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On the Cartesian product of an arbitrarily partitionable graph and a traceable graph

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A graph G of order n is called arbitrarily partitionable (AP, for short) if, for every sequence $\tau = (n_1, \ldots, n_k)$ of positive integers that sum up to n, there exists a partition $(V_1, \ldots, V_k)$ of the vertex set $V(G)$ such that each set V_i induces a connected subgraph of order n_i. A graph G is called AP+1 if, given a vertex $u \in V(G)$ and an index $q \in \{1, \ldots, k\}$, such a partition exists with $u \in V_q$. We consider the Cartesian product of AP graphs. We prove that if G is AP+1 and H is traceable, then the Cartesian product $G \Box H$ is AP+1. We also prove that $G \Box H$ is AP, whenever G and H are AP and the order of one of them is not greater than four.

Keywords: partitions of graphs, Cartesian product of graphs, traceable graphs

1 Introduction

Let $G = (V, E)$ be a graph of order n, and $\tau = (n_1, \ldots, n_k)$ a sequence of positive integers. The sequence τ is *admissible for* G if $n_1 + \ldots + n_k = n$. Such an admissible sequence τ is said to be *realizable in* G if we can partition V into k parts $(V_1, \ldots, V_k)$ such that $|V_i| = n_i$ and the subgraph $G[V_i]$ induced by V_i is connected, for every $i = 1, \ldots, k$. Note that in fact the ordering of τ is irrelevant, i.e., if τ is realizable in G, then it is also realizable after any permutation of its elements. We say that G is *arbitrarily partitionable* (AP, for short) if every admissible sequence τ for G is realizable in G. Furthermore, for a given integer $k \in \{1, \ldots, n\}$, a graph G is called k-AP if every admissible sequence τ for G of length at most k is realizable in G. Note that every connected graph is 1-AP. Arbitrarily partitionable graphs, sometimes also called *arbitrarily vertex decomposable*, have been intensively studied for ten years (examples of papers are listed in the bibliography).

The notion of AP graphs was introduced by Barth et al. [1] (and independently by M. Horňák and M. Woźniak [10]) to model the following problem. Suppose that we own a connected network of n resources,

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and we want to divide it among several customers. Every customer declares the number of resources that should be attributed to him. These numbers sum up to n. Moreover, each customer wants to receive a connected subnetwork of resources, and every resource must belong to exactly one subnetwork. Hence, we want our network to be partitionable as described, no matter how many subnetworks are requested (this number is at most n) and no matter what their orders are (but they sum up to n). Notice that any network with an underlying AP graph has this property.

Consider now an augmented version of this notion.

Let $\tau = (n_1, ..., n_k)$ be a sequence admissible for a graph G, let $(n_{i_1}, ..., n_{i_r})$ be a subsequence of length r of τ, and let $\{v_1, v_2, \ldots, v_r\}$ be a set of r vertices of G. The sequence τ is said to be *realizable in G under r vertex restrictions* $(v_1, n_{i_1}), \ldots, (v_r, n_{i_r})$ if there exists a realization $(V_1, \ldots, V_k)$ of τ in G such that v_p belongs to a part V_{i_p} for every $p = 1, \ldots, r$. A graph G is called *arbitrarily partitionable under r vertex restrictions* (AP+r, for short) if, for each $r' < r$, every admissible sequence for G of length at least r' is realizable in G under any r' vertex restrictions. Naturally, a graph is called k-AP+r if, for each $r' < r$, every admissible sequence of length l, with $r' \le l \le k$, is realizable in G under r' vertex restrictions.

The problem of arbitrary partitioning graphs into connected subgraphs under vertex restrictions was already considered in late 70s by Lovász [13], and independently by Győri [7]. They both proved the following result.

