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# *Two player game variant of the Erdős-Szekeres problem*

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The classical Erdős-Szekeres theorem states that a convex $k$-gon exists in every sufficiently large point set. This problem has been well studied and finding tight asymptotic bounds is considered a challenging open problem. Several variants of the Erdős-Szekeres problem have been posed and studied in the last two decades. The well studied variants include the empty convex $k$-gon problem, convex $k$-gon with specified number of interior points and the chromatic variant.

In this paper, we introduce the following two player game variant of the Erdős-Szekeres problem: Consider a two player game where each player playing in alternate turns, place points in the plane. The objective of the game is to avoid the formation of the convex k-gon among the placed points. The game ends when a convex k-gon is formed and the player who placed the last point loses the game. In our paper we show a winning strategy for the player who plays second in the convex 5-gon game and the empty convex 5-gon game by considering convex layer configurations at each step. We prove that the game always ends in the 9th step by showing that the game reaches a specific set of configurations.

**Keywords:** Erdős-Szekeres problem, Ramsey theory, convex $k$-gon, empty convex $k$-gon

## 1 Introduction

The *Erdős-Szekeres Problem* is defined as follows:

> *For any integer k, $k \geq 3$, determine the smallest positive integer N(k) such that any planar point set in general position that has at least N(k) points contains k points that are the vertices of a convex k-gon.*

In 1935 Erdős and Szekeres proved the finiteness of $N(k)$ using Ramsey theory [10]. There has been a series of improvements to bound the value of $N(k)$ and the current best known bounds are

$$2^{k-2} + 1 \leq N(k) \leq \binom{2k-5}{k-2} + 1,$$

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see [9, 18, 28]. Erdős and Szekeres conjectured that the current lower bound is tight. This conjecture has been proved for $k \leq 6$. $N(4) = 5$ was a geometric observation by Esther Klien. $N(5) = 9$ was shown by Kalbfeisch *et al.* [26]. $N(6) = 17$ has been proved by Szekeres and Peters using an combinatorial model of planar configurations [27]. See survey [23, 4] for a detailed description about the history of the problem and its variants.

Erdős asked the *Empty Convex $k$-gon Problem* which is defined as follows:

> *For any integer k, $k \geq 3$, determine the smallest positive integer $H(k)$ such that any planar point set in general position that has at least $H(k)$ points contains k points that are the vertices of a an empty convex $k$-gon, i.e., the vertices of a convex $k$-gon containing no points in its interior.*

It is easy to see that $H(4) = 5$. $H(5) = 10$ was proved by Harboth [14]. Gerken [13] and Nicholes [24] proved independently the existence of a empty hexagon. The current best bounds on $H(6)$ are $30 \leq H(6) \leq 463$ [19, 25]. Horton showed that $H(k)$ does not exist for any $k$, $k \geq 7$ [16].

The Erdős-Szekeres Problem has been studied in higher dimensions [10, 9, 29]. Other related variants that are well studied are the convex $k$-gon with specified number of interior points [2, 3, 11, 17, 30] and the chromatic variant [8, 1].

We introduce a two player game variant of Erdős-Szekeres Problem:
*Consider a two player game where each player playing in alternate turns, place points in the plane. The game ends when a convex k-gon is formed and the player who placed the last point loses the game.*

The goal of both the players is to avoid losing the game. In the scenario where a player is bound to lose the game, his goal is to prolong the game as much as possible.

Since computational efficiency is not an issue in our context, we assume that both players follow an optimal strategy that achieves their goal.

Since $N(k)$ is finite, we know that the game will end in atmost $N(k)$ number of steps. Can the game end before $N(k)$ steps?

Define $N_G(k)$ as the minimum number of steps for the game to end when both the players follow their optimal strategy.

In this paper we focus on finding the exact value of $N_G(k)$. We denote the player who plays first as player 1 and the player who plays second as player 2.

We also consider the two player game for the empty convex $k$-gon and correspondingly define $H_G(k)$. It is easy to see that $N_G(3) = H_G(3) = 3$, $N_G(4) = H_G(4) = 5$.

Combinatorial two player games have been well studied in the Maker/Breaker setting which is defined as follows:

> *Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the game (X, $\mathcal{F}$), two players (Maker and Breaker) take turns in claiming one previously unclaimed element of X. The objective of the Maker is to claim some $f \in \mathcal{F}$ i.e claim all the elements of f. The objective of the Breaker is to prevent the Maker from claiming any $f \in \mathcal{F}$.*

Chvátal and Erdős introduced the Maker/Breaker-type game [7] for spanning trees, Hamiltonian cycles on graphs. There are a lot of results on Maker/Breaker-type games where $(X, \mathcal{F})$ denotes random graphs, non planar graphs, near-perfect matchings, $k$-connected spanning subgraphs. There has also been a long line of research that studies the biased version of the various Maker/Breaker-type games [15, 5, 6, 12, 20, 21].

A complimentary variant of the Maker/Breaker game is the Avoider/Enforcer game where one player wants to avoid the formation of a structure and the other player wants to force the formation of that structure [22, 5].

The two player game variant of the Erdős-Szekeres Problem that we study in this paper is different from the Maker/Breaker, Avoider/Enforcer games since in our game the objective of both the players is the same, i.e., avoiding the formation of convex $k$-gon.

In this paper, we focus on the Erdős-Szekeres two player game for $k = 5$. We show a winning strategy for player 2 and prove that $N_G(5) = 9$ and $H_G(5) = 9$. i.e., the game will end in the 9th step.

We consider convex layer configurations at each step and give a strategy for player 2 such that the game will reach a specific set of configurations until the 8th step and finally we argue in the 9th step that a convex 5-gon or an empty convex 5-gon is formed.

We further show that the convex 5-gon and empty convex 5-gon game always ends in the 9th step, i.e., there is no strategy for player 2 to end the game earlier.

The paper is organized as follows. Section 2 contains the preliminaries and definitions that we will be using in the rest of the paper. Section 3 describes the proof for the convex 5-gon game. Section 4 describes the proof for the empty convex 5-gon game.

## 2 Preliminaries and Definitions

We assume in the rest of this paper that our point set $P$ is in general position, i.e., no 3 points of the point set are collinear. We denote the convex hull of a point set $P$ as $conv(P)$ and the vertices of $conv(P)$ as $CH(P)$.

**Definition 2.1** ***Convex k-gon:*** *is a convex polygon with k vertices.*

**Definition 2.2** ***Empty Convex k-gon:*** *is a convex k-gon with no points in its interior.*

**Definition 2.3** ***Points in convex position:*** *A point set $P$ is said to be in convex position if $CH(P) = P$.*

**Definition 2.4** ***Type of a point set P:*** *A point set $P$ is called $type(i_1, i_2, ...i_k)$, $|P| = \sum |i_k|$, if $P_1 = CH(P)$ is of size $i_1$, $P_2 = CH(P \setminus P_1)$ is of size $i_2 \ldots$*

The $type$ of point set $P$ describes the sizes of the different convex layers of $P$. We denote $P_1$ as the first convex layer and $P_2$ as the second convex layer.

**Definition 2.5** ***U(i,j) of point set P:*** *Point set having $i$ points of the first convex layer of $P$ and $j$ points of the second convex layer of $P$.*

**Definition 2.6** ***Type 1 Beam:*** *$A : BC$ denotes the region of the plane formed by deleting triangle $ABC$ from the convex region in the plane bounded by the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (see Fig. 1).*

**Definition 2.7** ***Type 2 Beam:*** *$AB : CD$ denotes the region of the plane formed by deleting convex 4-gon $ABCD$ from the convex region in the plane bounded by the segment $\overline{AB}$ and the rays $\overrightarrow{AD}$ and $\overrightarrow{BC}$ (see Fig. 2).*

Let $A, B, C, D$ be 4 points in convex position.

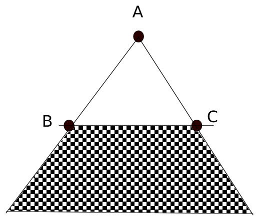



**Fig. 1:** Type 1 beam

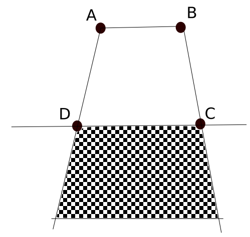



**Fig. 2:** Type 2 beam

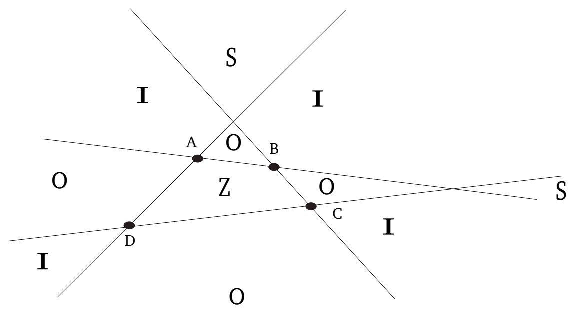



**Fig. 3:** Regions of a convex 4-gon

**Definition 2.8** ***Regions of convex 4-gon:*** *The $ABCD$ convex 4-gon divides the plane into 4 types of regions $I, O, S, Z$ (see Fig. 3). $O$ region is the region such that any point in it along with $ABCD$ forms a convex 5-gon. $I$ and $Z$ regions are the regions such that any point in these regions along with $ABCD$ forms a point set of $type(4,1)$. $I$ region is outside the convex 4-gon and $Z$ region is inside the convex 4-gon. $S$ region is the region such that any point in it along with $ABCD$ forms a point set of $type(3,2)$.*

**Definition 2.9** ***Regions of a configuration:*** *A configuration divides the plane into 4 types of regions $I, O, S, Z$. A region is called an $O$ region of a configuration if it is an $O$ region with respect to some convex 4-gon. The complement of the union of all the $O$ regions of a configuration is further divided into $I, S, Z$ regions. A region is called an $I$ region of a configuration if it is an $I$ region with respect to some convex 4-gon. Similarly we define $S$ and $Z$ regions of the configuration.*

**Definition 2.10** ***Feasible regions of a configuration for the convex 5-gon game:*** *A feasible region of a configuration for the convex 5-gon game is a region which is not contained in any of the $O$ regions of the convex 4-gon's of the configuration.*

**Definition 2.11** ***Feasible regions of a configuration for the empty convex 5-gon game:*** *A feasible region of a configuration for the empty convex 5-gon game is a region which is not contained in any of the $O$ regions of the empty convex 4-gon's of the configuration.*

We define the following set of point configurations that will be used in our proofs (see Fig. 4 and 24).

