the literature. For instance, it is known that interval graphs, proper interval graphs [14], strongly chordal graphs [13], circular-arc graphs [15][12], cocomparability graphs [11] etc. are closed under taking powers.

Subclasses of bipartite graphs, like chordal bipartite graphs, are not closed under powering since the m-th power of a bipartite graph need not be even bipartite. Chandran et al. in [4] introduced the notion of *bipartite powering* to retain the bipartitedness of a bipartite graph while taking power. The *m-th bipartite power* $G^{[m]}$ of a bipartite graph G is the bipartite graph obtained from G by adding edges (u, v) where $d_G(u, v)$ is odd and less than or equal to m. Note that $G^{[m]} = G^{[m+1]}$ for each odd m. It was shown in [4] that the m-th bipartite power of a tree is chordal bipartite. The intention there was to construct chordal bipartite graphs of high boxicity. The fact that the chordal bipartite graph under consideration was obtained as a bipartite power of a tree was crucial for proving that its boxicity was high. Since trees are a subclass of chordal bipartite graphs, a natural question that came up was the following: is it true that the m-th bipartite power of every chordal bipartite graph is chordal bipartite? In this paper we answer this question in the affirmative. In fact, we prove a more general result.

Our Result

Let m, k be positive integers with $k \geq 4$. Let G be a bipartite graph. If G is k-chordal, then so is $G^{[m]}$. Note that the special case when $k = 4$ gives us the following result: chordal bipartite graphs are closed under bipartite powering.

2 Graph Preliminaries

Throughout this paper we consider only finite, simple, undirected graphs. For a graph G, we use $V(G)$ to denote the set of vertices of G. Let $E(G)$ denote its edge set. For every $x, y \in V(G)$, $d_G(x, y)$ represents the distance between x and y in G. For every $u \in V(G)$, $N_G(u)$ denotes its *open neighborhood* in G, i.e. $N_G(u) = \{v \mid (u, v) \in E(G)\}$. A path P on the vertex set $V(P) = \{v_1, v_2, \ldots, v_n\}$ (where $n \geq 2$) has its edge set $E(P) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\}$. Such a path is denoted by $v_1 v_2 \ldots v_n$. If $v_i, v_j \in V(P)$, $v_i P v_j$ is the path $v_i v_{i+1} \ldots v_j$. The length of a path P is the number of edges in it and is denoted by $||P||$. A cycle C with vertex set $V(C) = \{v_1, v_2, \ldots, v_n\}$, and edge set $E(C) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(v_n, v_1)\}$ is denoted as $C = v_1 v_2 \ldots v_n v_1$. We use $||C||$ to denote the length of cycle C.

3 Holes in Bipartite Powers

Let H be a bipartite graph. Let $\mathcal{B}(H)$ be a family of graphs constructed from H in the following manner: $H' \in \mathcal{B}(H)$ if corresponding to each vertex $v \in V(H)$ there exists a nonempty bag of vertices, say B_v, in H' such that (a) for every $x \in B_u$, $y \in B_v$, $(x, y) \in E(H')$ if and only if $(u, v) \in E(H)$, and (b) vertices within each bag in H' are pairwise non-adjacent. Below we list a few observations about H and every H' (, where $H' \in \mathcal{B}(H)$):

Observation 1. *H' is bipartite.*

Observation 2. *H is an induced subgraph of H'.*

Observation 3. *Let k be an integer such that $k \geq 4$. If H is k-chordal, then so is H'.*

Proof: Any hole of size greater than 4 in H' cannot have more than one vertex from the same bag, say B_v, as such vertices have the same neighborhood. Hence, the vertices of a hole (of size greater than 4) in H' belong to different bags and thus there is a corresponding hole of the same size in H. □

Theorem 4. *Let m, k be positive integers with $k \geq 4$. Let G be a bipartite graph. If G is k-chordal, then so is $G^{[m]}$.*

Proof: We prove this by contradiction. Let p denote the size of a largest induced cycle, say $C = u_0u_1 \ldots u_{p-1}u_0$, in $G^{[m]}$. Assume $p > k$. Then, $p \geq 6$ (since $k \geq 4$ and $G^{[m]}$ is bipartite). Between each u_{i-1} and u_i, where $i \in \{0, \ldots, p-1\}$, there exists a shortest path of length not more than m in G [(i)]. Let P_i be one such shortest path between u_{i-1} and u_i in G.