Theorem 1 *[7, 13] A graph G is k-connected if and only if G is k-$AP + k$.*

It is easily seen that each traceable graph is AP, so each condition implying the existence of a Hamiltonian path in a graph also implies that the graph is AP. Therefore, by weakening some known conditions for traceability we can expect to obtain sufficient conditions for being AP. For example, the well-known Ore's theorem states that if G is a graph of order n such that the degree sum of every pair of nonadjacent vertices is at least $n - 1$, then G is traceable. In [8] Horňák et al. showed that a connected graph G of order $n \ge 20$ such that the degree sum of every pair of nonadjacent vertices is at least $n - 5$ is AP if and only if it admits a perfect matching or a quasi-perfect matching (i.e., a matching omitting exactly one vertex). This is clearly an improvement of the above-mentioned Ore's condition for being AP.

Recall that given two graphs G and H, the *Cartesian product of G and H*, denoted by $G\Box H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

It is well known that if two graphs G and H are traceable then their Cartesian product $G\Box H$ is also traceable. So one can formulate a conjecture.

Conjecture 2 *If G and H are AP graphs, then the Cartesian product $G\Box H$ is also AP.*

Here we prove the following.

Theorem 3 *The Cartesian product of two AP graphs is also AP, whenever at least one of these graphs is of order at most four.*

Conjecture 2 seems to be very hard, so we formulate the following weaker conjecture by assuming traceability of H.

Conjecture 4 *Let G be an AP graph, and let H be a traceable graph. Then $G\Box H$ is AP.*

We also confirm this conjecture for small orders of H.

Theorem 5 *If G is an AP graph, then the Cartesian product $G\Box H$ is AP, whenever H is a traceable graph of order at most four.*

It is easy to check that all AP graphs of order at most four are traceable (the smallest nontraceable AP graphs have order five). Hence, Theorem 3 is an immediate consequence of Theorem 5.

We finally totally prove an AP+1 version of Conjecture 2.

Theorem 6 *If G is an AP+1 graph and H is traceable, then the Cartesian product $G\Box H$ is AP+1.*

2 Proof of Theorem 5

If a and b are integers with $a < b$, then $[a, b]$ will denote the set of integers $\{a, a+1, \dots, b-1, b\}$. Let $\tau = (n_1, ..., n_k)$ be any sequence. We will denote by $\|\tau\|$ the sum $n_1 + ... + n_k$ and by $|\tau|$ the length k of τ. Let σ be a permutation of the set $[1, k]$, and let $r \in [1, k-1]$. Then we say that τ *is partitioned into two sequences* $\tau^1 = (n_{\sigma(1)}, \dots, n_{\sigma(r)})$ and $\tau^2 = (n_{\sigma(r+1)}, \dots, n_{\sigma(k)})$, and we use the notation $\tau^2 = \tau - \tau^1$ (here the sequence τ^2 is determined up to the ordering of its elements). One can recursively extend this definition to a partition of τ into several sequences.

Clearly, it suffices to prove both Theorem 5 and Theorem 6 for H being a path P_l of order $l = |H|$. Thus, in both proofs, we focus on the Cartesian products of the form $G\Box P_l$. Let $V(P_l) = \{p_1, ..., p_l\}$. The Cartesian product $G\Box\{p_i\}$, where $i \in [1, l]$ and $\{p_i\}$ is identified with the graph K_1, is called the *i-th layer of G in $G\Box P_l$* and denoted by G^i. For each $u \in V(G)$, the vertex $(u, p_i) \in V(G\Box P_l)$ is called the *i-th layer of u in $G\Box P_l$*, and we will write $(u, p_i) = u^i$. Similarly, the *i-th layer of a vertex subset $S \subseteq V(G)$ in $G\Box P_l$*, denoted by S^i, is the set of the i-th layers of all vertices of S.

The following simple lemma justifies the next proposition which is crucial in our proof of Theorem 5.

Lemma 7 *Let $l \geq 2$ be an integer, and $\tau = (n_1, ..., n_k)$ be a sequence of positive integers such that $\|\tau\| \equiv 0 \pmod l$. If $k > l$, then τ can be partitioned into two nonempty subsequences τ^1 and τ^2 such that $\|\tau^1\| \equiv 0 \pmod l$ and $\|\tau^2\| \equiv 0 \pmod l$.*

Proof: If τ contains an element n_i such that $n_i \equiv 0 \pmod l$, then if suffices to consider $\tau^1 = (n_i)$ and $\tau^2 = \tau - (n_i)$. Suppose then that $n_i \not\equiv 0 \pmod l$, for every $i \in [1, k]$.