**Definition 2.12** ***Configuration 4:*** *A 4 point set forming a parallelogram.*

**Definition 2.13** ***Configuration 5.1:*** *A 5 point set of $type(4,1)$ where the 4 points in the first convex layer form a parallelogram.*

**Definition 2.14** ***Configuration 5.2:*** *A 5 point set of $type(4,1)$ where the 3 points in the first convex layer with the 1 point in the interior form a parallelogram.*

**Definition 2.15** ***Configuration 6.1:*** *A 6 point set of $type(4,2)$ where 4 points in the first convex layer form a parallelogram and the 2 points inside the parallelogram are symmetrically placed in opposite triangles formed by diagonals of parallelogram.*

**Definition 2.16** ***Configuration 6.2:*** *A 6 point set of $type(4,2)$ where 4 points in the first convex layer form a trapezoid and the 2 points inside the trapezoid are symmetrically placed in opposite triangles formed by the diagonals.*

**Definition 2.17** ***Configuration 7.1:*** *A 7 point set of $type(3,4)$ where the 3 points of the first convex layer are in different $I$ regions of the parallelogram formed by the 4 points in the second convex layer.*

**Definition 2.18** ***Configuration 7.2.1:*** *A 7 point set of $type(4,3)$ that does not have a convex 5-gon.*

**Definition 2.19** ***Configuration 7.2.2:*** *A 7 point set of $type(4,3)$ that has a non-empty convex 5-gon.*

**Definition 2.20** ***Configuration 8:*** *An 8 point set of $type(4,4)$ where the 4 points of the first convex layer are placed such that each point lies in different $I$ region of the convex 4-gon of the second convex layer.*

# 3 Game for the convex 5-gon

In this section, we show that the second player has a winning strategy in the convex 5-gon game and the game ends in 9 moves.

### *Overview of our proof and player 2's strategy to win the convex 5-gon game.*

In our game, player 1 plays in the odd steps (1st, 3rd, ...) and player 2 plays in the even steps (2nd, 4th, ...). In player 1's turn, we argue that any point added without forming an convex 5-gon will always result in specific configurations. In player 2's turn we show a feasible region (*Definition* 2.10) for placing the point that results in specific configurations that are favorable for player 2.

W.l.o.g., assume that the first 3 points of the game forms a triangle. We now describe the winning strategy for player 2:

Player 2 will place the point in the 4th step such that the resultant point set forms a parallelogram (configuration 4). In the 6th step, we show that there exists a feasible region in both configuration 5.1 and 5.2 where the 6th point is placed, so that it will reach configuration 6.1 or configuration 6.2. Similarly in the 8th step we show that there will exist a feasible region in configuration 7.1 and 7.2.1 where the 8th point is placed such that the resultant point set is configuration 8.

In player 1's turn we show that any point added by player 1 without forming an convex 5-gon results in configuration 5.1 or 5.2 (in the 5th step) and in configuration 7.1 or 7.2.1 (in the 7th step).

Finally, we argue that any point added by player 1 to configuration 8 will result in the formation of a convex 5-gon and player 2 will always win in the 9th step.

Any convex 5-gon game follows a path in the game tree shown in Fig. 4.

We further show that there is no better strategy for player 2 to end the game earlier (in the 5th or 7th step) by arguing that all configurations of 4 points and 6 points have a feasible region.

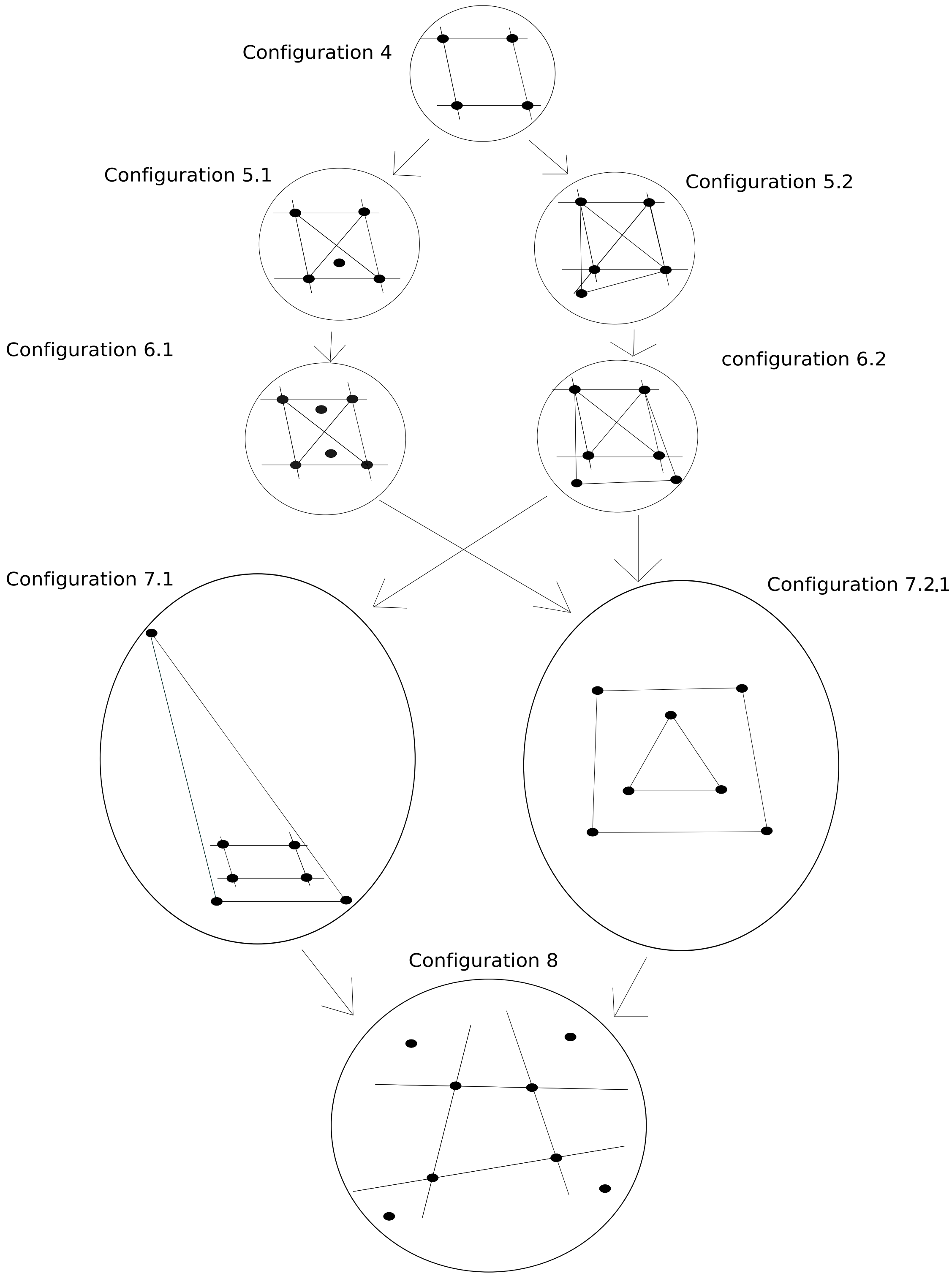



**Fig. 4:** Game tree for the convex 5-gon

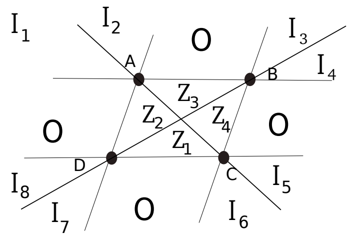



**Fig. 5:** Division of parallelogram regions

### 3.1 Proof for $N_G(5) = 9$

We make the following observations on the number of points that can be contained in type 1 and type 2 beams without forming an empty convex 5-gon. We will assume that the points in the beam are in convex position with the points that form the beam. Thus, if the point set does not contain a convex 5-gon, type 1 beam has atmost 1 point and type 2 beam does not contain any point.

**Lemma 3.1** *A strategy for player 2 exists such that convex 5-gon game will always reach either configuration 6.1 or configuration 6.2 at the end of 6th step.*

**Proof:** A triangle is formed by the first 3 points of the game. The 4th point is placed in such a manner that the resultant point set forms a parallelogram $ABCD$ (configuration 4). For the 5th step, we divide the regions of the parallelogram into $I, O, Z$ regions as shown in Fig. 5. The feasible regions where a point can be placed are the $I$ and $Z$ regions.

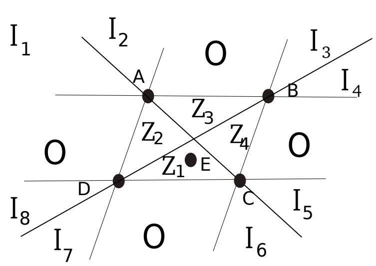



**Fig. 6:** Configuration 5.1

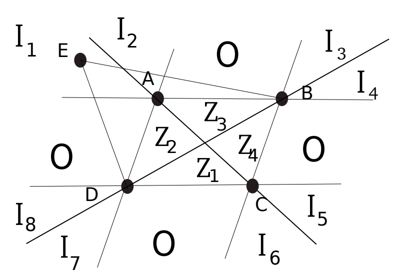



**Fig. 7:** Configuration 5.2

If the 5th point, say $E$, is placed in the interior of the parallelogram i.e., $Z$ region, the resultant point set forms configuration 5.1 (see Fig. 6). If $E$ is placed in $I$ region the resultant point set forms configuration 5.2 (see Fig. 7).

In the 6th step we show a feasible region in configuration 5.1 and configuration 5.2 where a point is added such that the resultant point set formed is either configuration 6.1 or configuration 6.2.

If the 5 point set formed is configuration 5.1, then let us suppose that $E$ has been placed in $Z_1$ region (see Fig. 6), then the 6th point, say $F$ is placed in $Z_3$ such that $EF$ is parallel to $AD$ and this forms configuration 6.1 (see Fig. 8). Let us call this as $\{Z_1, Z_3\}$ point placement. By symmetry $\{Z_2, Z_4\}$ point placement is similar and they form configuration 6.1.

