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Efficient open domination in graph products

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A graph G is an efficient open domination graph if there exists a subset D of $V(G)$ for which the open neighborhoods centered in vertices of D form a partition of $V(G)$. We completely describe efficient open domination graphs among lexicographic, strong, and disjunctive products of graphs. For the Cartesian product we give a characterization when one factor is K_2.

Keywords: efficient open domination; graph products; total domination

1 Introduction and preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use standard notations $N_G(v)$ for the *open neighborhood* $\{u : uv \in E(G)\}$ and $N_G[v]$ for the *closed neighborhood* $N_G(v) \cup \{v\}$ for a graph G. By $u \sim v$ we denote adjacency of u and v from $V(G)$. Throughout the article we consider only simple graphs.

The domination number $\gamma(G)$ of a graph G is a classical invariant in graph theory. It is the minimum cardinality of a set S for which the union of closed neighborhoods centered in vertices of S cover the whole vertex set of G. Such a set S is called a *dominating set* of G. Hence each vertex of G is either in S or adjacent to a vertex in S. In other words, we can say that vertices of S control each vertex outside of S. A classical question in such a situation is: who controls the vertices of S? One possible solution to this dilemma is the total domination. A set $D \subseteq V(G)$ is a *total dominating set* of G if every vertex of G is adjacent to a vertex of D. (Hence, also vertices of D are controlled by D.) The *total domination number* of a graph G is the minimum cardinality of a total dominating set of G and is denoted by $\gamma_t(G)$. A total dominating set D of cardinality $\gamma_t(G)$ is called $\gamma_t(G)$-*set*.

The natural question for a graph G is whether we can find a total dominating set D for which its open neighborhoods not only cover $V(G)$ but also form a partition of $V(G)$, which means that $N_G(u) \cap$

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$N_G(v) = \emptyset$ for every different $u, v \in D$. The problem has been presented under the names: total perfect codes [3], efficient open domination [7] and exact transversals [4]. In the present work we follow the terminology of efficient open domination. A graph G is an *efficient open domination graph* if there exists a set D, called an *efficient open dominating set*, for which $\bigcup_{v\in D} N_G(v) = V(G)$ and $N_G(u)\cap N_G(v) = \emptyset$ for every pair u and v of distinct vertices of D. It is easy to see that paths P_n are efficient open domination graphs if and only if $n \not\equiv 1 \pmod 4$, while cycles C_n are efficient open domination graphs if and only if $n \equiv 0 \pmod 4$.

The problem of establishing whether a graph G is an efficient open domination graph is an NP-complete problem, see [16] (or [8]). This issue has been studied under its different names, nevertheless it has received not enough attention. Efficient open domination trees have been characterized recursively in [8]. Various properties of efficient open domination graphs were presented in [7]. The efficient open domination graphs among Cayley graphs were studied in [20] and the efficient open domination grid graphs in [4, 5, 14]. Moreover, the efficient open domination graphs among direct product graphs were characterized in [1]. However we have not found the following basic connection in the literature.

Observation 1.1 *If G is an efficient open domination graph with an efficient open dominating set D, then $\gamma_t(G) = |D|$.*

Proof. If D is an efficient open dominating set of G, then D is also a total dominating set of G and $\gamma_t(G) \le |D|$ follows. On the other hand, an arbitrary vertex of D has at least one neighbor in every $\gamma_t(G)$-set D', since $\bigcup_{v\in D'} N_G(v) = V(G)$. Moreover, these neighbors must be different, since $\bigcup_{v\in D} N_G(v)$ form a partition of $V(G)$. Hence $\gamma_t(G) \ge |D|$ and the equality follows. □

From this observation immediately follows that all efficient open dominating sets of a graph G have the same cardinality. Another simple fact holds for regular graphs, since vertices from an efficient open dominating set are given by disjoint pairs of adjacent vertices and all the vertices have the same number of neighbors.

Observation 1.2 *Let G be an r-regular graph of order n. If G is an efficient open domination graph, then $n \equiv 0 \pmod{2r}$.*

A similar approach is also known for dominating sets under the name 1-perfect graphs. That is, G is 1-*perfect graph* if there exists a set $P \subseteq V(G)$ for which $V(G) = \bigcup_{v\in P} N_G[v]$ and $N_G[u] \cap N_G[v] = \emptyset$ for every pair u and v of distinct vertices of P. Set P is called a *perfect code* of G. The name arose from codes, since one can discover and repair one error in such graphs.