For every $j \in [1, k]$, let $s_j = \sum_{i=1}^{j} n_i$ be the sum of the first j elements of τ. If there exist two indices r and t with $r < t$, such that $s_r \equiv s_t \pmod l$, then it is easy to see that $\tau^1 = (n_{r+1}, \dots, n_t)$ and $\tau^2 = \tau - \tau^1$ satisfy our conditions. Since the number k of elements of the sequence $s = (s_1, \dots, s_k)$ is greater than l, the number of distinct residues modulo l, there must exist two elements of s congruent modulo l. This ends the proof. $\Box$

Proposition 8 *Let $l \geq 2$ be a fixed integer such that for every connected graph H, the Cartesian product $H\Box P_l$ is l-AP. Then $G\Box P_l$ is AP for every AP graph G.*

Proof: Let G be an AP graph and τ a sequence admissible for $G\Box P_l$. If τ has length at most l, then τ is realizable in $G\Box P_l$ according to our assumption since G is connected. Let us now suppose that τ has length at least $l+1$. By repeatedly applying Lemma 7, the sequence τ can be partitioned into subsequences $\tau^1, ..., \tau^r$ such that $\|\tau^i\| \equiv 0 \pmod l$ and $|\tau^i| \leq l$ for every $i \in [1, r]$.

For every $i \in [1, r]$, let $\alpha_i = \frac{\|\tau^i\|}{l}$. As $\|\tau^i\| \equiv 0 \pmod{l}$, it follows that for each i, the number α_i is an integer. Clearly, the sequence $\alpha = (\alpha_1, ..., \alpha_r)$ sums up to the order of G and hence, because G is AP, admits a realization $(V_1, ..., V_r)$ in G. Put $U_i = V(G[V_i] \square P_l)$ for $i = 1, \dots, r$. Hence $(U_1, \dots, U_r)$ is a partititon of the vertex set of $G \square P_l$, and each U_i has exactly $\alpha_i l = \|\tau^i\|$ vertices. Since each subgraph $G[V_i]$ is connected, we infer by assumptions that each subgraph induced by U_i admits a realization of τ^i within it. To obtain a global realization of τ in $G \square P_l$, we take realizations of all the sequences $\tau^1, \dots, \tau^r$. □

In view of Proposition 8, to prove Conjecture 4 it suffices to show the following.

Conjecture 9 *If G is a connected graph, then $G \square P_l$ is l-AP for every integer $l \geq 1$.*

In what follows, we prove that the assumptions of Proposition 8 hold for every $l \in [2, 4]$. We will assume in this section that every sequence $\tau = (n_1, \dots, n_k)$ is nonincreasing, i.e., $n_1 \geq \dots \geq n_k$.

Proposition 10 *If G is a connected graph and $l \geq 2$, then $G \square P_l$ is 2-AP.*

Proof: If G is connected, then clearly $G \square P_l$ is 2-connected. The conclusion follows from Theorem 1. □

Consequently, to prove that the Cartesian product of a connected graph G and a path P_l of order $l > 2$ is l-AP, it suffices to consider sequences admissible for $G \square P_l$ which are of length greater than two.

Proposition 11 *If G is a connected graph, then $G \square P_3$ is 3-AP.*

Proof: Suppose that the sequence $\tau = (n_1, n_2, n_3)$ is admissible for $G \square P_3$, with $n_1 > n = |G|$ and $n_3 < n$ (otherwise $n_1 = n_2 = n_3 = n$ and $(V(G^1), V(G^2), V(G^3))$ is an obvious realization of τ in $G \square P_3$). We distinguish two cases depending on the value of n_2.