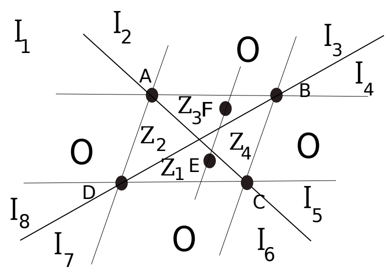



**Fig. 8:** Configuration 6.1

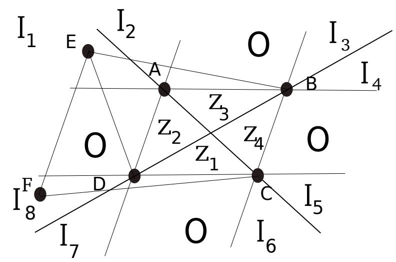



**Fig. 9:** Configuration 6.2

Now we argue for configuration 5.2. Let us suppose that $E$ has been placed in $I_1$ region (see Fig. 7). The 6th point, say $F$, is placed in $I_8$ region such that $EF$ is parallel to $AD$ (see Fig. 9). Let us call this as $\{I_1, I_8\}$ point placement. By symmetry $\{I_2, I_3\}, \{I_4, I_5\}, \{I_6, I_7\}$ point placements are similar and they form configuration 6.2. □

**Lemma 3.2** *Any point added by player 1 to configuration 6.1 or configuration 6.2 without forming an convex 5-gon, results in either configuration 7.1 or configuration 7.2.1.*

**Proof:** We consider 2 cases corresponding to configuration 6.1 and configuration 6.2. We will now describe the $O$ regions of configuration 6.1 and 6.2.

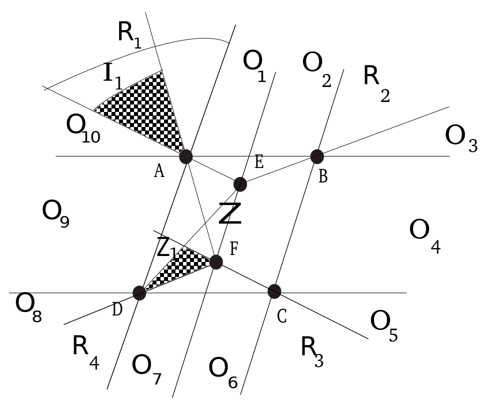



**Fig. 10:** Configuration 6.1 divided into regions

**Case 1 (configuration 6.1):** Let us consider all the convex 4-gons of configuration 6.1 (see Fig. 10). The convex 4-gons of configuration 6.1 are of the form $U(4,0), U(3,1), U(2,2)$.

*U(4,0) of configuration 6.1:* $ABCD$ is the only convex 4-gon of this form. Consider $ABCD$ convex 4-gon. The regions that are contained in the $O$ regions of $ABCD$ are $O_1$, $O_2$, $O_4$, $O_6$, $O_7$, $O_9$.

*U(3,1) of configuration 6.1:* $ABCF$, $BCDE$, $CDAE$, $DABF$ are the convex 4-gons of this form. The regions that are contained in $O$ regions of $ABCF$, $BCDE$, $CDAE$, $DABF$ are $O_k$ where $k = 1, 2, ..10$. $R_1 \setminus I_1$ is also contained in $O$ region of $ABCF$. Fig. 10 shows $I_1$ in $R_1$ region. Symmetrically, there are $I_k$ in other $R_k$ regions. So $R_k \setminus I_k$ where $k = 2, 3, 4$ are also contained in the $O$ regions $BCDE$, $CDAE$, $DABF$ convex 4-gons.

*U(2,2) of configuration 6.1:* $EFDA$, $EFCB$, $BEDF$, $AECF$ are the convex 4-gons of this form and these are empty. Consider $EFDA$ empty convex 4-gon. The regions that are contained in the $O$ regions of $EFDA$ are $O_1$, $O_7$, $O_8$, $O_9$, $O_{10}$. Similarly the regions that are contained in the $O$ regions of $EFCB$ empty convex 4-gon are $O_2$, $O_3$, $O_4$, $O_5$, $O_6$. The $O$ regions of $BEDF$, $AECF$ empty convex 4-gons does not contain the entire region of $Z$ ($Z$ region of $ABCD$ convex 4-gon) and $R_k$ where $k = 1, 2, 3, 4$.

Thus, the feasible regions of configuration 6.1 (the regions that are not contained in any $O$ region of convex 4-gons) are portions of $R_k$ where $k = 1, 2, 3, 4$ and $Z$ region. From Fig. 10, we can verify that region $I_1$ of $R_1$ and $Z_1$ of $Z$ is not contained in any of the $O$ regions of the convex 4-gons. Fig. 10 shows $I_1$ in $R_1$ region and $Z_1$ in $Z$ region. Symmetrically, there are feasible regions $I_k$ in $R_k$ region and $Z_k$ in $Z$ region where $k = 2, 3, 4$. Any point added in $I_k$ or $Z_k$ region will result in configuration 7.2.1.

**Case 2 (configuration 6.2):** Let us consider the convex 4-gons of configuration 6.2 (see Fig. 11). The convex 4-gons of configuration 6.2 are of the form $U(4,0), U(3,1), U(2,2)$.

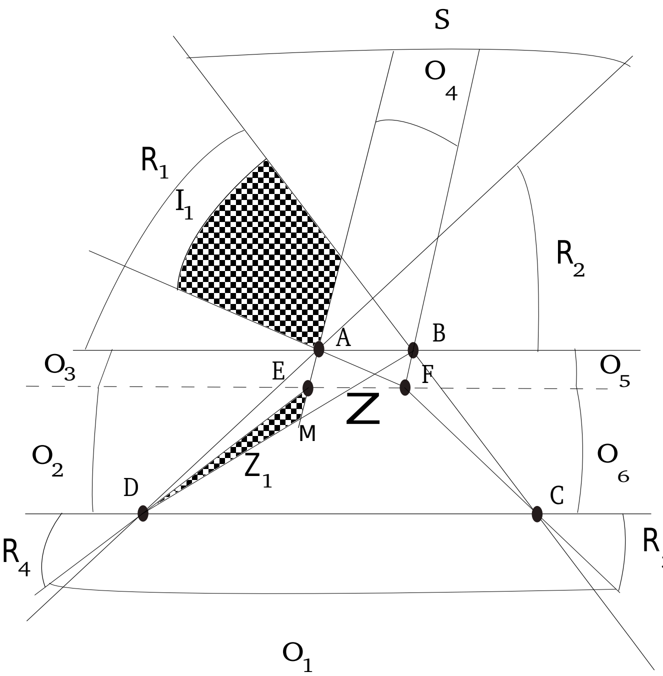



**Fig. 11:** Configuration 6.2 divided into regions

*U(4,0) of configuration 6.2:* $ABCD$ is the only convex 4-gon of this form. Consider $ABCD$ convex 4-gon. The regions that are contained in the $O$ regions of $ABCD$ are $O_2$, $O_3$, $O_5$, $O_6$.

*U(3,1) of configuration 6.2:* $AFCD$, $BEDC$, $ABCE$, $ABFD$ are the convex 4-gons of this form. The regions that are contained in $O$ regions of $AFCD$, $BEDC$, $ABCE$, $ABFD$ are $O_2$, $O_3$, $O_5$, $O_6$. $R_1 \setminus I_1$ is also contained in $O$ region of $AFCD$. Fig. 11 shows $I_1$ in $R_1$ region. Symmetrically, there are $I_k$

in other $R_k$ regions. So $R_k \setminus I_k$ where $k = 2, 3, 4$ are also contained in the $O$ regions of $ABCE$,$BEDC$, $ABFD$ convex 4-gons.

*U(2,2) of configuration 6.2:* $EFCD$, $ABFE$, $AFCE$, $BEDF$ are the convex 4-gons of this form and these are empty. Consider $EFCD$ empty convex 4-gon. The regions that are contained in the $O$ regions of $EFCD$ are $O_2$, $O_6$, $O_1$. Similarly the regions contained in the $O$ regions of $ABFE$ empty convex 4-gon are $O_3$, $O_5$, $O_4$. The O-regions of $AFCE$, $BEDF$ empty convex 4-gons do not contain the entire regions of $Z$ ($Z$ region of $ABCD$ convex 4-gon) and $R_k$ where $k = 2, 3, 4$.

Thus, the feasible regions of configuration 6.2 (the regions that are not contained in any $O$ region of convex 4-gons) is $S \setminus O_4$ and portions of $R_k$ and $Z$ regions. From Fig. 11, we can verify that region $I_1$ of $R_1$ is not contained in any of the $O$ regions of these convex 4-gons. Fig. 11 shows $I_1$ in $R_1$ region and $Z_1$ in $Z$ region. $Z_1$ is the region formed by triangle $EDM$ where $M$ is the intersection point of $\overrightarrow{AE}$ and $\overrightarrow{BD}$. Symmetrically, there are feasible regions $I_k$ in $R_k$ region and $Z_k$ in $Z$ region where $k = 2, 3, 4$.

If the point is placed in $I_k$ of $R_k$ or $Z_k$ of $Z$ where $k = 1, 2, 3, 4$, the resultant point set is configuration 7.2.1. If the point is placed in $S \setminus O_4$ the resultant point set is configuration 7.1. □

Now we prove that given a set of 7 points in configuration 7.1 or 7.2.1 there exists a feasible region where a point is added such that the resultant point set is configuration 8.

**Lemma 3.3** *A (4,3) configuration where the 4 points of the outer convex layer is of the form (I,O,O,O) (3 points each in different O regions and 1 point in a I region of the inner triangle) contains a convex 5-gon.*

**Proof:**

Let the inner triangle be $ABC$ and the 3 points in the $O$ regions of the triangle be $F$, $G$, $H$. Let the point in the $I$ region be $E$ and w.l.o.g let this $I$ region be in between the $O$ regions which contain the points $F$ and $H$ (see Fig. 12). We claim that the pair of rays $(\overrightarrow{BF}, \overrightarrow{CH})$ does not intersect.