Several graph products have been investigated in the last few decades and a rich theory involving the structure and recognition of classes of these graphs has emerged, cf. [10]. The most studied graph products are the Cartesian product, the strong product, the direct product and the lexicographic product which are also called *standard products*. One standard approach to graph products is to deduce properties of a product with respect to (the same) properties of its factors. See a short collection of these types involving total domination and perfect codes in [6, 9, 11, 12, 13, 15, 17].

The domination related problems on the Cartesian product seems to be the most problematic among standard products. We just mention the famous Vizing's conjecture: $\gamma(G\Box H) \ge \gamma(G)\gamma(H)$, which is probably the most challenging problem in the area of domination (see the latest survey on Vizing's conjecture [2]). The efficient open domination is no exception.

In the next section we completely describe the efficient open domination graphs among strong, lexicographic, and disjunctive products. After that we introduce zig-zag graphs and explore their structure. They are used in the following section where we completely describe graphs for which their Cartesian product with K_2 is an efficient open domination graph. We end with a section on grid-like graphs.

2 The lexicographic, strong, and disjunctive products

The *lexicographic product* $G \circ H$ (sometimes also denoted by $G[H]$) of graphs G and H is a graph with $V(G \circ H) = V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent in $G \circ H$ whenever $gg' \in E(G)$ or ($g = g'$ and $hh' \in E(H)$). For a fixed $h \in V(H)$ we call $G^h = \{(g,h) \in V(G \circ H) : g \in V(G)\}$ a *G-layer* in $G \circ H$. Symmetrically H-layers ${}^gH$ for a fixed $g \in V(G)$ are defined. Notice that the subgraph of $G \circ H$ induced by a G-layer or an H-layer is isomorphic to G or H, respectively. The map $p_G : V(G \circ H) \to V(G)$ defined by $p_G((g,h)) = g$ is called a *projection map onto* G. Similarly, we define p_H as the *projection map onto* H. Projections are defined as maps between vertices, but frequently it is more comfortably to see them as maps between graphs. In this case we observe the subgraphs induced by $A \subseteq V(G \circ H)$ and $p_X(A)$ for $X \in \{G,H\}$. The lexicographic product is clearly not commutative, nevertheless it is associative [10].

Theorem 2.1 *Let G and H be graphs. The lexicographic product $G \circ H$ is an efficient open domination graph if and only if either*

(i) *G is a graph without edges and H is an efficient open domination graph, or*

(ii) *G is an efficient open domination graph and H contains an isolated vertex.*

Proof. If G is a graph without edges on n vertices, then $G \circ H$ is isomorphic to n copies of H. If in addition H is an efficient open domination graph, then also n copies of H form an efficient open domination graph. Now, let G be an efficient open domination graph, let D_G be one efficient open dominating set and let h_0 be an isolated vertex of H. We will show that $D_G \times \{h_0\}$ is an efficient open dominating set of $G \circ H$. For this, notice that $N_{G \circ H}(g,h_0) = N_G(g) \times V(H)$ and $\bigcup_{g \in D_G} N_{G \circ H}(g,h_0) = V(G \times H)$. If $g,g' \in D_G$ and $g \neq g'$, then $N_{G \circ H}(g,h_0) \cap N_{G \circ H}(g',h_0) \neq \emptyset$ implies that $N_G(g) \cap N_G(g') \neq \emptyset$, which is a contradiction. Therefore, $G \circ H$ is an efficient open domination graph.

Conversely, let $G \circ H$ be an efficient open domination graph with an efficient open dominating set D. Let $(g,h),(g',h') \in D$ be adjacent vertices. Suppose first that there exists such an edge with $g \neq g'$. If $h'' \in N_H(h)$, then $(g,h'') \in N_{G \circ H}(g,h) \cap N_{G \circ H}(g',h')$, which is a contradiction. Hence h (and by symmetry also h') is an isolated vertex of H. Since H contains an isolated vertex, it follows that G has no isolated vertices, otherwise $G \circ H$ would contain isolated vertices, which is impossible for an efficient open domination graph. Thus, the fact $(g_1,h_1),(g_2,h_2) \in D$, implies that h_1 and h_2 are isolated vertices of H (notice that it can happen $h_1 = h_2$). If $g_1 \in N_G(g) \cap N_G(g')$ for some $g,g' \in p_G(D)$, then ${}^{g_1}H \subseteq N_{G \circ H}(g,h) \cap N_{G \circ H}(g',h')$ for $(g,h),(g',h') \in D$, which is a contradiction. Also $\bigcup_{g_1 \in p_G(D)} N_G(g_1) = V(G)$, since $\bigcup_{(g_1,h_1) \in D} N_{G \circ H}(g_1,h_1) = V(G \circ H)$ and D is an efficient open dominating set. Thus, G is an efficient open domination graph (with an efficient open dominating set $p_G(D)$).