- If $n > n_2 \geq n_3$, let $T_1, T_2 \subset V(G)$ be subsets of vertices such that $G[T_1]$ is a tree of order n_2, and $G[T_2]$ is a tree of order n_3. Observe then that the partition $(V(G \square P_3) \setminus (T_1^1 \cup T_2^3), T_1^1, T_2^3)$ is a correct realization of τ in $G \square P_3$. Indeed, $G \square P_3 - (T_1^1 \cup T_2^3)$ is connected since every vertex from $V(G^1) \cup V(G^3) \setminus (T_1^1 \cup T_2^3)$ has a neighbour in $V(G^2)$.
- If $2n > n_2 \geq n$, then $(V(G^1) \cup S_1, V(G^3) \cup S_2, T^2)$ is a realization of τ, where $T \subset V(G)$ is chosen is such a way that $G[T]$ is a tree of order n_3, and S_1 and S_2 are subsets of vertices from $V(G^2) \setminus T^2$ chosen arbitrarily with $|S_1| = n_1 - n$ and $|S_2| = n_2 - n$. Notice that each vertex of S_1 or S_2 has a neighbour both in $V(G^1)$ and $V(G^3)$, so the subgraphs $(G \square P_3)[V(G^1) \cup S_1]$ and $(G \square P_3)[V(G^3) \cup S_2]$ are connected. □

Proposition 12 *If G is a connected graph, then $G \square P_4$ is 4-AP.*

Proof:
Suppose $\tau = (n_1, n_2, n_3)$ and $\|\tau\| = 4n$. We distinguish three cases.

- If $n > n_2 \geq n_3$, then, for the same reasons as in Proposition 11, $(V(G \square P_4) \setminus (T_1^1 \cup T_2^4), T_1^1, T_2^4)$ is a realization of τ, where $T_1, T_2 \subset V(G)$ are chosen in such a way that they induce trees of orders n_2 and n_3, respectively.

- If $2n > n_2 \geq n > n_3$, then we can deduce a realization (V_1, V_2, V_3) of $(n_1, n_2 - n, n_3)$ in $(G\Box P_4)[V(G^2) \cup V(G^3) \cup V(G^4)]$ using Proposition 11. Observe then that it can be extended to a realization of τ in $G\Box P_4$ by simply considering $(V_1, V_2 \cup V(G^1), V_3)$.

- $2n > n_2 \geq n_3 \geq n$. Notice that $2n \geq n_1$. Let $T_2 \subseteq T_1 \subset V(G)$ be subsets of vertices such that $G[T_1]$ and $G[T_2]$ are trees of orders $n_2 - n$ and $n_3 - n$, respectively. Consider now a sequence $(V(G^1) \cup (V(G^2) \setminus T_1^2), T_1^2 \cup T_2^3, V(G^4) \cup (V(G^3) \setminus T_2^3))$ of subsets of $V(G\Box P_4)$. Notice that these three subsets induce connected subgraphs with the desired orders since every vertex of $V(G^2) \setminus T_1^2$ has a neighbour in $V(G^1)$, every one from $V(G^3) \setminus T_2^3$ is adjacent to a vertex in $V(G^4)$, and T_1^2 and T_2^3 have some adjacent vertices. Thus, $(V(G^1) \cup (V(G^2) \setminus T_1^2), T_1^2 \cup T_2^3, V(G^4) \cup (V(G^3) \setminus T_2^3))$ is a realization of τ.

Now, consider the case where $\tau = (n_1, n_2, n_3, n_4)$, with $n_1 > n$ and $n_4 < n$ (otherwise $n_1 = n_2 = n_3 = n_4 = n$, and $(V(G^1), V(G^2), V(G^3), V(G^4))$ is a trivial realization of τ in $G\Box P_4$). Depending on the values of n_2 and n_3, we claim that a realization of τ in $G\Box P_4$ can always be found.