Let H be the subgraph induced on the vertex set $\bigcup_{i=0}^{p-1} V(P_i)$ in G. As mentioned in the beginning of this section, construct a graph H' from H, where $H' \in \mathcal{B}(H)$, in the following manner: for each $v \in V(H)$, let $|B_v| = |\{P_i \mid 0 \leq i \leq p-1, v \in V(P_i)\}|$ i.e., let B_v have as many vertices as the number of paths in $\{P_0 \ldots P_{p-1}\}$ that share vertex v in H. For each $i \in \{0, \ldots, p-1\}$, let $Q'_i = u_{i-1}Q_i$ be a shortest path between u_{i-1} and u_i in H' such that no two paths Q_i and Q_j (where $i \neq j$) share a vertex [(i)]. From our construction of H' from H it is easy to see that such paths exist. Let $Q_i = v_{i,1}v_{i,2} \ldots v_{i,r_i}u_i$, where $r_i = ||Q_i|| \geq 0$. Thus, $Q'_i = u_{i-1}v_{i,1}v_{i,2} \ldots v_{i,r_i}u_i$. Clearly, $||Q'_i|| = ||P_i|| \leq m$. The reader may also note that the cycle C $(= u_0u_1 \ldots u_{p-1}u_0)$ which is present in $G^{[m]}$ will be present in $H^{[m]}$ and thereby in $H'^{[m]}$ too.

In order to prove the theorem, it is enough to show that there exists an induced cycle of size at least p in H'. Then by combining Observation 3 and the fact that H is an induced subgraph of G, we get $k \geq \mathcal{C}(G) \geq \mathcal{C}(H) \geq \mathcal{C}(H') \geq p$ contradicting our assumption that $p > k$. Hence, in the rest of the proof we show that $\mathcal{C}(H') \geq p$.

Consider the following drawing of the graph H'. Arrange the vertices $u_0, u_1, \ldots, u_{p-1}$ in that order on a circle in clockwise order. Between each u_{i-1} and u_i on the circle arrange the vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,r_i}$ in that order in clockwise order. Recall that these vertices are the internal vertices of path Q'_i.

Claim 4.1. In this circular arrangement of vertices of H', each vertex has an edge (in H') with both its left neighbor and right neighbor in the arrangement.

Let $x_1, x_2 \in V(H')$, where $x_1 \in V(Q_i)$, $x_2 \in V(Q_j)$. We define the *clockwise distance from x_1 to x_2*, denoted by $clock\_dist(x_1, x_2)$, as the minimum non-negative integer s such that $j = i + s$. Similarly, the *clockwise distance from x_2 to x_1*, denoted by $clock\_dist(x_2, x_1)$, is the minimum non-negative integer s' such that $i = j + s'$. Let $x, y, z \in V(H')$. We say $y <_x z$ if scanning the vertices of H' in clockwise direction along the circle starting from x, vertex y is encountered before z. Let $x \in V(Q_i)$. Vertex y is called the *farthest neighbor of x before z* if $y \in N_{H'}(x)$, $y \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$, $y <_x z$, and for every other $w \in N_{H'}(x)$ either $z <_x w$ or $w \notin V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$ or both.

Claim 4.2. There always exists a vertex which is the farthest neighbor of x before z, unless $(x, z) \in E(H')$ and $z \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$.

[(i)] throughout this proof expressions involving subscripts of u, P, Q, and Q' are to be taken modulo p. Every such expression should be evaluated to a value in $\{0, \ldots, p-1\}$. For example, consider a vertex u_i, where $i < p$ Then, $p + i = i$.

Let $\{A, B\}$ be the bipartition of the bipartite graph H'. We categorize the edges of H' as follows: an edge $(x, y) \in E(H')$ is called an *l-edge*, if $l = \min(clock\_dist(x, y), clock\_dist(y, x))$.

