A variant of Niessen's problem on degree sequences of graphs

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Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two sequences of nonnegative integers satisfying the condition that $b_1 \geq b_2 \geq \cdots \geq b_n$, $a_i \leq b_i$ for $i = 1, 2, \ldots, n$ and $a_i + b_i \geq a_{i+1} + b_{i+1}$ for $i = 1, 2, \ldots, n-1$. In this paper, we give two different conditions, one of which is sufficient and the other one necessary, for the sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ such that for every $(c_1, c_2, \ldots, c_n)$ with $a_i \leq c_i \leq b_i$ for $i = 1, 2, \ldots, n$ and $\sum\limits_{i=1}^{n} c_i \equiv 0$ (mod 2), there exists a simple graph G with vertices $v_1, v_2, \ldots, v_n$ such that $d_G(v_i) = c_i$ for $i = 1, 2, \ldots, n$. This is a variant of Niessen's problem on degree sequences of graphs (Discrete Math., 191 (1998), 247–253).

Keywords: graph; degree sequence; Niessen's problem

1 Introduction

A sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The following well-known theorem due to Erdős and Gallai [2] gives a characterization of π that is graphic.

Theorem 1.1 (Erdős and Gallai [2]) *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a sequence of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ and $\sum\limits_{i=1}^{n} d_i \equiv 0$ (mod 2). Then π is graphic if and only if for $t = 0, 1, \ldots, n$, we have*

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}. \tag{1}$$

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative integers with $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. Motivated by Theorem 1.1, Niessen [4] posed the following problem.

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Problem 1.1 (Niessen [4, Problem 297]) *Give a simple characterization of the sequences* $(a_1, a_2, \ldots, a_n)$ *and* $(b_1, b_2, \ldots, b_n)$ *(like Theorem 1.1) such that there exists* $(c_1, c_2, \ldots, c_n)$ *with* $a_i \leq c_i \leq b_i$ *for* $i = 1, 2, \ldots, n$ *and* $\sum\limits_{i=1}^{n} c_i \equiv 0$ *(mod 2) so that* $(c_1, c_2, \ldots, c_n)$ *is graphic.*

Motivated by Problem 1.1, we now propose the following problem.

Problem 1.2 *Give a simple characterization of the sequences* $(a_1, a_2, \ldots, a_n)$ *and* $(b_1, b_2, \ldots, b_n)$ *such that every* $(c_1, c_2, \ldots, c_n)$ *with* $a_i \leq c_i \leq b_i$ *for* $i = 1, 2, \ldots, n$ *and* $\sum\limits_{i=1}^{n} c_i \equiv 0$ *(mod 2) is graphic.*

Cai et al. [1] gave a solution to Problem 1.1. They defined for $t = 0, 1, \ldots, n$

$$J_t = \{i | i \geq t+1 \text{ and } b_i \geq t+1\}$$

and

$$\varepsilon(t) = \begin{cases} 1 & \text{if } a_i = b_i \text{ for all } i \in J_t \text{ and } \sum\limits_{i \in J_t} b_i + t|J_t| \equiv 1 \text{ (mod 2)}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2 (Cai et al. [1]) *Let* $A_n = (a_1, a_2, \ldots, a_n)$ *and* $B_n = (b_1, b_2, \ldots, b_n)$ *be two sequences of nonnegative integers satisfying the condition that* $a_1 \geq a_2 \geq \cdots \geq a_n$*,* $a_i \leq b_i$ *for* $i = 1, 2, \ldots, n$ *and* $a_i = a_{i+1}$ *implies that* $b_i \geq b_{i+1}$ *for* $i = 1, 2, \ldots, n-1$*. Then there exists* $(c_1, c_2, \ldots, c_n)$ *with* $a_i \leq c_i \leq b_i$ *for* $i = 1, 2, \ldots, n$ *and* $\sum\limits_{i=1}^{n} c_i \equiv 0$ *(mod 2) so that* $(c_1, c_2, \ldots, c_n)$ *is graphic if and only if for* $t = 0, 1, \ldots, n$*, we have*

$$\sum_{i=1}^{t} a_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t, b_i\} - \varepsilon(t). \tag{2}$$