- $n > n_2 \geq n_3$. Thus, $n_1 \geq n$. Let $T_3 \subseteq T_2 \subseteq T_1 \subset V(G)$ be subsets inducing subtrees of G having orders n_4, n_3, and n_2, respectively. Notice then that $(V(G\Box P_4) \setminus (T_1^1 \cup T_2^2 \cup T_3^3), T_1^1, T_2^2, T_3^3)$ is a realization of τ in $G\Box P_4$ since every vertex from $V(G^1) \setminus T_1^1$ has a neighbour in $V(G_2) \setminus T_2^2$, every vertex from $V(G_2) \setminus T_2^2$ is adjacent to a vertex of $V(G_3) \setminus T_3^3$, and, finally, every vertex of $V(G_3) \setminus T_3^3$ has a neighbour in $V(G^4)$.

- $n_2 \geq n > n_3$. Notice that $3n > n_1 + n_3 \geq 2n$ and $n < n_2 + n_4 \leq 2n$. Let $T_1, T_2 \subset V(G)$ be subsets inducing subtrees of G of orders n_3 and n_4, respectively. Consider a sequence $(V(G^2) \cup (V(G^1) \setminus T_1^1) \cup S_1, V(G^4) \cup S_2, T_1^1, T_2^3)$, where the subsets S_1 and S_2 are chosen arbitrarily from $V(G^3) \setminus T_2^3$ in such a way that $S_1 \cap S_2 = \emptyset$, $|S_1| = n_1 + n_3 - 2n$ and $|S_2| = n - |S_1| - n_4 = n_2 - n$. Observe that this is a realization of τ in $G\Box P_4$ since every vertex of $(V(G^1) \setminus T_1^1) \cup (S_1 \cup S_2)$ has a neighbour in $V(G_2)$ and all vertices of S_2 are adjacent to some vertices of $V(G^4)$.

- If $n_2 \geq n_3 \geq n$, let $T \subset V(G)$ be a subset of vertices inducing a subtree of G with order n_4, $W \subseteq (V(G) \setminus T)$ be $n_3 - n$ arbitrary vertices of G, and $u \in V(G)$ be a vertex of G which is not an cut vertex. Consider now the realization $(V(G^1) \cup V(G^2) \cup V(G^3)) \setminus (S_1 \cup S_2 \cup T^3 \cup W^3), S_1 \cup S_2, V(G^4) \cup W^3, T^3)$ of τ in $G\Box P_4$, where S_1 and S_2 are obtained in the following way. We start by putting u^1 and u^2 into S_1, and we start building a subtree of $V(G^2)$ rooted in u^2 iteratively, that is we take an arbitrary vertex s^2 joined to $V(G^2) \setminus S_1$ and add it to S_1. If s^3 does not belong to W^3, then s^3 is added to S_2. The procedure goes on until $|S_1| + |S_2| \geq n_2$. Observe that once the procedure is over, we can only have $|S_1| + |S_2| \in \{n_2, n_2 + 1\}$. In the case where $|S_1| + |S_2| = n_2 + 1$, we can remove the vertex u^1 from S_1 so that $S_1 \cup S_2$ is a subset with size n_2. In any case, the four subgraphs induced by the realization are connected since $(G\Box P_4)[S_1]$ induces a tree, every vertex from S_2 has a neighbour in S_1, and any vertex from $V(G^3) \setminus (S_2 \cup T^3 \cup W^3)$ is adjacent to one in $V(G^2) \setminus S_2$ which is itself adjacent to one in $V(G^1)$. □

As a consequence of Propositions 8, 10, 11 and 12, we can state the following corollary, which in turn immediately implies Theorems 3 and 5.

Corollary 13 *If a graph G is AP, and $l \leq 4$, then $G\Box P_l$ is AP.* □

3 Proof of Theorem 6

In this section, we also use the terminology and notation on the Cartesian product $G\Box P_l$ which have we introduced in the initial paragraphs of Section 2. Clearly, it suffices to prove that $G\Box P_l$ is AP+1, with $l = |H|$.

Let $\tau = (n_1, ..., n_k)$ be a sequence which sums up to nl, the order of $G\Box P_l$. Let (u_0^j, n_q) be a vertex restriction. Without loss of generality, we may assume that $q = 1$. In what follows, we explain how to find a realization of τ in $G\Box P_l$ such that the j-th layer of a vertex u_0 in $G\Box P_l$ belongs to a part of order n_1 in this realization.