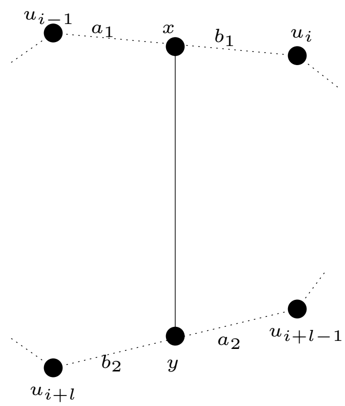

Fig. 1: $x \in V(Q_i), y \in V(Q_{i+l})$ and let $(x, y) \in E(H')$ be an l-edge, where $l > 2$. The dotted line between u_{i-1} and u_i indicate the path Q_i. Similarly, the dotted line between u_{i+l-1} and u_{i+l} indicate the path Q_{i+l}.

Claim 4.3. H' cannot have an l-edge, where $l > 2$.

Proof: Suppose H' has an l-edge, where $l > 2$, between $x \in Q_i$ and $y \in Q_{i+l}$ (see Fig. 1). Let $a_1 = ||u_{i-1}Q_i'x||$, $b_1 = ||xQ_i'u_i||$, $a_2 = ||u_{i+l-1}Q_{i+l}'y||$ and $b_2 = ||yQ_{i+l}'u_{i+l}||$. We consider the following two cases:
Case 1*:* l is even
In this case u_{i-1} and u_{i+l-1} will be on the same side of the bipartite graph H'. Without loss of generality, let $u_{i-1}, u_{i+l-1} \in A$. Then, $u_i, u_{i+l} \in B$. We know that, for every $w_1, w_2 \in V(H'^{[m]})$ with $w_1 \in A$ and $w_2 \in B$, if $(w_1, w_2) \notin E(H'^{[m]})$ then $d_{H'}(w_1, w_2) \geq m + 2$ (recalling m and $d_{H'}(w_1, w_2)$ are odd integers). Therefore, we have $a_1 + 1 + b_2 \geq d_{H'}(u_{i-1}, u_{i+l}) \geq m + 2$. Similarly, $b_1 + 1 + a_2 \geq d_{H'}(u_i, u_{i+l-1}) \geq m + 2$. Summing up the two inequalities we get, $(a_1 + b_1) + (a_2 + b_2) \geq 2m + 2$. This implies that either $||Q_i'||$ or $||Q_{i+l}'||$ is greater than m which is a contradiction.
Case 2*:* l is odd
(proof is similar to the above case and hence omitted).

Hence we prove the claim. □

We find a cycle $C' = z_0z_1 \ldots z_qz_0$ in H' using Algorithm 3.1 [(i)]. Please read the algorithm before proceeding further. .

[(i)] throughout this proof expressions involving subscripts of z are to be taken modulo $q + 1$. Every such expression should be evaluated to a value in $\{0, \ldots, q\}$. For example, consider a vertex z_a, where $a < q + 1$. Then, $q + 1 + a = a$.

Algorithm 3.1 Finding Cycle C' in H' such that $||C'|| \geq ||C||$

1. $l \leftarrow \max_{l'}(H'$has an l'-edge$)$. Without loss of generality assume that this l-edge is between a vertex in Q_0 and a vertex in Q_l
2. Scan the vertices of Q_0 in clockwise direction to find the first vertex z_0, where $z_0 \in V(Q_0)$, which has an l-edge to a vertex in Q_l.
3. Scan the vertices of Q_l in clockwise direction to find the last vertex in Q_l which is a neighbor of z_0 in H'. Call it z_1.
4. Find the farthest neighbor of z_1 before z_0. Call it z_2. /* refer proof of Claim 4.4 for a proof of existence of such a z_2*/
5. $s \leftarrow 2$.

while $(z_s, z_0) \notin E(H')$ **do**

6. Find the farthest neighbor of z_s before z_0. Call it z_{s+1}. /* such a neighbor exists by Claim 4.2*/
7. $s \leftarrow s + 1$.

end while

8. $q \leftarrow s$.
9. Return cycle $C' = z_0 z_1 \ldots z_q z_0$.

Claim 4.4. There always exists a farthest neighbor of z_1 before z_0.

Proof: Note that $z_0 \in Q_0$ and $z_1 \in Q_l$, where $l \leq 2$ (by Claim 4.3). Recalling that $||C|| = p \geq 6$, we have $z_0 \notin V(Q_l) \cup V(Q_{l+1}) \cup V(Q_{l+2})$. Hence by Claim 4.2, the claim is true. □

Claim 4.5. The while loop in Algorithm 3.1 terminates after a finite number of iterations.