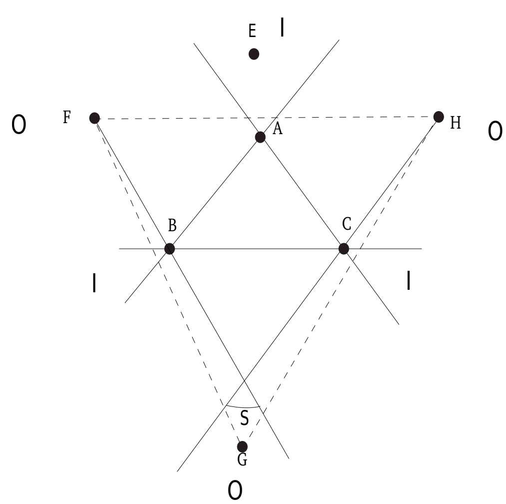



**Fig. 12:** (I,O,O,O) (4,3) convex layer configuration

Consider the $FBCH$ convex 4-gon. Now $G$ should always lie in the $S$ region of this convex 4-gon or else the 3 points $F$, $G$, $H$ would not be containing the triangle $ABC$ inside. To form an $S$ region below $BC$, the rays $\overrightarrow{FB}$ and $\overrightarrow{HC}$ intersects below $BC$. Hence the pair of rays $(\overrightarrow{BF}, \overrightarrow{CH})$ do not intersect.

Now $E$ always lies in the $O$ region of $FBCH$ convex 4-gon or else the point set is not a (4,3) convex layer configuration. Thus $EFBCH$ forms an convex 5-gon. □

**Lemma 3.4** *There exists a feasible region in configuration 7.1 and 7.2.1 such that a point added by player 2 in this feasible region results in configuration 8.*

**Proof:** We consider 2 cases corresponding to configuration 7.1 and configuration 7.2.1.

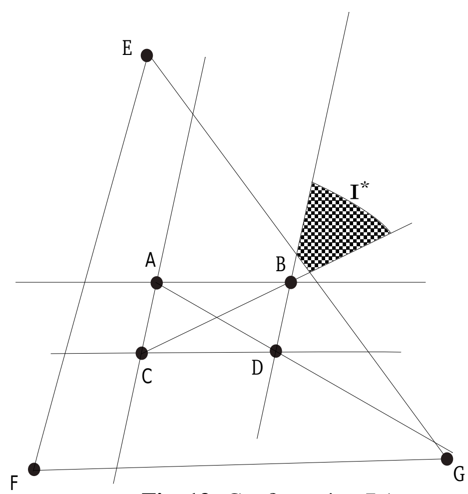



**Fig. 13:** Configuration 7.1

**Case 1 (configuration 7.1):** Consider the region $I^*$ as shown in Fig. 13. We will show that this is a feasible region, i.e., $I^*$ is not contained in any $O$ region of convex 4-gons.

Let us consider all the convex 4-gons of configuration 7.1. The convex 4-gons of configuration 7.1 are of the form $U(2,2)$, $U(1,3)$, $U(0,4)$.

*U(0,4) of configuration 7.1:* $ABCD$ is the only convex 4-gon of this form and has $I^*$ contained in its $I$ region.

*U(1,3) of configuration 7.1:* $EBDA$, $FDBC$, $ADGC$, $ABGC$, $DCEB$, $ABDF$ are the convex 4-gons of this form and these have $I^*$ contained in their $I$ regions.

*U(2,2) of configuration 7.1:* $FCDG$, $EACF$, $ABGF$, $BDFE$, $ADGF$, $FCBG$ are the convex 4-gons of this form and these have $I^*$ in their $I$ regions.

Hence $I^*$ is feasible region where a point is added and the resultant point set that is formed is configuration 8.

**Case 2 (configuration 7.2.1)**: Configuration 7.2.1 is a point set of (4,3) convex layer configuration without a convex 5-gon. First, we give the following characterization of configuration 7.2.1:

*An $I$ region or an $O$ region of the inner triangle cannot have more than 1 point (see Fig. 14).*

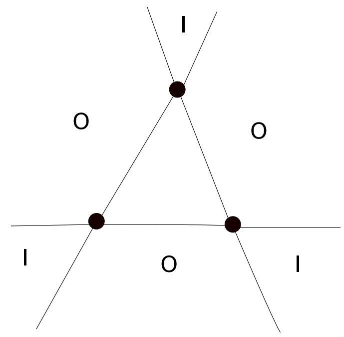



**Fig. 14:** Division of triangle into regions

Consider an $O$ region. If an $O$ region has 2 points then there is an empty convex 5-gon being formed by these 2 points with the 3 points of the triangle. Hence an $O$ region cannot have more than 1 point. Consider an $I$ region. If an $I$ region has 2 points then no other $I$ region or the adjacent $O$ regions has any points because the 2 points in this $I$ region along with the third point in the $I$ region or in the adjacent $O$ region will form an empty convex 5-gon with the side of the triangle. Thus if an $I$ region has 2 points then the $O$ region that is opposite has to have the other 2 points which leads to the formation of an empty convex 5-gon.

Based on the above constraints, the only possible (4,3) convex layer configurations are $(I, I, O, O)$ (2 points each in different $I$ regions, 2 points each in different $O$ regions) and $(I, O, O, O)$ (3 points each in different $O$ regions, 1 point in a $I$ region). By Lemma 3.3, $(I, O, O, O)$ contains a convex 5-gon and hence will not be considered.

**(I,I,O,O) (2 points each in different $I$ regions, 2 points each in different $O$ regions):**

Let us assume that point $G$ lies in region $X_1$ (see Fig. 15). The case when the point $G$ is in $X_2$ can be argued in a similar fashion (see Fig. 17). We have 2 cases corresponding to $EBG$ being a anti-clockwise turn or clockwise turn.

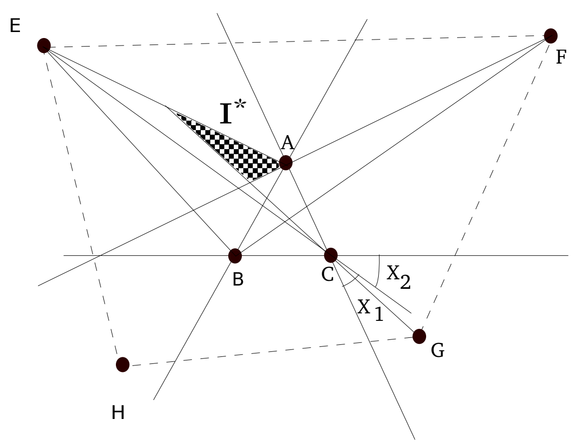



**Fig. 15:** Configuration 7.2.1 (I,I,O,O) G is in X1 and EBG is anticlockwise turn

**Type 1 (when $G$ lies in $X_1$ and $EBG$ is a anti-clockwise turn):** We will show that $I^*$ (triangle region bounded by the $CG$, $FA$ and $EA$) is a feasible region (see Fig. 15). Consider the convex 4-gons of this configuration. The convex 4-gons are either of the type $U(2,2)$, $U(1,3)$ or $U(3,1)$.

*U(1,3) of Type1 Configuration 7.2.1 (I,I,O,O):* $EACB$, $FABC$ are the 2 convex 4-gons of this type. $I^*$ region is in the interior of $EACB$ and it is in the $I$ region of $FABC$. Thus $I^*$ is feasible for these convex 4-gons.

*U(2,2) of Type1 Configuration 7.2.1 (I,I,O,O):* $EABH$, $FACG$, $ECGB$, $EACH$, $FABG$, $BCFE$, $AHCF$, $EBGA$ and $HBFC$ or $AFBH$ based upon the position of $H$ in the $I$ region are the convex 4-gons of this type. It is easy to see that $I^*$ region is in the interior of $EABH$, $EACH$, $BCFE$, $EBGA$ and $I^*$ region is in the $I$ region of $FACG$, $ECGB$, $FABG$, $AHCF$. If $FAH$ is an anti clockwise turn and $FCH$ is clockwise turn, then either $HBFC$ or $AFBH$ is the convex 4-gon formed based upon the position of $H$ in the $I$ region of the triangle and both these convex 4-gons have $I^*$ in their $I$ region. Thus $I^*$ is feasible for these convex 4-gons.

*U(3,1) of Type1 Configuration 7.2.1 (I,I,O,O):* $EAGH$, $FAHG$, $FBHG$, $ECGH$, $EHCF$, $EBGF$ are the convex 4-gons of this type and these 4-gons have $I^*$ either in their $I$ region or in their $Z$ region. If $FAH$ is an clockwise turn then $EHAF$ is a convex 4-gon of this type having $I^*$ region inside. If $FAH$ is an anti clockwise turn and $FCH$ is anticlockwise turn then $FCHG$ is the convex 4-gon is of this type and has $I^*$ contained in its $I$ region. Thus $I^*$ is feasible for these convex 4-gons.

Any point added to $I^*$ results in configuration 8.

**Type 2 (when $G$ lies in $X_1$ and $EBG$ is a clockwise turn):** Consider the convex 4-gons of this configuration. The convex 4-gons are either of the type $U(2,2)$, $U(1,3)$ or $U(3,1)$ (see Fig. 16).

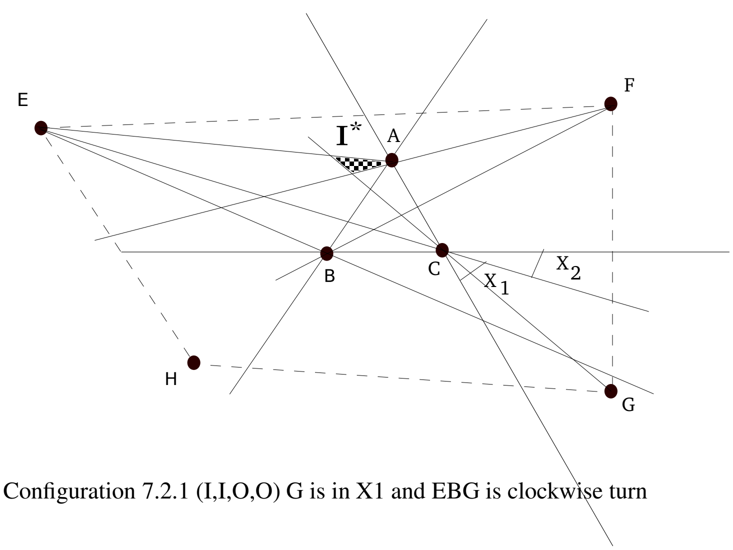
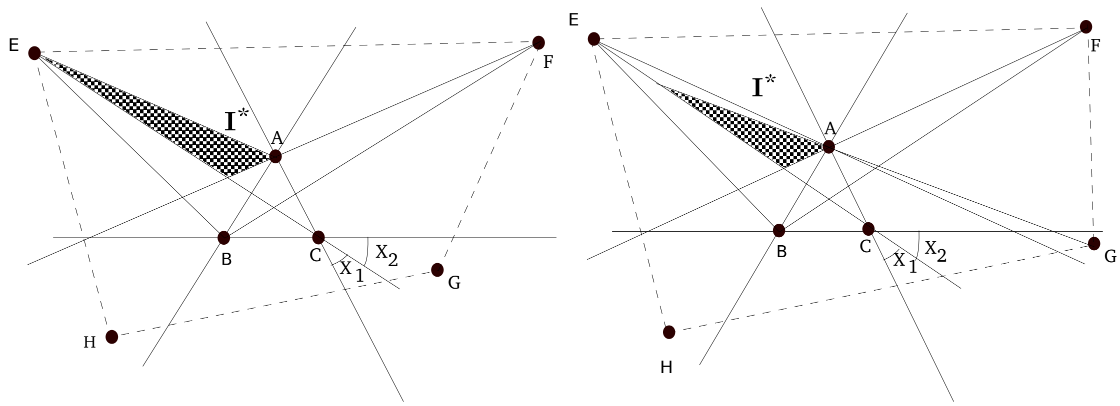



**Fig. 16:** Configuration 7.2.1 (I,I,O,O) G is in X1 and EBG is clockwise turn

*U(3,1) of Type2 Configuration 7.2.1 (I,I,O,O):* The convex 4-gons are the same as U(3,1) of Type1 Configuration 7.2.1 (I,I,O,O) except that we have $EBGH$ instead of $EBGF$. $EBGH$ has $I^*$ contained in its $I$ region.