Note that if $l = 1$, then $G\Box P_1$ is isomorphic to G, and thus it is AP+1, by assumption. Hence, suppose that $l \geq 2$. We distinguish two cases depending on the cardinality of n_1. In each case, the proof goes in two stages. Roughly, we first modify the sequence τ by dividing some of its terms n_i into two or more addends, so as τ could be partitioned into l sequences $\tau^1, \ldots, \tau^l$ such that usually $\|\tau^i\| = n$, $i \in [1, l]$. Next, for all i, we construct a realization of τ^i in the i-th layer G^i of $G\Box P_l$ in such a way that parts whose sizes result from the splitting of a same element of τ yield a connected graph when glued together.

Case 1: $n_1 < n$.

Consider the sequence $\tau' = (n_2, \ldots, n_k)$. For each $i \in [1, l]$, we define a *capacity* $\varphi(i)$ of the i-th layer G^i of $G\Box P_l$ as follows:

$$\varphi(i) = \begin{cases} n - n_1 & \text{if } i = j, \\ n & \text{otherwise.} \end{cases}$$

Now, we will recursively define l sequences $\tau^1, ..., \tau^l$ such that every τ^i sums up to $\varphi(i)$, as follows. If there exists $p \in [2, k]$ such that $\sum_{i=2}^{p} n_i = \varphi(1)$, then we set $\tau^1 = (n_2, \ldots, n_p)$, $\tau'' = (n_{p+1}, \ldots, n_k)$, and $\alpha(1) = 0$ (for each $i \in [1, l-1]$, the number $\alpha(i)$ will indicate whether the last term of τ^i originated from dividing a term of τ). Otherwise, let $p \in [2, k]$ be such that $\sum_{i=2}^{p-1} n_i < \varphi(1)$ and $\sum_{i=2}^{p} n_i > \varphi(1)$. Let $\eta_1 = \varphi(1) - \sum_{i=2}^{p-1} n_i$ and $\eta_2' = n_p - \eta_1$. We then set $\tau^1 = (n_2, \ldots, n_{p-1}, \eta_1)$, $\tau'' = (\eta_2', n_{p+1}, \ldots, n_k)$, and $\alpha(1) = 1$.

We repeat this procedure with the sequence τ'' instead of τ' for the graph $G\Box P_{l-1}$, where P_{l-1} denotes the path $p_2 \ldots p_l$, in order to obtain a sequence τ^2 and the number $\alpha(2)$, and so on. After l repetitions of this procedure, we obtain the desired sequences $\tau^1, ..., \tau^l$ such that $\|\tau^i\| = \varphi(i)$ for $i \in [1, l]$. Denote $\tau^i = (n_1^i, \ldots, n_{k_i}^i),\ i \in [1, l]$.

We begin our construction of a realization of τ by choosing a realization $(V_1, V_1^j, \ldots, V_{k_j}^j)$ in G^j of the sequence $(n_1, n_1^j, \ldots, n_{k_j}^j)$ such that $u_0^j \in V_1$. We can choose it since this sequence sums up to n, and G^j is AP+1.

Next, for $i = j-1, j-2, \ldots, 1$, we proceed as follows. If $\alpha(i) = 0$, then we take any realization $(V_1^i, \ldots, V_{k_i}^i)$ of τ^i in G^i. Otherwise, when $\alpha(i) = 1$, we choose any vertex $u^{i+1} \in V_1^{i+1}$ and take a realization of τ^i in G^i such that $u^i \in V_{k_i}^i$. This is possible since G is AP+1. Note that the subgraph induced by $V_{k_i}^i \cup V_1^{i+1}$ in $G\Box P_l$ is connected due to the edge $u^i u^{i+1}$ of the Cartesian product. Naturally, if $j = 1$, then we omit this step and go directly to the step discribed below.