Proof: From Claim 4.1, we know that each vertex has an edge (in H') with both its left neighbor and right neighbor in the circular arrangement. Each time when Step 6 of Algorithm 3.1 is executed, a vertex z_{s+1} is chosen such that z_{s+1} is the farthest neighbor of z_s before z_0. Since H' is a finite graph, there will be a point of time in the execution of the algorithm when in Step 6 it picks a z_{s+1} such that $(z_{s+1}, z_0) \in E(H')$. □

From Claim 4.5, we can infer that C' is a cycle.

Claim 4.6. C' is an induced cycle in H'.

Proof: Suppose C' is not an induced cycle. Then there exists a chord (z_a, z_b) in C'. Since (z_a, z_b) is a chord, we have $b \neq a - 1$ or $b \neq a + 1$. Let $l = \max_{l'}(H'$ has an l'-edge$)$. Let $z_a \in V(Q_i)$, $z_b \in V(Q_j)$. We know that $\min(clock\_dist(z_a, z_b), clock\_dist(z_b, z_a)) \leq l$. Without loss of generality, assume $clock\_dist(z_a, z_b) \leq l \leq 2$ (from Claim 4.3). That is, $j - i \leq l \leq 2$ and (z_a, z_b) is a $(j - i)$-edge. If $z_a = z_0$, then $z_b \neq z_1$ and the algorithm exits from the while loop, when $q = b$, thus returning a cycle $z_0 \ldots z_b z_0$. But in such a cycle (z_b, z_0) is not a chord. Therefore, $z_a \neq z_0$. Similarly, $z_b \neq z_0$. We know that $z_{a+1} \neq z_b$, $z_{a+1} <_{z_a} z_b$, and $z_{a+1} \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$. Since $j - i \leq 2$, $z_b \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$. If $z_b <_{z_a} z_0$, then it contradicts the fact that z_{a+1} is the farthest neighbor of z_a before z_0. Therefore, $z_0 <_{z_a} z_b$. Then, either $z_b = z_1$ or $z_1 <_{z_a} z_b$. Recall that $l = \max_{l'}(H'$has an l'-edge$)$, and (z_0, z_1) is an l-edge with $z_0 \in V(Q_0)$ and $z_1 \in V(Q_l)$. Since (i) (z_a, z_b) is a $(j - i)$-edge, where $j - i \leq l$, (ii) $z_0 <_{z_a} z_b$, and (iii) $z_b = z_1$ or $z_1 <_{z_a} z_b$, we have $l \geq j - i = clock\_dist(z_a, z_b) \geq clock\_dist(z_0, z_b) \geq clock\_dist(z_0, z_1) = l$. Hence, $j - i = l$ and

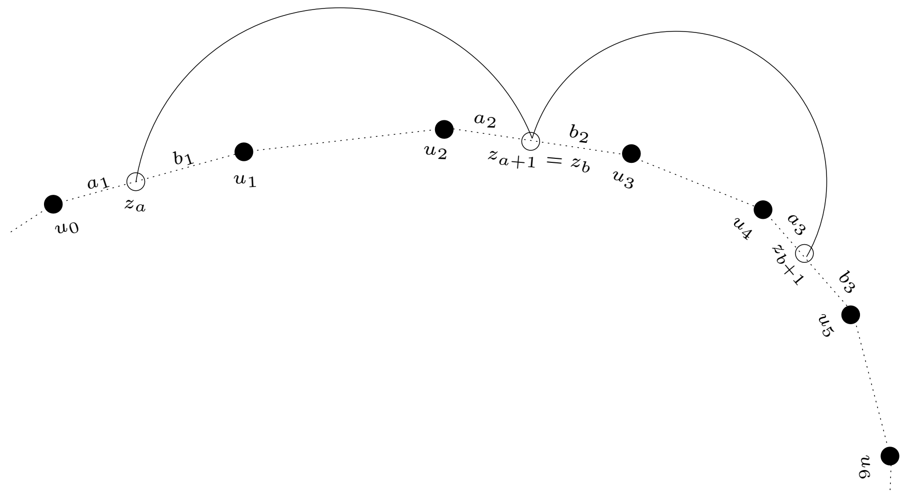

Fig. 2: Figure illustrates the case when path P defined in Claim 4.8 is a trivial path. The dotted lines between each u_{i-1} and u_i indicate the path Q'_i. Each continuous arc corresponds to an edge in the cycle $C' = z_0 \ldots z_q z_0$.