Cai et al. [1] also showed that Theorem 1.2 reduces to Theorem 1.1 when $a_i = b_i = d_i$ for $i = 1, 2, \ldots, n$. The purpose of this paper is to give a necessary condition and a sufficient condition (with bounds differing by at most one) on the sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ such that every $(c_1, c_2, \ldots, c_n)$ with $a_i \leq c_i \leq b_i$ for $i = 1, 2, \ldots, n$ and $\sum\limits_{i=1}^{n} c_i \equiv 0$ (mod 2) is graphic. This is a partial solution to Problem 1.2. They are Theorem 1.3 and Theorem 1.4 below. We define for $t = 0, 1, \ldots, n$

$$\xi(t) = \begin{cases} 1 & \text{if } a_i < b_i \text{ for some } i \in J_t \text{ or } \sum\limits_{i \in J_t} b_i + t|J_t| \equiv 1 \text{ (mod 2)}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.3 *Let* $A_n = (a_1, a_2, \ldots, a_n)$ *and* $B_n = (b_1, b_2, \ldots, b_n)$ *be two sequences of nonnegative integers satisfying the condition that* $b_1 \geq b_2 \geq \cdots \geq b_n$*,* $a_i \leq b_i$ *for* $i = 1, 2, \ldots, n$ *and* $a_i + b_i \geq a_{i+1} + b_{i+1}$ *for* $i = 1, 2, \ldots, n-1$*. If every* $(c_1, c_2, \ldots, c_n)$ *with* $a_i \leq c_i \leq b_i$ *for* $i = 1, 2, \ldots, n$ *and* $\sum\limits_{i=1}^{n} c_i \equiv 0$ *(mod 2) is graphic, then for* $t = 0, 1, \ldots, n$*, we have*

$$\sum_{i=1}^{t} b_i \leq \begin{cases} t(t-1) + \sum\limits_{i=t+1}^{n} min\{t, a_i\} - \xi(t) + 2 & \text{if } a_i < b_i \text{ for some } i, \\ t(t-1) + \sum\limits_{i=t+1}^{n} min\{t, a_i\} - \xi(t) & \text{if } a_i = b_i \text{ for each } i. \end{cases} \tag{3}$$

Theorem 1.4 *Let $A_n = (a_1, a_2, \dots, a_n)$ and $B_n = (b_1, b_2, \dots, b_n)$ be two sequences of nonnegative integers satisfying the condition that $b_1 \geq b_2 \geq \cdots \geq b_n$, $a_i \leq b_i$ for $i = 1, 2, \dots, n$ and $a_i + b_i \geq a_{i+1} + b_{i+1}$ for $i = 1, 2, \dots, n-1$. If for $t = 0, 1, \dots, n$, we have*

$$\sum_{i=1}^{t} b_i \leq \begin{cases} t(t-1) + \sum\limits_{i=t+1}^{n} min\{t, a_i\} - \xi(t) + 1 & \text{if } a_i < b_i \text{ for some } i, \\ t(t-1) + \sum\limits_{i=t+1}^{n} min\{t, a_i\} - \xi(t) & \text{if } a_i = b_i \text{ for each } i, \end{cases} \tag{4}$$

then every $(c_1, c_2, \dots, c_n)$ with $a_i \leq c_i \leq b_i$ for $i = 1, 2, \dots, n$ and $\sum\limits_{i=1}^{n} c_i \equiv 0$ (mod 2) is graphic.

The condition (3) of Theorem 1.3 is not sufficient, as can be seen by taking $A_5 = (4, 3, 2, 1, 1)$ and $B_5 = (4, 3, 3, 2, 2)$, which satisfy $b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5$, $a_i \leq b_i$ for $i = 1, 2, 3, 4, 5$ and $a_i + b_i \geq a_{i+1} + b_{i+1}$ for $i = 1, 2, 3, 4$. It is easy to check that (3) holds for $t = 0, 1, 2, 3, 4, 5$. However, $(c_1, c_2, c_3, c_4, c_5) = (4, 3, 3, 1, 1)$ with $\sum\limits_{i=1}^{5} c_i = 12$ and $a_i \leq c_i \leq b_i$ for $i = 1, 2, 3, 4, 5$ is not graphic.

The condition (4) of Theorem 1.4 is not necessary: take $A_5 = (4, 3, 1, 2, 1)$ and $B_5 = (4, 3, 3, 2, 2)$, which satisfy $b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5$, $a_i \leq b_i$ for $i = 1, 2, 3, 4, 5$ and $a_i + b_i \geq a_{i+1} + b_{i+1}$ for $i = 1, 2, 3, 4$. It is easy to see that every $(c_1, c_2, c_3, c_4, c_5)$ with $a_i \leq c_i \leq b_i$ for $i = 1, 2, 3, 4, 5$ and $\sum\limits_{i=1}^{5} c_i \equiv 0$ (mod 2) is graphic. However, (4) does not hold for $t = 2$.