*U(1,3) of Type2 Configuration 7.2.1 (I,I,O,O):* This is same as U(1,3) of Type1 Configuration 7.2.1 (I,I,O,O).




(a) EAG is a clockwise turn (b) EAG is an anticlockwise turn

**Fig. 17:** Configuration 7.2.1 (I,I,O,O) G is in X2

*U(2,2) of Type2 Configuration 7.2.1 (I,I,O,O):* This is a subset of U(2,2) of Type1 Configuration 7.2.1 (I,I,O,O) ($ECGB$ is a convex 4-gon for type1 but it is not convex 4-gon for type 2). □

Lemmas 3.1, 3.2 and 3.4 show a player 2 strategy that forces the game to reach configuration 8. Since $N(5) = 9$ the game ends in the 9th step and player 2 wins the game.

We now argue that there is no strategy for player 2 to end the game earlier, i.e., the game will always reach the 9th step. We call a point set as bad configuration for the convex 5-gon game if the point set has no convex 5-gon and any point added to the point set forms an convex 5-gon. Note that the game ends in the $i$th step only if the point set reached in the $i-1$th step is a bad configuration.

It is easy to see that the game cannot end in the 5th step since no 4 point set is a bad configuration.

**Lemma 3.5** *There is no strategy for player 2 such that the game ends in the 7th step.*

**Proof:** We prove this by showing that there are no 6 point sets which are bad configurations. When the point set contains 6 points it is easy to see that (3,3) and (4,2) are the only valid convex layer configurations for the convex 5-gon game. To show that a (3,3), (4,2) convex layer configurations are not bad configurations we show a feasible region where a point is added without forming an convex 5-gon.

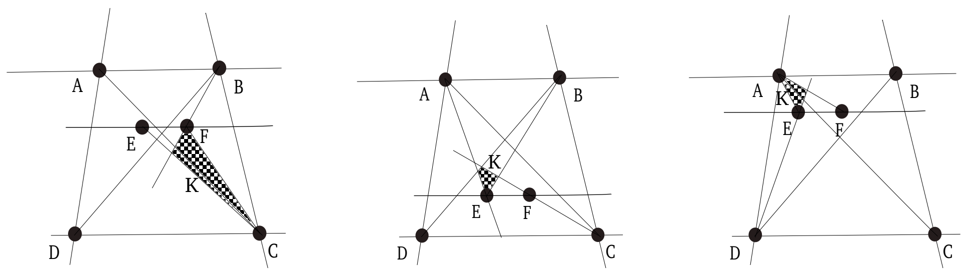
*(4,2) Convex layer configuration:* Consider the line joining $EF$ of the second convex layer (see Fig. 18(a)). This line divides the first convex layer into 2 parts. If either of the parts contains 3 points of the first convex layer then they form an empty convex 5-gon with the 2 points of the second convex layer. So the only other possible option is that both the parts contain 2 points.

Based upon the position of $E$, $F$ in the diagonal triangles of the first convex layer, (4,2) convex layer configurations are divided into 3 cases.

Case 1: $E$, $F$ are in opposite triangles, see Fig. 18(a).

Case 2: $E$, $F$ are in same triangle, see Fig. 18(b).

Case 3: $E$, $F$ are in adjacent triangles, see Fig. 18(c).




(a) Case 1 (b) Case 2 (c) Case 3

**Fig. 18:** Cases of (4,2) configuration

In each of the above cases a feasible region $K$ exists where a point can be added without forming a convex 5-gon (see shaded region in Fig. 18). Thus (4,2) convex layer configurations are not bad configurations.

*(3,3) Convex layer configuration:* We make the following observations on the placement of the three points of the first convex layer in the $I$ and $O$ regions of the triangle formed by the second convex layer (see Fig. 14). If an $I$ region has 3 points then the triangle in the second convex layer will not be contained in the triangle of the first convex layer. Hence an $I$ region cannot have 3 points. Similarly an $O$ region also cannot have 3 points. If an $O$ region has 2 points then the opposite $I$ region has to have the third point. In this case, there is an empty convex 5-gon being formed by these 2 points with the 3 points of the triangle. Hence an $O$ region cannot have more than 1 point.

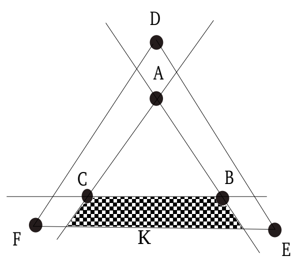



**Fig. 19:** (I,I,I) configuration

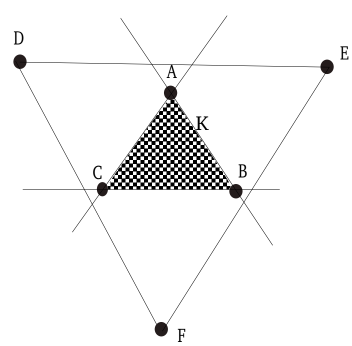



**Fig. 20:** (O,O,O) configuration

The only possible (3,3) point configurations are the following:
$(I.I, I)$ (3 points in different $I$ regions) Fig. 19.
$(O, O, O)$ (3 points in different $O$ regions) Fig. 20.
$(I, I, O)$ (2 points are in different $I$ regions and 1 point in an $O$ region) Fig. 21.

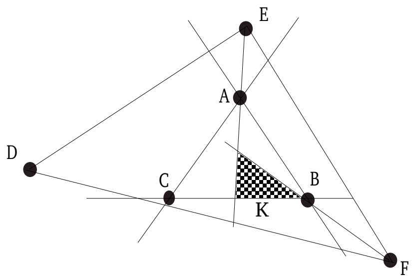



**Fig. 21:** (I,I,O) configuration

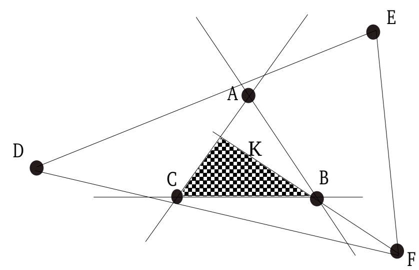



**Fig. 22:** (O,O,I) configuration

$(O, O, I)$ (2 points are in different $O$ regions and 1 point in an $I$ region) Fig. 22.
$(2I, O)$ (2 points in an $I$ region and 1 point in the opposite $O$ region) Fig. 23.

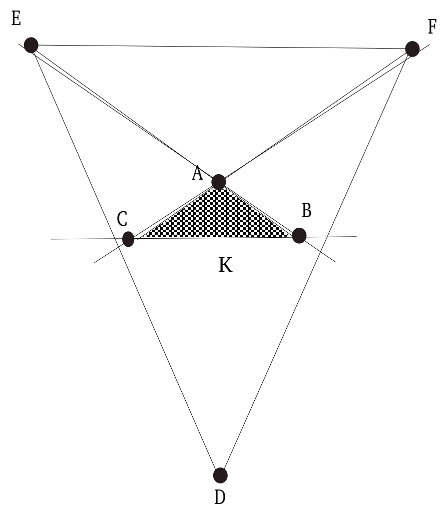



**Fig. 23:** (2I,O) configuration

In each of the above cases a feasible region $K$ exists where a point can be added without forming a convex 5-gon (see shaded region in Fig. 19, 20, 21, 22, 23). Thus (3,3) convex layer configurations are not bad configurations. □

**Theorem 3.1** *The convex 5-gon game always ends in the 9th step.*

**Proof:** The game does not end in the 5th step because no 4 point sets are bad configurations. By Lemma 3.1, the game will reach either configuration 6.1 or 6.2. Since all 6 point sets are not bad configurations (Lemma 3.5), the game will not end in the 7th step. By Lemma 3.2 the game will reach either configuration 7.1 or 7.2.1. By Lemma 3.4, the game will reach configuration 8 and finally since $N(5) = 9$ the game ends in the 9th step and player 2 wins the game. □

# 4 Game for the empty convex 5-gon

In this section, we show that the second player has a winning strategy in the empty convex 5-gon game and the game ends in 9 moves.

### *Overview of our proof and player 2's strategy to win the empty convex 5-gon game.*

In our game, player 1 plays in the odd steps (1st, 3rd, ...) and player 2 plays in the even steps (2nd, 4th, ...). In player 1's turn, we argue that any point added without forming an empty convex 5-gon will always result in specific configurations. In player 2's turn we show a feasible region (*Definition* 2.11) for placing the point that results in specific configurations that are favorable for player 2.

W.l.o.g., assume that the first 3 points of the game forms a triangle. We now describe the winning strategy for player 2:

Player 2 will place the point in the 4th step such that the resultant point set forms a parallelogram (configuration 4). In the 6th step, we show that there exists a feasible region in both configuration 5.1 and 5.2 where the 6th point is placed, so that it will reach configuration 6.1 or configuration 6.2. Similarly in the 8th step we show that there will exist a feasible region in configuration 7.1, 7.2.1 and 7.2.2 where the 8th point is placed such that the resultant point set is configuration 8.