(z_a, z_b) is an l-edge. We know that (z_0, z_1) is also an l-edge with $z_0 \in V(Q_0)$ and $z_1 \in V(Q_l)$. Since $z_0 <_{z_a} z_b$ and $z_b = z_1$ or $z_1 <_{z_0} z_b$, we get $z_a \in V(Q_0)$ and $z_b \in V(Q_l)$. From Step 2 of the algorithm we know that z_0 is the first vertex (in a clockwise scan) in Q_0 which has an l-edge to a vertex in Q_l. This implies that, since $z_0 <_{z_a} z_b$, $z_a = z_0$ which is a contradiction. Hence we prove the claim. □

What is left now is to show that $q + 1 \geq p$, i.e., $||C'|| \geq ||C||$, where $C' = z_0 \ldots z_q z_0$ and $C = u_0 \ldots u_{p-1} u_0$. In order to show this, we state and prove the following claims.

Claim 4.7. For every $j \in \{0, \ldots, p-1\}$, $(V(Q_j) \cup V(Q_{j+1})) \cap V(C') \neq \emptyset$.

Proof: Suppose the claim is not true. Find the minimum j that violates the claim. Clearly, $j \neq 0$ as $z_0 \in V(Q_0)$. We claim that $z_q \in V(Q_{j-1})$. Suppose $z_q \notin V(Q_{j-1})$. Let $a = \max\{i \mid z_i \in V(Q_{j-1})\}$ (note that, since $j \neq 0$, by the minimality of j, $(V(Q_{j-1}) \cup V(Q_j)) \cap V(C') \neq \emptyset$ and therefore $V(Q_{j-1}) \cap V(C') \neq \emptyset$). Since $z_a \neq z_q$, by the maximality of a, we have $z_{a+1} \notin V(Q_{j-1})$. From our assumption, $(V(Q_j) \cup V(Q_{j+1})) \cap V(C') = \emptyset$ and therefore $z_{a+1} \notin V(Q_{j-1}) \cup V(Q_j) \cup V(Q_{j+1})$. Thus $z_a \neq z_q$ and z_{a+1} is not the farthest neighbor of z_a before z_0. This is a contradiction to the way z_{a+1} is chosen by Algorithm 3.1. Hence, $z_q \in V(Q_{j-1})$. We know that $(z_q, z_0) \in E(H')$ with $z_q \in V(Q_{j-1})$ and $z_0 \in V(Q_0)$. Since $l = \max_{l'}(H'$ has an l'-edge), we have $\min(clock\_dist(z_q, z_0), clock\_dist(z_0, z_q)) \leq l$. That is, $j \geq p + 1 - l$ or $j \leq 1 + l$. As $l \leq 2$ (by Claim 4.3), we have $j = p - 1$ or $j \leq 1 + l$. Since $z_0 \in V(Q_0)$, $(V(Q_{p-1}) \cup V(Q_0)) \cap V(C') \neq \emptyset$ and hence $j \neq p - 1$. Therefore, $j \leq 1 + l$. Since $z_0 \in V(Q_0)$ and $z_1 \in V(Q_l)$ (recall $l \leq 2$), we get $j = 1 + l$. We know that, for every $z_a, z_b \in V(C')$, if $a < b$ then $z_a <_{z_0} z_b$. Therefore, $z_1 <_{z_0} z_q$. We have $z_1 \in V(Q_l)$. Since $j = 1 + l$, we also have $z_q \in V(Q_l)$. Thus, we have $z_1, z_q \in V(Q_l)$ and $z_1 <_{z_0} z_q$. But this contradicts the fact that z_1 is the last vertex in Q_l encountered in a clockwise scan that has z_0 as its neighbor. □

Claim 4.8. Let $(z_a, z_{a+1}), (z_b, z_{b+1}) \in E(C')$ be two 2-edges, where $a < b$. Let P, P' denote the clockwise $z_{a+1} - z_b$, $z_{b+1} - z_a$ paths respectively in C'. Both P and P' contain at least one 0-edge.