Theorems 1.3 and 1.4 show that the left hand sides of (3) and (4) are equal and the right hand sides of (3) and (4) with bounds differing by one if $a_i < b_i$ for some i and they coincide if $a_i = b_i$ for each i. For $a_i = b_i = d_i$ for $i = 1, \dots, n$, it is easy to get that $\xi(t) = \varepsilon(t)$ for $t = 0, 1, \dots, n$. Thus, (3) and (4) imply that

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\} - \varepsilon(t) \text{ for } t = 0, 1, \dots, n,$$

that is (2) for $t = 0, 1, \dots, n$. Therefore, Theorems 1.3 and 1.4 reduce to Theorem 1.1 when $a_i = b_i = d_i$ for $i = 1, 2, \dots, n$.

2 Proofs of Theorem 1.3–1.4

Proof of Theorem 1.3. If $a_i = b_i$ for $i = 1, 2, \dots, n$ and $\sum\limits_{i=1}^{n} a_i \equiv 0$ (mod 2), then $(a_1, a_2, \dots, a_n)$ is graphic and $\sum\limits_{i=1}^{t} a_i \leq t(t-1) + \sum\limits_{i=t+1}^{n} min\{t, a_i\}$ for $t = 0, 1, \dots, n$ by Theorem 1.1. Since $\xi(t) = \varepsilon(t) = 0$ for $t = 0, 1, \dots, n$ (see [1]), we have that

$$\sum_{i=1}^{t} a_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t, a_i\} - \xi(t) \text{ for } t = 0, 1, \dots, n.$$

If there is an j so that $a_j < b_j$, we let $C_n = (c_1, \dots, c_n) = (b_1, \dots, b_t, a_{t+1}, \dots, a_n)$, where $0 \leq t \leq n$. If $\sum\limits_{i=1}^{n} c_i \equiv 0$ (mod 2), then C_n is graphic and $\sum\limits_{i=1}^{t} b_i \leq t(t-1) + \sum\limits_{i=t+1}^{n} min\{t, a_i\}$. It follows from

$\xi(t) \le 1$ that $\sum_{i=1}^{t} b_i \le t(t-1) + \sum_{i=t+1}^{n} min\{t, a_i\} - \xi(t) + 2$. Assume that $\sum_{i=1}^{n} c_i \equiv 1$ (mod 2). We now consider two cases according to whether $1 \le j \le t$ or $t+1 \le j \le n$.

If $1 \le j \le t$, we let $c_j^* = b_j - 1$, $c_i^* = b_i$ for all $i \in \{1, \ldots, t\} \backslash \{j\}$ and $c_i^* = a_i$ for all $i \ge t+1$, then $\sum_{i=1}^{n} c_i^* \equiv 0$ (mod 2), and hence $C_n^* = (c_1^*, \ldots, c_n^*)$ is graphic. Thus $\sum_{i=1}^{t} b_i - 1 \le t(t-1) + \sum_{i=t+1}^{n} min\{t, a_i\}$, implying that $\sum_{i=1}^{t} b_i \le t(t-1) + \sum_{i=t+1}^{n} min\{t, a_i\} - \xi(t) + 2$.

If $t+1 \le j \le n$, let $c_j^* = a_j + 1$, $c_i^* = b_i$ for all $1 \le i \le t$ and $c_i^* = a_i$ for all $i \in \{t+1, \ldots, n\} \backslash \{j\}$, then $\sum_{i=1}^{n} c_i^* \equiv 0$ (mod 2), and hence $C_n^* = (c_1^*, \ldots, c_n^*)$ is graphic. Thus

$$\begin{aligned}
\sum_{i=1}^{t} b_i &\le t(t-1) + \sum_{\substack{i=t+1 \\ i \ne j}}^{n} min\{t, a_i\} + min\{t, a_j + 1\} \\
&\le t(t-1) + \sum_{i=t+1}^{n} min\{t, a_i\} + 1 \\
&\le t(t-1) + \sum_{i=t+1}^{n} min\{t, a_i\} - \xi(t) + 2.
\end{aligned}$$