In player 1's turn we show that any point added by player 1 without forming an empty convex k-gon results in configuration 5.1 or 5.2 (5th step) and in configuration 7.1, 7.2.1 or 7.2.2 (7th step).

Finally, we argue that any point added by player 1 to configuration 8 will result in the formation of a empty convex 5-gon and player 2 will always win in the 9th step.

Any empty convex 5-gon game follows a path in the game tree shown in Fig. 24.

The feasible regions of the convex 5-gon game is a subset of the feasible regions of the empty convex 5-gon game (by *Definitions* 2.10, 2.11). Thus in the even step (player 2's turn to place the point) for configurations without a convex 5-gon (configurations 5.1, 5.2, 7.1, 7.2.1), the proof for the convex 5-gon game also applies for the empty convex 5-gon game. For configurations with a convex 5-gon (configurations 7.2.2) we give a separate proof.

We further show that there is no better strategy for player 2 to end the game earlier (in the 5th or 7th step) by arguing that all configurations of 4 points and 6 points have a feasible region.

## *4.1 Proof for $H_G(5) = 9$*

**Lemma 4.1** *The game for the empty convex 5-gon will always reach either configuration 6.1 or configuration 6.2 at the end of 6th step.*

**Proof:** From Lemma 3.1 it is easy to see that the game for the empty convex 5-gon will always reach either configuration 5.1 or configuration 5.2 at the end of the 5th step.

Since the feasible regions for the convex 5-gon game is a subset of the feasible regions for the empty convex 5-gon game, by Lemma 3.1 the game will reach either configuration 6.1 or configuration 6.2 at the end of 6th step. □

**Lemma 4.2** *Any point added by player 1 to either configuration 6.1 or configuration 6.2 without forming an empty convex 5-gon, results in either configuration 7.1, 7.2.1 or 7.2.2.*

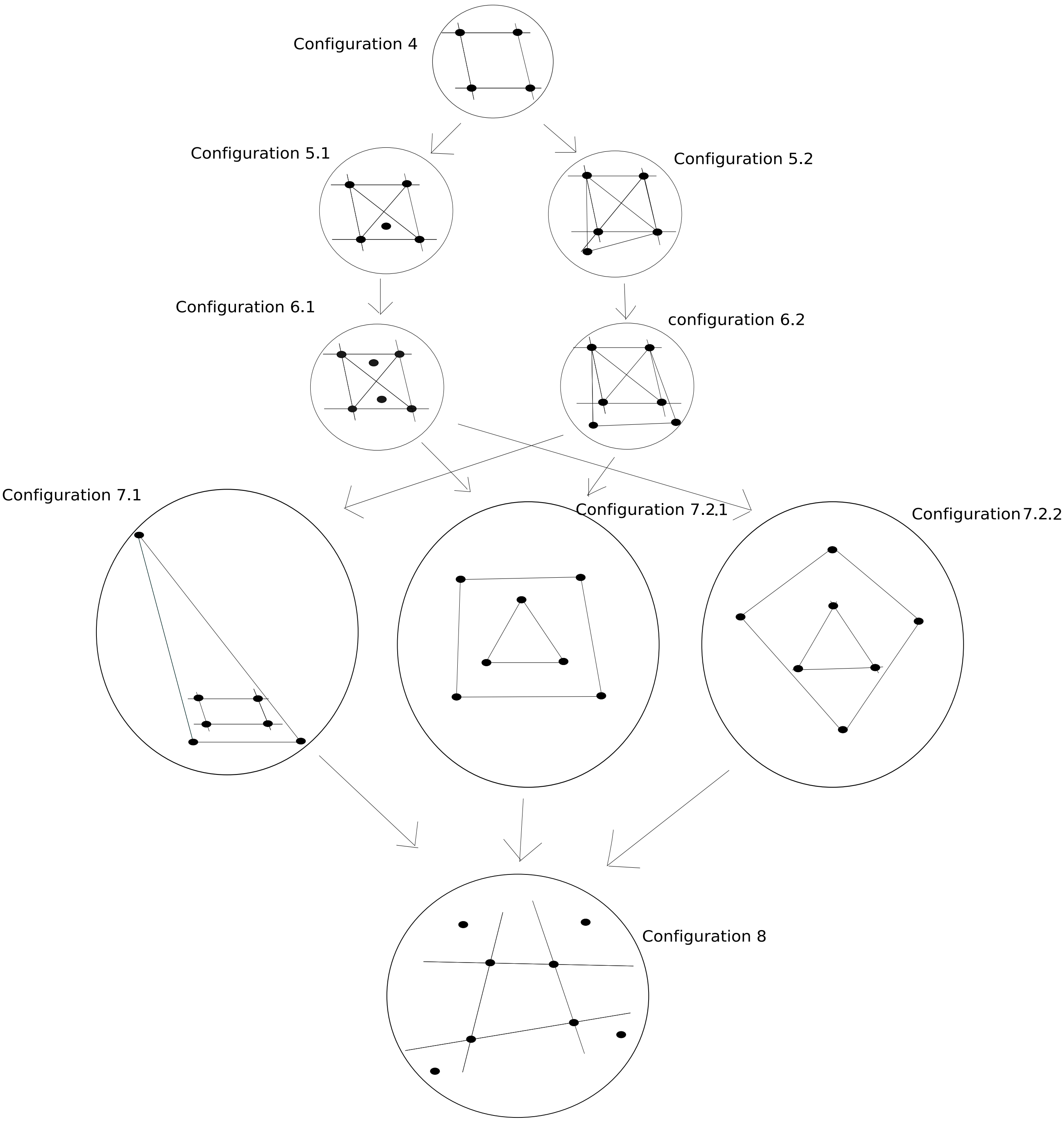



**Fig. 24:** Game tree for the empty convex 5-gon

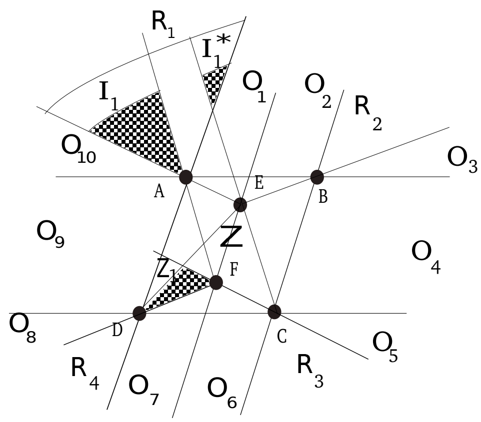



**Fig. 25:** Configuration 6.1 divided into regions

**Proof:** We consider 2 cases corresponding to configuration 6.1 and configuration 6.2. We will now describe the $O$ regions of configuration 6.1 and 6.2.

**Case 1 (configuration 6.1):** Let us consider all the empty convex 4-gons of configuration 6.1 (see Fig. 25). The convex 4-gons of configuration 6.1 that are formed by $U(4,0)$, $U(3,1)$ are not empty, so they are not considered. The empty convex 4-gons are of the form $U(2,2)$ of configuration 6.1.

*U(2,2) of configuration 6.1:* $EFDA$, $EFCB$, $BEDF$, $AECF$ are the empty convex 4-gons of this form. Consider $EFDA$ empty convex 4-gon. The regions that are contained in the $O$ regions of $EFDA$ are $O_1$, $O_7$, $O_8$, $O_9$, $O_{10}$. Similarly the regions that are contained in the $O$ regions of $EFCB$ empty convex 4-gon are $O_2$, $O_3$, $O_4$, $O_5$, $O_6$. The $O$ regions of $BEDF$, $AECF$ empty convex 4-gons does not contain the entire region of $Z$ ($Z$ region of $ABCD$ convex 4-gon) and $R_k$ where $k = 1, 2, 3, 4$.

Thus, the only feasible regions of configuration 6.1 (the regions that are not contained in any $O$ region of empty convex 4-gons) are portions of $R_k$ where $k = 1, 2, 3, 4$ and $Z$ region. From Fig. 25, we can verify that region $I_1$, $I_1^*$ of $R_1$ and $Z_1$ of $Z$ is not contained in any of the $O$ regions of the empty convex 4-gons. Fig. 25 shows $I_1$, $I_1^*$ only in $R_1$ region and $Z_1$ in $Z$ region. Symmetrically, there are feasible regions $I_k$, $I_k^*$ in $R_k$ region and $Z_k$ in $Z$ region where $k = 2, 3, 4$. Any point added in $I_k$ or $Z_k$ region will result in configuration 7.2.1. Any point added in $I_k^*$ region will result in configuration 7.2.2.

**Case 2 (configuration 6.2):** Let us consider the empty convex 4-gons of configuration 6.2 (see Fig. 26). The convex 4-gons of configuration 6.2 that are formed by $U(4,0)$, $U(3,1)$ are not empty, so they are not considered. The empty convex 4-gons are of the form $U(2,2)$ of configuration 6.2 (see Fig. 26).

*U(2,2) of configuration 6.2:* $EFCD$, $ABFE$, $AFCE$, $BEDF$ are the empty convex 4-gons of this form. Consider $EFCD$ empty convex 4-gon. The regions that are contained in the $O$ regions of $EFCD$ are $O_2$, $O_6$, $O_1$. Similarly the regions contained in the $O$ regions of $ABFE$ empty convex 4-gon are $O_3$, $O_5$, $O_4$. The O-regions of $AFCE$, $BEDF$ empty convex 4-gons do not contain the entire regions of $Z$ and $R_k$ where $k = 1, 2, 3, 4$.

Thus, the feasible regions of configuration 6.2 (the regions that are not contained in any $O$ region of empty convex 4-gons) is $S \setminus O_4$ and portions of $R_k$ and $Z$ regions. From Fig. 26, we can verify that region $I_1$, $I_1^*$ of $R_1$ is not contained in any of the $O$ regions of these convex 4-gons. Fig. 26 shows $I_1$, $I_1^*$ only in

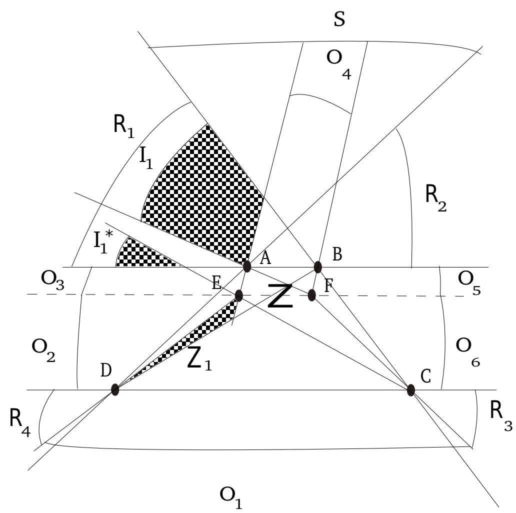



**Fig. 26:** Configuration 6.2 divided into regions

$R_1$ region and $Z_1$ in $Z$ region. Symmetrically, there are feasible regions $I_k$, $I_k^*$ in $R_k$ region and $Z_k$ in $Z$ region where $k = 2, 3, 4$.