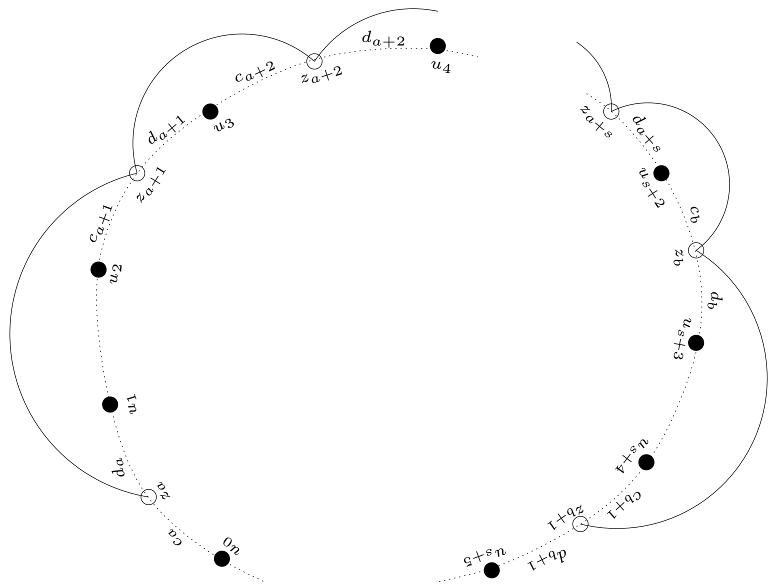

Fig. 3: Figure illustrates the case when path P defined in Claim 4.8 is $P = z_{a+1}z_{a+2}\dots z_{a+1+s}$, where $s \geq 1$ and $z_{a+1+s} = z_b$. The dotted lines between each u_{i-1} and u_i indicate the path Q'_i. Each continuous arc corresponds to an edge in the cycle $C' = z_0 \dots z_q z_0$.

Proof: Consider the path P (proof is similar in the case of path P'). Path P is a non-trivial path only if $z_{a+1} \neq z_b$. Suppose $z_{a+1} = z_b$ (see Fig. 2). Let $z_a \in V(Q_f)$. For the sake of ease of notation, assume $f = 1$ (the same proof works for any value of f). Let $a_1 = ||u_0Q'_1z_a||$, $b_1 = ||z_aQ'_1u_1||$, $a_2 = ||u_2Q'_3z_b||$, $b_2 = ||z_bQ'_3u_3||$, $a_3 = ||u_4Q'_5z_{b+1}||$, and $b_3 = ||z_{b+1}Q'_5u_5||$. We know that, for every $w_1, w_2 \in V(H'^{[m]})$ with $w_1 \in A$ and $w_2 \in B$, if $(w_1, w_2) \notin E(H'^{[m]})$ then $d_{H'}(w_1, w_2) \geq m + 2$. Since $(u_0, u_3) \notin E(H'^{[m]})$, $(u_1, u_4) \notin E(H'^{[m]})$ and $(u_2, u_5) \notin E(H'^{[m]})$, we have $a_1 + b_2 \geq m + 1$, $b_1 + a_3 \geq m$, and $a_2 + b_3 \geq m + 1$. Adding the three inequalities and by applying an easy averaging argument we can infer that either $a_1 + b_1 = ||Q_1|| > m$, $a_2 + b_2 = ||Q_3|| > m$, or $a_3 + b_3 = ||Q_5|| > m$ which is a contradiction. Therefore P is a non-trivial path i.e., $z_{a+1} \neq z_b$. Assume P does not contain any 0-edge. Let $P = z_{a+1}z_{a+2}\dots z_{a+1+s}$, where $s \geq 1$, $a + 1 + s = b$, and $(z_{a+1}, z_{a+2})\dots(z_{a+s}, z_{a+1+s})$ are 1-edges (see Fig. 3). Since $(u_0, u_3) \notin E(H'^{[m]})$, $(u_1, u_4) \notin E(H'^{[m]})$, we have $c_a + d_{a+1} \geq m + 1$ and $d_a + d_{a+2} \geq m$ (please refer Fig. 3 for knowing what $c_a, d_a, \dots, c_{b+1}, d_{b+1}$ are). Summing up the two inequalities, we get $d_{a+1} + d_{a+2} \geq 2m + 1 - (c_a + d_a)$. We know that, for each $i \in \{0, \dots p - 1\}$, $||Q'_i|| \leq m$. Therefore, we have $c_a + d_a \leq m$. Hence, $d_{a+1} + d_{a+2} \geq m + 1$. Since $(c_{a+1} + d_{a+1}) + (c_{a+2} + d_{a+2}) \leq 2m$, we get