For every $i = j+1, \ldots, l$, if $\alpha(i-1) = 0$, then we take any realization $(V_1^i, \ldots, V_{k_i}^i)$ of τ^i in G^i. Otherwise, when $\alpha(i-1) = 1$, we choose a vertex $u^{i-1} \in V_{k_{i-1}}^{i-1}$, and take any realization of τ^i such that $u^i \in V_1^i$. Again, this is possible since G is AP+1. Also the subgraph $(G\Box P_l)[V_{k_{i-1}}^{i-1} \cup V_1^i]$ is connected.

Thus, we obtained a realization

$$\mathcal{R} = (V_1, V_1^1, \ldots, V_{k_1}^1, V_1^2, \ldots, V_{k_l}^l)$$

of the sequence $(n_1, n_1^1, \ldots, n_{k_1}^1, n_1^2, \ldots, n_{k_l}^l)$. Suppose that in the procedure of defining the sequences $\tau^1, ..., \tau^l$, a certain term n_ν of τ has been divided into $s \geq 2$ addends $n_{k_i}^i, \ldots, n_1^{i+r-1}$. If $s \geq 3$, then of course, $k_{i+1} = \ldots = k_{i+s-2} = 1$. Denote $V_\nu = V_{k_i}^i \cup V_1^{i+1} \cup \ldots \cup V_1^{i+s-1}$. By our construction, the subgraph $(G \Box P_l)[V_\nu]$ is connected. We then replace the subsequence $V_{k_i}^i, V_1^{i+1}, \ldots, V_1^{i+s-1}$ in $\mathcal{R}$ by V_ν. If we do the same for all divided terms of τ, then we obtain a realization of τ in $G \Box P_l$ satisfying the vertex restriction (u_0^j, n_1).

Case 2: $n_1 \geq n$.

Let $d = \lfloor \frac{n_1}{n-1} \rfloor$ and $r \equiv n_1 \pmod{(n-1)}$. In order to make our description more legible, we distinguish two subcases.

Subcase 2.1: $d < j$. We define the capacity function φ for each i-th layer G^i:

$$\varphi(i) = \begin{cases} 1 & \text{if } i \in [j-d+1, j], \\ n-r & \text{if } i = j-d, \\ n & \text{otherwise.} \end{cases}$$

Then we consider the sequence $\tau' = (n_2, \ldots, n_k)$, and we define the sequences $\tau^1, \ldots, \tau^l$, as well as numbers $\alpha(1), \ldots, \alpha(l)$, according to the function φ exactly in the same way as in Case 1.

Now, we start to construct a realization $(V_1, \ldots, V_k)$ of the sequence τ with choosing a vertex $x \in V(G) \setminus \{u_0\}$, and defining the set V_1 of n_1 vertices as

$$V_1 = \bigcup_{i=j-d+1}^{j} (V(G^i) \setminus \{x^i\}) \cup W,$$

where W is a set of r vertices of G^{j-d}. When $r \neq 0$, we may additionally require, as G is AP+1, that $(W, V_1^{j-d}, \ldots, V_{k_{j-d}}^{j-d})$ is a realization of the sequence $(r, n_1^{j-d}, \ldots, n_{k_{j-d}}^{j-d})$ satisfying the vertex restriction $(x^{j-d}, n_{k_{j-d}}^{j-d})$. When $r = 0$, we consider any realization $(V_1^{j-d}, \ldots, V_{k_{j-d}}^{j-d})$ of τ^{j-d}, such that $x^{j-d} \in V_{k_{j-d}}^{j-d}$ if $\alpha(j-d-1) = 1$.

Then, we consecutively produce realizations of $\tau^{j-d-2}, \tau^{j-d-3} \ldots, \tau^1$, following the same rules as in Case 1. That is, for every $i = j-d-2, \ldots, 1$, when $\alpha(i) = 1$, we choose a vertex $u^{i+1} \in V_1^{i+1}$, and using the fact that G is AP+1, we find a realization of τ^i in G^i such that $u^i \in V_{k_i}^i$. For $i \in [j-d+1, j]$, we take $\{x^i\}$ as a realization of the sequence $\tau^i = (1)$. Next, for $i = j+1, \ldots, l$, we proceed exactly in the same way as in Case 1.