The proof of Theorem 1.3 is completed. □

The proof of Theorem 1.4 depends on a factor theorem due to Niessen [3]. Let G be a simple graph and let $g, f : V \to Z^+$ such that $g(v) \le f(v)$ for all $v \in V$, where $V = V(G)$ is the vertex set of G and Z^+ denotes the nonnegative integers. An (g, f)-*factor* of G is a spanning subgraph F such that $g(v) \le d_F(v) \le f(v)$ for all $v \in V$. An (f, f)-factor is called an f-*factor*. If there exists a function $h : V \to Z^+$ with $g(v) \le h(v) \le f(v)$ for all $v \in V$ and $\sum_{v \in V} h(v) \equiv 0$ (mod 2), then G is said to have *all* (g, f)-*factors* if and only if G has an h-factor for every h described above. Let $U, W \subseteq V$ be disjoint sets and $e_G(U, W)$ denote the number of edges of G joining U to W.

Theorem 2.1 (Niessen [3]) *G has all (g, f)-factors if and only if*

$$\sum_{v \in S} g(v) - \sum_{v \in T} f(v) + \sum_{v \in T} d_{G \backslash S}(v) - \omega(S, T) \ge \begin{cases} -1 & \textit{if } g < f \textit{ for some } v \\ 0 & \textit{if } g = f \textit{ for each } v \end{cases}$$

for all disjoint sets $S, T \subseteq V$, where $\omega(S, T)$ denotes the number of components C of $G - (S \cup T)$ such that there exists $v \in V(C)$ with $g(v) < f(v)$ or $\sum_{v \in V(C)} f(v) + e_G(V(C), T) \equiv 1$ (mod 2).

Proof of Theorem 1.4. By Theorem 2.1, we take $G = K_n$ (the complete graph on n vertices), where $V(K_n) = \{v_1, \ldots, v_n\}$. Let $g(v_i) = a_i$ and $f(v_i) = b_i$ for $i = 1, 2, \ldots, n$. It is easy to see that G has all (g, f)-factors if and only if every $(c_1, c_2, \ldots, c_n)$ with $a_i \le c_i \le b_i$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} c_i \equiv 0$ (mod 2) is graphic. Therefore, we only need to verify that

$$\Delta(S, T) := \sum_{v_i \in S} a_i - \sum_{v_i \in T} b_i + t(n - 1 - s) - \omega(S, T) \ge \begin{cases} -1 & \text{if } a_i < b_i \text{ for some } i \\ 0 & \text{if } a_i = b_i \text{ for each } i \end{cases}$$

for any two disjoint subsets S and T of $V(K_n)$, where $s = |S|$ and $t = |T|$.

Set $R = V(K_n) \setminus (S \cup T)$ and $r = |R|$, then we can get that

$$\omega(S,T) = \begin{cases} 1 & \text{if } a_i < b_i \text{ for some } v_i \in R \text{ or } \sum\limits_{v_i \in R} b_i + rt \equiv 1 \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let S and T be two disjoint subsets of $V(K_n)$ satisfying

(a) $\Delta(S,T)$ is minimized;

(b) subject to (a), $|S| + |T|$ is minimized;

(c) subject to (a) and (b), $|T \cap \{v_1, \ldots, v_t\}|$ is maximized, where $t = |T|$.

Now we claim that $T = \{v_1, v_2, \ldots, v_t\}$. To justify it, assume the contrary: $v_i \notin T$ and $v_j \in T$ for some i and j with $i \leq t < j$. We consider two cases according to whether $v_i \in R$ or $v_i \in S$.

Case 1. $v_i \in R$.

Set $T' = (T \cup \{v_i\}) \setminus \{v_j\}$. Then $R' = (R \cup \{v_j\}) \setminus \{v_i\}$. It follows from (c) that $\Delta(S,T') - \Delta(S,T) \geq 1$. So $b_j - b_i + \omega(S,T) - \omega(S,T') \geq 1$. Since $b_i \geq b_j$, we have that $b_i = b_j$, $\omega(S,T) = 1$ and $\omega(S,T') = 0$.

If $a_j < b_j$, then $\omega(S,T') = 1$, a contradiction. If $a_i < b_i$ and $a_j = b_j$, then $a_i < a_j$, and hence $a_i + b_i < a_j + b_j$, which is impossible. If $a_i = b_i$ and $a_j = b_j$, then $\omega(S,T) = \omega(S,T')$, a contradiction.

Case 2. $v_i \in S$.