$$c_{a+1} + c_{a+2} \leq m - 1 \tag{1}$$

Since $(u_{s+2}, u_{s+5}) \notin E(H'^{[m]})$, $(u_{s+1}, u_{s+4}) \notin E(H'^{[m]})$, we have,

$$\begin{aligned} c_b + d_{b+1} &\geq m + 1 \\ c_{a+s} + c_{b+1} &\geq m \end{aligned}$$

Summing up the two inequalities, we get

$$c_b + c_{a+s} \geq 2m + 1 - (c_{b+1} + d_{b+1})$$

Since $b = a + s + 1$ and $c_{b+1} + d_{b+1} \leq m$, we get

$$c_{a+s+1} + c_{a+s} \geq m + 1 \tag{2}$$

Substituting for $s = 1$ in Inequality 2, we get $c_{a+2} + c_{a+1} \geq m + 1$. But this contradicts Inequality 1. Hence $s > 1$. Suppose $s = 2$. Since $(u_2, u_5) \notin E(H'^{[m]}))$, we have $c_{a+1} + d_{a+3} \geq m$. Adding this with Inequality 2, we get $c_{a+1} + c_{a+2} \geq (2m + 1) - (c_{a+3} + d_{a+3}) \geq m + 1$. But this contradicts Inequality 1. Hence $s > 2$. Since $(u_s, u_{s+3}) \notin E(H'^{[m]})), \ldots, (u_2, u_5) \notin E(H'^{[m]}))$, we have the following inequalities:-

$$\begin{aligned} c_{a+s-1} + d_{a+s+1} &\geq m \\ \vdots \quad &\vdots \quad \vdots \\ c_{a+1} + d_{a+3} &\geq m \end{aligned}$$

Adding the above set of inequalities and applying the fact that $c_i + d_i \leq m$, $\forall i \in \{0, \ldots q\}$, we get $c_{a+1} + c_{a+2} + d_{a+s} + d_{a+s+1} \geq 2m$. Adding this with Inequality 2, we get $c_{a+1} + c_{a+2} \geq (3m + 1) - (c_{a+s+1} + d_{a+s+1}) - (c_{a+s} + d_{a+s}) \geq m + 1$. But this contradicts Inequality 1. Hence we prove the claim. □

Claim 4.9. For every $j, j' \in \{0, \ldots, p - 1\}$, where $j < j'$ and $(V(Q_j) \cup V(Q_{j'})) \cap V(C') = \emptyset$, there exist $i, i' \in \{0, \ldots, p - 1\}$, where only i satisfies $j < i < j'$, such that $|V(Q_i) \cap V(C')| \geq 2$ and $|V(Q_{i'}) \cap V(C')| \geq 2$.

Proof: By Claim 4.7, (i) $j' \neq j + 1$ or $j' \neq j - 1$, and (ii) there exist $r, r' \in \{0, \ldots, q\}$ such that (z_r, z_{r+1}) is a 2-edge with its endpoints on Q_{j-1} and Q_{j+1} and $(z_{r'}, z_{r'+1})$ is a 2-edge with its endpoints on $Q_{j'-1}$ and $Q_{j'+1}$. By Claim 4.8, we know that if P, P' denote the clockwise $z_{r+1} - z_{r'}$, $z_{r'+1} - z_r$ paths respectively in C', then both P and P' contains at least one 0-edge. This proves the claim. □

In order to show that the size of cycle C' $(= z_0 \ldots z_q z_0)$ is at least p, we consider the following three cases:-

Case $|\{Q_j \in \{Q_0 \ldots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| = 0$: In this case, for every $j \in \{0, \ldots p - 1\}$, Q_j contributes to $V(C')$ and therefore $||C'|| \geq p = ||C||$.