If the point is placed in $I_k$, $I_k^*$ of $R_k$ or $Z_k$ of $Z$ where $k = 1, 2, 3, 4$, the resultant point set is either configuration 7.2.1 or configuration 7.2.2. If the point is placed in $S \setminus O_4$ the resultant point set is configuration 7.1. □

Now we prove that given a set of 7 points in configuration 7.1, 7.2.1 or 7.2.2 there exists a feasible region where a point is added such that the resultant point set is configuration 8.

**Lemma 4.3** *There exists a feasible region in configuration 7.1, 7.2.1 and 7.2.2 such that a point added by player 2 in this feasible region results in configuration 8.*

**Proof:** The feasible regions for the convex 5-gon game is a subset of the feasible regions for the empty convex 5-gon game. By Lemma 3.4, there exists a feasible region such that a point added by player 2 in this feasible region results in configuration 8.

Let us now consider configuration 7.2.2 (which has non-empty convex 5-gon). First we will give the following characterization of configuration 7.2.2. *An $I$ region or an $O$ region of the inner triangle cannot have more than 1 point (see Fig. 14).*

Consider an $O$ region. If an $O$ region has 2 points then there is an empty convex 5-gon being formed by these 2 points with the 3 points of the triangle. Hence an $O$ region cannot have more than 1 point. Consider an $I$ region. If an $I$ region has 2 points then no other $I$ region or the adjacent $O$ regions has any points because the 2 points in this $I$ region along with the third point in the $I$ region or in the adjacent $O$ region will form an empty convex 5-gon with the side of the triangle. Thus if an $I$ region has 2 points then the $O$ region that is opposite has to have the other 2 points which leads to the formation of an empty convex 5-gon.

Based on the above constraints, the only possible (4,3) convex layer configuration is $(I, O, O, O)$ (3 points each in different $O$ regions, 1 point in a $I$ region).

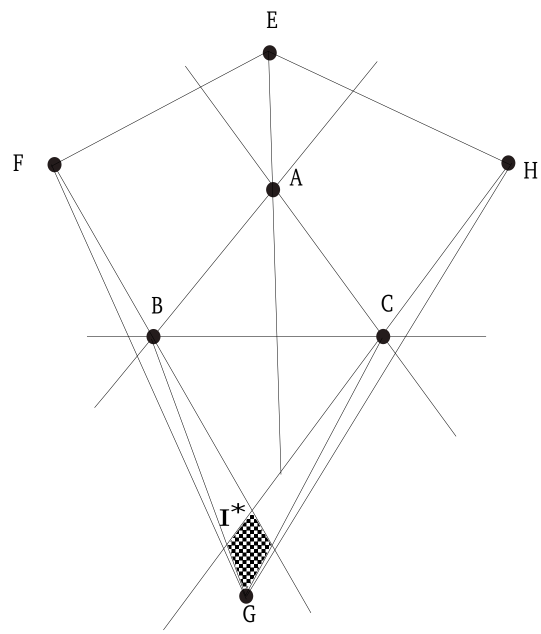



**Fig. 27:** Configuration 7.2.2 (I,O,O,O)

**(I,O,O,O) (3 points each in different $O$ regions, 1 point in a $I$ region):** Let us consider all the empty convex 4-gons of this configuration (see Fig. 27). The regions that are not contained in the $O$ regions of empty convex 4-gons are precisely the regions where a point is added without forming an empty convex 5-gon. We will show that there exists a region $I^*$ in these feasible regions where a point is placed forming configuration 8. The convex 4-gons that are formed by $U(4,0)$ of configuration 7.2.2 are not empty, so they are not considered. The remaining empty convex 4-gons of configuration 7.2.2 are of the form $U(2,2)$, $U(1,3)$, $U(3,1)$. It is important to note that $GHE$ should be an anticlockwise turn and $GFE$ a clockwise turn, otherwise we do not have a convex 4-gon in the first convex layer. *U(3,1) of configuration 7.2.2 (I,O,O,O): $EHAF$* is the empty convex 4-gon of this type and it has $I^*$ contained in its $I$ region. Thus $I^*$ is feasible for these convex 4-gons.

*U(1,3) of configuration 7.2.2 (I,O,O,O): $FACB$, $HABC$, $ACGB$* are the empty convex 4-gons of this type. $FACB$, $HABC$ have $I^*$ contained in there $I$ regions and $ACGB$ has $I^*$ inside. Thus $I^*$ is feasible for these convex 4-gons.

*U(2,2) of configuration 7.2.2 (I,O,O,O): $EABF$, $EHCA$* are the empty convex 4-gons of this type and these have $I^*$ contained in there $I$ regions. Thus $I^*$ is feasible for these convex 4-gons.

Any point added to $I^*$ results in configuration 8.

□

We will now show that any point added to configuration 8 forms an empty convex 5-gon and hence the empty convex 5-gon game also ends in the 9th step. First we prove the following lemma.

**Lemma 4.4** *A (4,3,2) convex layer configuration has an empty convex 5-gon.*

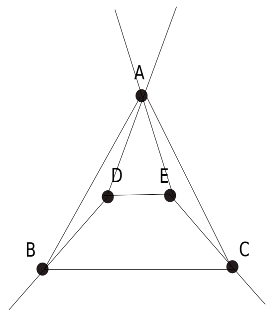


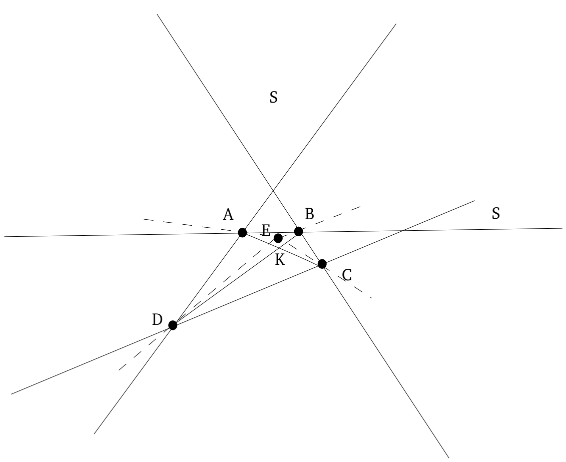

**Fig. 28:** (3,2) configuration

**Proof:** Let $F$, $G$, $H$, $J$ be the 4 points of $CH(P)$. Let $A$, $B$, $C$ be the points in the second convex layer and $D$, $E$ be the points of the inner most convex layer (see Fig. 28).

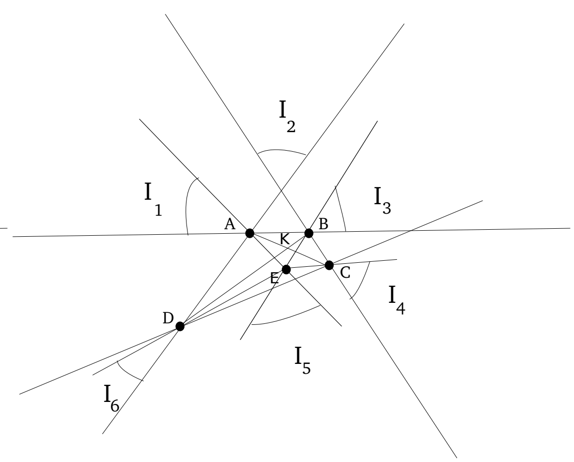
The points $F, G, H, J$ lie in the union of beams formed by the beams $DE : CB$, $D : AB$ and $E : AC$. Since beam $DE : CB$ is type 2, if there exists a point in its beam region then there exists an empty convex 5-gon. Thus, one of the type 1 beams $D : AB$ or $E : AC$ beams has atleast 2 points, which forms an empty convex 5-gon. $\square$

**Lemma 4.5** *A (4,4,1) convex layer configuration has an empty convex 5-gon.*

**Proof:** Let $F$, $G$, $H$, $J$ be the points of $CH(P)$. Let $A$, $B$, $C$, $D$ be the points of the second convex layer. Let $K$ be the intersection of the diagonals of the convex 4-gon $ABCD$. Let $E$ be the point inside the convex 4-gon $ABCD$. The union of beams $E : AB$, $E : BC$, $E : CD$, $E : DA$ contain the points $F$, $G$, $H$, $J$.




(a) Case 1




(b) Case 2

**Fig. 29:** Cases for (4,4,1) configuration

Depending upon the position of point $E$ inside the $ABCD$ convex 4-gon the proof is divided into 4 cases.

**Case 1:** When $E$ is in the triangle $ABK$ (see Fig. 29(a)). In this case $E : CD$ beam is contained in the union of $AE : CD$ and $EB : CD$ beams. Therefore any point in the $E : CD$ beam forms an empty convex 5-gon. If $E : CD$ beam does not have any point, one of the 3 type 1 beams $E : BC$, $E : AB$, $E : DA$ has atleast 2 points (among $F$, $G$, $H$, $J$) which forms an empty convex 5-gon.

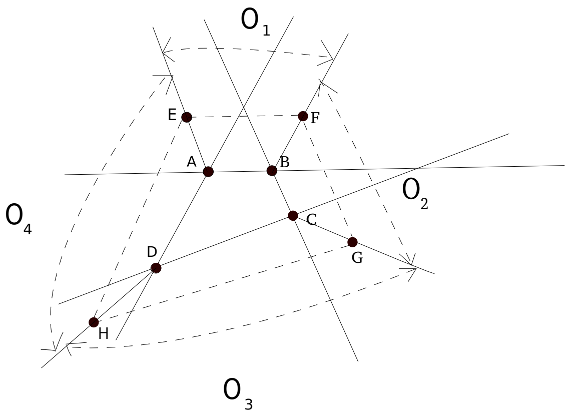
**Case 2:** When $E$ is in the triangle $KCD$ (see Fig. 29(b)). In this case $I_1$, $I_2$, $I_3$, $I_4$, $I_5$, $I_6$ are the only feasible regions outside the $ABCD$ convex 4-gon for the placement of $F$, $G$, $H$, $J$ points. The remaining outer regions are present in the $O$ region of some convex 4-gon and hence not feasible. The $E : CD$ beam contains $I_4$, $I_5$, $I_6$ regions. Since $E : CD$ beam is type 1 beam it can have atmost 1 point. Therefore $I_1$, $I_2$, $I_3$ regions combined should have 3 points. From the figure we can see that the union of $I_1$, $I_2$, $I_3$ regions has 3 points which forms an empty convex 5-gon with the points $A$, $B$.