The same arguments as in Case 1 show that we can glue corresponding elements of the realization $(V_1, V_1^1, \ldots, V_{k_1}^1, V_1^2, \ldots, V_{k_l}^l)$ of the sequence $(n_1, n_1^1, \ldots, n_{k_1}^1, n_1^2, \ldots, n_{k_l}^l)$ to obtain a realization $(V_1, \ldots, V_k)$ of τ.

Subcase 2.2: $d \geq j$. If $n_1 = nl$, then $k = 1$ and $\tau = (nl)$ is trivially realizable in $G \Box P_l$. For $k \geq 2$, we fix any vertex $x \in V(G) \setminus \{u_0\}$ and define

$$V_1 = \bigcup_{i=1}^{r} V(G^i) \cup \bigcup_{i=r+1}^{d} (V(G^i) \setminus \{x^i\}).$$

Clearly, $u_0^j \in V_1$. Then we consider the capacity function

$$\varphi(i) = \begin{cases} 1 & \text{if } i \in [r+1, d], \\ n & \text{if } i \in [d+1, n], \end{cases}$$

and we define sequences $\tau^{r+1}, \ldots, \tau^l$ and their realizations according to the same rules as in Case 1. It is easy to see that we can obtain a required realization $(V_1, \ldots, V_k)$ of τ in the same way as previously. □

References

[1] D. Barth, O. Baudon and J. Puech, Network sharing: a polynomial algorithm for tripodes, Discrete Appl. Math. 119 (2002), 205-216.

[2] D. Barth, and H. Fournier: A degree bound on decomposable trees. Discrete Mathematics 306 (2006), 469-477.

[3] D. Barth, H. Fournier, and R. Ravaux: On the shape of decomposable trees. Discrete Math. 309 (2009), 3882-3887.

[4] O. Baudon, J. Bensmail, J. Przybyło, M. Woźniak, Partitioning powers of traceable or Hamiltonian graphs, submitted.

[5] O. Baudon, J. Przybyło, and M. Woźniak, On minimal arbitrarily partitionable graphs. Inf. Process. Lett. 112 (2012), 697-700.

[6] H.J. Broersma, D. Kratsch, and G.J. Woeginger, Fully decomposable split graphs, Lecture Notes in Computer Science 5874 (2009) 4105-112.

[7] E. Győri, On division of graphs to connected subgraphs, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I, pp. 485-494, Colloq. Math. Soc. János Bolyai, 18, North-Holland, Amsterdam-New York, 1978

[8] M. Horňák, A. Marczyk, I. Schiermeyer, and M. Woźniak, Dense arbitrarily vertex decomposable graphs, Graphs and Combin. 28 (2012), 807-821.

[9] M. Horňák, Zs. Tuza, and M. Woźniak, On-line arbitrarily vertex decomposable trees, Discrete Appl. Math. 155 (2007), 1420-1429.

[10] M. Horňák, and M. Woźniak, Arbitrarily vertex decomposable trees are of maximum degree at most six, Opuscula Mathematica 23 (2003), 49-62.

[11] R. Kalinowski, M. Pilśniak, M. Woźniak, and I.A. Zioło, Arbitrarily vertex decomposable suns with few rays, Discrete Math. 309 (2009), 3726-3732.

[12] R. Kalinowski, M. Pilśniak, M. Woźniak, and I.A. Zioło, On-line arbitrarily vertex decomposable suns, Discrete Math. 309 (2009), 6328-6336.

[13] L. Lovász, A homology theory for spanning trees of a graph, Acta Mathematica Academiae Scientiarum Hungaricae 30 (1977), 241-251.

[14] A.Marczyk, A note on arbitrarily vertex decomposable graphs, Opuscula Mathematica 26 (2006), 109-118.

[15] A. Marczyk, An ore-type condition for arbitrarily vertex decomposable graphs, Discrete Math. 309 (2009), 3588-3594.