Case $|\{Q_j \in \{Q_0 \ldots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| = 1$: Let Q_j be that only path (among $Q_0 \ldots Q_{p-1}$) that does not contribute to $V(C')$. Then we claim that there exists a $Q_{j'}$, where $j' \neq j$, such that $V(C') \cap V(Q_{j'}) \geq 2$. Suppose the claim is not true then it is easy to see that $||C'|| = p - 1$ which is an odd number thus contradicting the bipartitedness of H'. Hence the claim is true. Now, by applying the claim it is easy to see that $||C'|| = \sum_j |V(C') \cap V(Q_j)| \geq p = ||C||$.

Case $|\{Q_j \in \{Q_0 \ldots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| > 1$: Scan vertices of H' starting from any vertex in clockwise direction. Claim 4.9 ensures that between every Q_j and $Q_{j'}$, which do not contribute to $V(C')$, encountered there exists a Q_i which compensates by contributing at least two vertices to $V(C')$. Therefore, $||C'|| \geq p = ||C||$. ■

4 Discussion

An interesting open question that naturally follows from our result is the following: given a graph G and positive integers k, m where $k \geq 4$, if $G^{[m]}$ is k-chordal, then is $G^{[m+2]}$ also k-chordal? As mentioned earlier, Brandstädt et al. in [3] showed a similar result in the context of *ordinary* graph powering. They showed that, for every graph G, if G^m is k-chordal, then so is G^{m+2}, where k, m are positive integers with $k \geq 3$. A straightforward extension of their proof technique doesn't seem to work in our context due to the bipartite nature of the powering that we consider.

References

[1] R. Balakrishnan and P. Paulraja. Powers of chordal graphs. *J. Aust. Math. Soc. Ser. A*, 35:211–217, 1983.

[2] H. L. Bodlaender and D. M. Thilikos. Treewidth for graphs with small chordality. *Discrete Applied Mathematics*, 79:45–61, 1997.

[3] Andreas Brandstädt, Van Bang Le, and Thomas Szymczak. Duchet-type theorems for powers of hhd-free graphs. *Discrete Mathematics*, 177(1-3):9–16, 1997.

[4] L. Sunil Chandran, Mathew C. Francis, and Rogers Mathew. Chordal bipartite graphs with high boxicity. *Graphs and Combinatorics*, 27(3):353–362, 2011.

[5] L. Sunil Chandran and L. Shankar Ram. On the number of minimum cuts in a graph. In *Proceedings of the 8th International computing and combinatorics conference, LNCS 2387*, pages 220–230, 2002.

[6] L. Sunil Chandran, C.R. Subramanian, and Vadin V. Lozin. Graphs of low chordality. To appear in Dicrete Mathematics and Theoretical Computer Science, 2005.

[7] Gerard J. Chang and George L. Nemhauser. The k-domination and k-stability problems on sun-free chordal graphs. *SIAM Journal on Algebraic and Discrete Methods*, 5(3):332–345, 1984.

[8] Yon Dourisboure. Compact routing schemes for generalised chordal graphs. *Journal of Graph Algorithms and Applications*, 9:277–297, 2005.

[9] Feodor F. Dragan. Estimating all pairs shortest paths in restricted graph families: a unified approach. *Journal of Algorithms*, 57(1):1 – 21, 2005.

[10] P. Duchet. Classical perfect graphs. *Ann. Discrete Math.*, 21:67–96, 1984.

[11] Carsten Flotow. On powers of m-trapezoid graphs. *Discrete Appl. Math.*, 63(2):187–192, 1995.

[12] Carsten Flotow. On powers of circular arc graphs and proper circular arc graphs. *Discrete Appl. Math.*, 69(3):199–207, 1996.

[13] A Lubiw. γ-free matrices. Master's thesis, Dept. of combinatorics and Optimization, University of Waterloo, 1982.

[14] A. Raychaudhuri. On powers of interval and unit interval graphs. *Congr.Numerantium*, 59:235–242, 1987.

[15] A. Raychaudhuri. On powers of strongly chordal and circular graphs. *Ars Combinatoria*, 34:147–160, 1992.

[16] Jeremy P. Spinrad. Finding large holes. *Information Processing Letters*, 39(4):227 – 229, 1991.