The case where $E$ is in the triangle $KBC$ is symmetrical to the case when it is in triangle $ABK$. The case where $E$ is in the triangle $KAD$ is symmetrical to the case when it is in triangle $KCD$. □

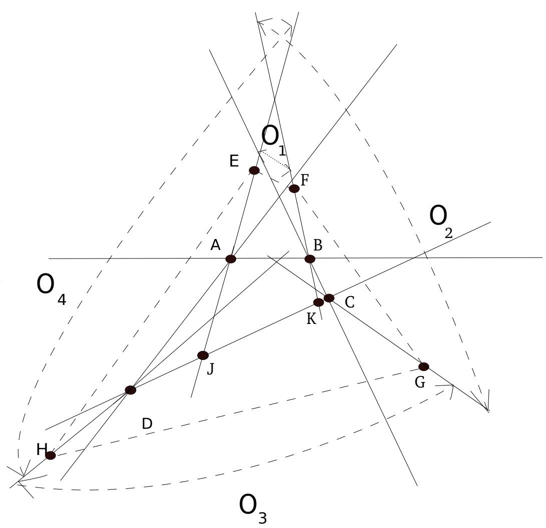
Let $E, F, G, H$ be the points of the first convex layer and $A, B, C, D$ be the points of the second convex layer of configuration 8. Without loss of generality, let us assume that rays $\overrightarrow{AB}$, $\overrightarrow{DC}$ intersect and rays $\overrightarrow{DA}$, $\overrightarrow{CB}$ intersect.

**Lemma 4.6** *The 4 type 2 beams AB : FE, BC : GF, DC : GH, AD : HE contain the entire outer region in configuration 8.*

**Proof:** We consider 2 cases depending on the intersections of the rays $\overrightarrow{AE}$, $\overrightarrow{BF}$, $\overrightarrow{CG}$, $\overrightarrow{DH}$.




(a) When no pair of rays intersect




(b) When one pair of rays intersect

**Fig. 30:** Cases according to the intersection of pairs of rays

**Case1:** When none of the rays $\overrightarrow{AE}$, $\overrightarrow{BF}$, $\overrightarrow{CG}$, $\overrightarrow{DH}$ intersect each other (see Fig. 30(a)).

The regions contained by $AB : FE$, $BC : GF$, $DC : GH$, $AD : HE$ beams are correspondingly $O_1$, $O_2$, $O_3$, $O_4$ beam regions which contain the entire outer region in configuration 8.

**Case2:** When some of the rays $\overrightarrow{AE}$, $\overrightarrow{BF}$, $\overrightarrow{CG}$, $\overrightarrow{DH}$ intersect each other (see Fig. 30(b)).

Now we prove that atmost one of the pairs of rays $(\overrightarrow{AE}, \overrightarrow{BF})$ or $(\overrightarrow{CG}, \overrightarrow{BF})$ can intersect in configuration 8. Assume that $(\overrightarrow{AE}, \overrightarrow{BF})$ intersect (see Fig. 30(b)). The case $(\overrightarrow{CG}, \overrightarrow{BF})$ intersecting is symmetrical.

Consider $EA$ and $FB$, extend them inside $ABCD$ convex 4-gon until they intersect with $CD$. Let the points of intersection be $J, K$. When $GC$ is extended inside the $ABCD$ convex 4-gon it has to intersect $BK$ before it intersects $AB$ or $AD$. Hence $(\overrightarrow{BF}, \overrightarrow{CG})$ do not intersect. By a similar reasoning, ray $\overrightarrow{HD}$ intersects $AJ$ before it intersects $AB$ or $BC$. Hence $(\overrightarrow{AE}, \overrightarrow{DH})$ do not intersect. The pair of rays $(\overrightarrow{CG}, \overrightarrow{DH})$ do not intersect because the pair of rays $(\overrightarrow{BC}, \overrightarrow{AD})$ do not intersect.

The regions contained by $AB : FE$, $BC : GF$, $DC : GH$, $AD : HE$ beams are correspondingly $O_1$, $O_2$, $O_3$, $O_4$ beam regions (see Fig. 30(b)) which contain the entire outer region in configuration 8. □

**Lemma 4.7** *Any point added to configuration 8 forms an empty convex 5-gon.*

**Proof:**

We consider 2 cases depending on whether the point is added inside $EFGH$ convex 4-gon or outside $EFGH$ convex 4-gon (see Fig. 31).

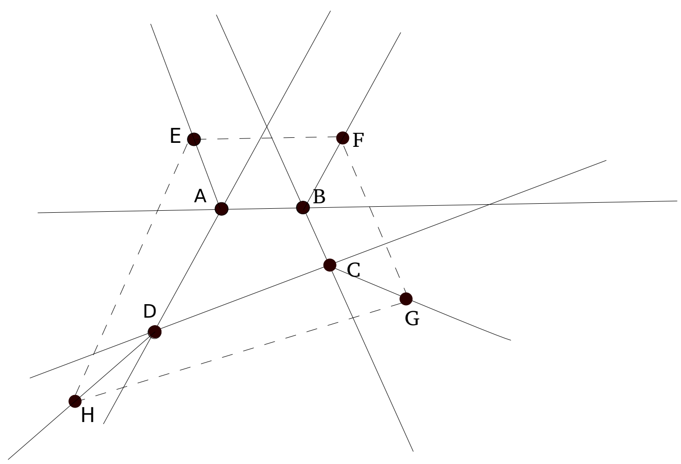



**Fig. 31:** Configuration 8

**Case1:** Placing the point inside $EFGH$ convex 4-gon:

The point added lies in $I$ region, $S$ region, $Z$ region or $O$ region of $ABCD$ convex 4-gon. If the point is placed in the $O$ region of $ABCD$ convex 4-gon then there exists an empty convex 5-gon. If point is placed in the $S$ region of $ABCD$ convex 4-gon then it forms a (4,3,2) convex layer configuration which contains an empty convex 5-gon (Lemma 4.4). If the point is placed in the $I$ or $Z$ region of the $ABCD$ convex 4-gon then it forms a (4,4,1) convex layer configuration which contains an empty convex 5-gon (Lemma 4.5).

**Case2:** Placing the point outside the $EFGH$ convex 4-gon:

By Lemma 4.6, the 4 beams $AB : FE$, $BC : GF$, $DC : GH$, $AD : HE$ contain the entire outer region. The added point lies in one of the 4 type 2 beams and forms an empty convex 5-gon. □

Lemmas 4.1, 4.2, 4.3 and 4.7 show a strategy for player 2 to win the game in the 9th step.

We now argue that there is no strategy for player 2 to end the game earlier, i.e., the game will always reach the 9th step. We call a point set as bad configuration for the empty convex 5-gon game if the point set has no empty convex 5-gon and any point added to the point set forms an empty convex 5-gon. Note that the game ends in the $i$th step only if the point set reached in the $i-1$th step is a bad configuration.

It is easy to see that the game cannot end in the 5th step since no 4 point set is a bad configuration.

**Lemma 4.8** *There is no strategy for player 2 such that the game ends in the 7th step.*

**Proof:** We prove this by showing that there are no 6 point sets which are bad configurations. When the point set contains 6 points it is easy to see that (3,3), (4,2) and (5,1) are the only valid convex layer configurations for the empty convex 5-gon game.

The feasible regions for the convex 5-gon game is a subset of the feasible regions for the empty convex 5-gon game. Thus by Lemma 3.5, there exists a feasible region for the (3,3), (4,2) convex layer configurations in the empty convex 5-gon game. Therefore (3,3), (4,2) convex layer configurations are not bad configurations for the empty convex 5-gon game.

To show that a (5,1) convex layer configuration is not a bad configuration we show a feasible region where a point is added without forming an empty convex 5-gon.

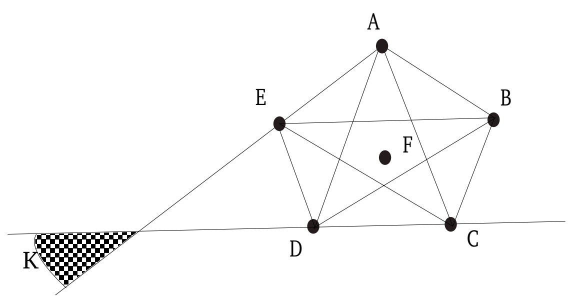



**Fig. 32:** (5,1) configuration

*(5,1) Convex layer configuration:* A feasible region $K$ exists where a point can be added without forming a empty convex 5-gon (see shaded region in Fig. 32). □

**Theorem 4.1** *The empty convex 5-gon game always ends in the 9th step.*

**Proof:** The game does not end in the 5th step because no 4 point sets are bad configurations. By Lemma 4.1, the game will reach either configuration 6.1 or 6.2. Since all 6 point sets are not bad configurations (Lemma 4.8), the game will not end in the 7th step. By Lemma 4.2 the game will reach either configuration 7.1, 7.2.1 or 7.2.2. By Lemma 4.3, the game will reach configuration 8 and finally from Lemma 4.7 the game ends in the 9th step and player 2 wins the game. □

## Conclusion

In our paper we have introduced the two player game variant of Erdős-Szekeres problem and proved that the game ends in the 9th step for the convex 5-gon and empty convex 5-gon game and player 2 wins in both the cases.

One natural question would be to analyze the game for higher values of $k$ i.e. determine $N_G(k)$ and $H_G(k)$ for $k > 5$. Our approach will be very tedious for higher values of $k$ as with the increase in the number of points in the point set, the number of point configurations increases exponentially.

We have shown that configuration 8 is a bad configuration for the empty convex 5-gon game. Another natural question is to determine whether there exists bad configurations for $k > 5$. More specifically, does there exist point configurations with the property that any point added to this configuration forms an empty convex $k$-gon or convex $k$-gon for $k > 5$. A negative result for the above question gives a lower bound for $N_G(k)$ and $H_G(k)$.

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