Set $S^\star = (S \cup \{v_j\}) \setminus \{v_i\}$ and $T^\star = (T \cup \{v_i\}) \setminus \{v_j\}$. By (c), we have that $\Delta(S^\star, T^\star) - \Delta(S,T) \geq 1$, i.e., $a_j - b_i + b_j - a_i \geq 1$, implying that $a_i + b_i < a_j + b_j$, which is impossible.

Therefore, we conclude that $T = \{v_1, v_2, \ldots, v_t\}$, as claimed. Thus, we obtain that

$$\begin{aligned} \Delta(S,T) &= \sum_{v_i \in S} a_i - \sum_{i=1}^{t} b_i + t(n-1-s) - \omega(S,T) \\ &= -\sum_{i=1}^{t} b_i + t(t-1) + \sum_{i=t+1}^{n} min\{t, a_i\} - \omega(S,T) \\ &\quad + \sum_{v_i \in S} (a_i - min\{t, a_i\}) + \sum_{v_i \in R} (t - min\{t, a_i\}). \end{aligned}$$

If $a_k = b_k$ for each k, then by (4) we have that

$$\Delta(S,T) \geq \xi(t) - \omega(S,T) + \sum_{v_i \in S} (a_i - min\{t, a_i\}) + \sum_{v_i \in R} (t - min\{t, a_i\}).$$

In this case, if $a_i > t$ for some $v_i \in S$ or $a_i < t$ for some $v_i \in R$ or $\xi(t) = 1$, then $\Delta(S,T) \geq 0$. If $a_i \leq t$ for all $v_i \in S$, $a_i \geq t$ for all $v_i \in R$ and $\xi(t) = 0$, then $\{v_i | i \in J_t\} \subseteq R$, $a_i = t$ for all $v_i \in R \setminus \{v_i | i \in J_t\}$ and $\sum\limits_{i \in J_t} b_i + t|J_t| \equiv 0 \ (mod\, 2)$, implying that

$$\begin{aligned} \sum_{v_i \in R} b_i + rt &= \sum_{i \in J_t} b_i + \sum_{v_i \in R \setminus \{v_i | i \in J_t\}} b_i + rt \\ &= \sum_{i \in J_t} b_i + t|R \setminus \{v_i | i \in J_t\}| + rt \\ &\equiv \sum_{i \in J_t} b_i + t|J_t| \ (mod\, 2) \\ &\equiv 0 \ (mod\, 2). \end{aligned}$$

Thus, $\omega(S,T)=0$ and $\Delta(S,T)\geq 0$.

If $a_k<b_k$ for some k, then by (4) we have that

$$\Delta(S,T)\geq \xi(t)-1-\omega(S,T)+\sum_{v_i\in S}(a_i-min\{t,a_i\})+\sum_{v_i\in R}(t-min\{t,a_i\}).$$

In this case, if $a_i>t$ for some $v_i\in S$ or $a_i<t$ for some $v_i\in R$ or $\xi(t)=1$, then $\Delta(S,T)\geq -1$. If $a_i\leq t$ for all $v_i\in S$, $a_i\geq t$ for all $v_i\in R$ and $\xi(t)=0$, then $a_i=b_i$ for all $i\in J_t$, $\{v_i|i\in J_t\}\subseteq R$, $a_i=b_i=t$ for all $v_i\in R\setminus\{v_i|i\in J_t\}$ and $\sum\limits_{i\in J_t}b_i+t|J_t|\equiv 0\ (mod\ 2)$, implying that $a_i=b_i$ for all $v_i\in R$ and $\sum\limits_{v_i\in R}b_i+rt\equiv\sum\limits_{i\in J_t}b_i+t|J_t|\ (mod\ 2)\equiv 0\ (mod\ 2)$. Thus, $\omega(S,T)=0$ and $\Delta(S,T)\geq -1$. The proof of Theorem 1.4 is completed. □

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References

[1] M.C. Cai, X.T. Deng and W.A. Zang. Solution to a problem on degree sequences of graphs. *Discrete Math.,* 219: 253–257, 2000.

[2] P. Erdős and T. Gallai. Graphs with prescribed degrees of vertices (Hungarian). *Mat. Lapok,* 11: 264–274, 1960.

[3] T. Niessen. A characterization of graphs having all (g,f)-factors. *J. Combin. Theory Ser.B,* 72: 152–156, 1998.

[4] T. Niessen. Research problems. *Discrete Math.,* 191: 247–253, 1998.