Now we can assume that all edges between vertices of D have the same first coordinate: $(g,h)(g,h')$. Thus, g is an isolated vertex of G, otherwise ${}^{g'}H \subseteq N_{G \circ H}(g,h) \cap N_{G \circ H}(g,h')$ for any neighbor g'

of g in G, which is not possible. Since $\{N_{G\circ H}(g,h) : (g,h) \in D\}$ forms a partition of $G \circ H$, every vertex (g_1,h_1) is in some $N_{G\circ H}(g,h)$. Again (g,h) is in some $N_{G\circ H}(g',h')$ and we have $g = g' = g_1$. Hence every vertex of G is an isolated vertex and suppose that there are n vertices in G. Every H-layer is isomorphic to H and $G \circ H$ is isomorphic to n copies of H. Since $G \circ H$ is an efficient open domination graph, every component of $G \circ H$ is such. Therefore, also H is an efficient open domination graph which ends the proof. □

We can easily generalize the construction of Theorem 2.1 (ii). Namely, let G be any efficient open domination graph with $V(G) = \{g_1,\dots,g_n\}$. Choose n arbitrary graphs $H_1,\dots,H_n$. Let $N_G(g_i) = \{g_{j_1}, g_{j_2},\dots,g_{j_{k_i}}\}$ for every $i \in \{1,\dots,n\}$. Connect g_i by an edge with every vertex of graphs $H_{j_1}, H_{j_2},\dots,H_{j_{k_i}}$ to obtain a new graph G^*. It is easy to see that an efficient open dominating set D of G is also an efficient open dominating set of G^*. Moreover, we can add to G^* arbitrary many edges between vertices of H_i and H_j as long as $g_ig_j \in E(G)$ and the obtained graph is still an efficient open domination graph. If we add all possible edges between H_i and H_j whenever $g_ig_j \in E(G)$ and all graphs $H_1,\dots,H_n$ are isomorphic to a graph H, then the new graph is isomorphic to $G \circ (H \cup \{h\})$ where h is an isolated vertex of $H \cup \{h\}$.

The *strong product* $G \boxtimes H$ of graphs G and H is a graph with $V(G \boxtimes H) = V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent in $G \boxtimes H$ whenever ($gg' \in E(G)$ and $h = h'$) or ($g = g'$ and $hh' \in E(H)$) or ($gg' \in E(G)$ and $hh' \in E(H)$). The commutativity of the strong product follows from the symmetry of the definition of adjacency and for associativity see [10].

Since $G \circ H \cong G \boxtimes H$ if G is a graph without edges, Theorem 2.1 (i) already gives a hint for the strong product. Surprisingly, these are the only graphs (up to the commutativity of the factors) among strong products which are efficient open domination graphs. This follows immediately from the fact that $|N_{G\boxtimes H}(g,h) \cap N_{G\boxtimes H}(g',h')| \geq 2$ for any two adjacent vertices (g,h) and (g',h'), where both g and h are not isolated vertices of G and H, respectively.

Proposition 2.2 *Let G and H be two graphs. The strong product $G \boxtimes H$ is an efficient open domination graph if and only if one factor is a graph without edges and the other is an efficient open domination graphs.*

The *disjunctive product* $G \oplus H$ of graphs G and H is a graph with $V(G \oplus H) = V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent in $G \oplus H$ whenever $gg' \in E(G)$ or $hh' \in E(H)$. Like the Cartesian and the direct product, the disjunctive product is also commutative. This notion of graph product was introduced by Ore [18], where it was called the *Cartesian sum* of graphs. The name disjunctive product is due to [19].

Theorem 2.1 (ii) gives a hint for a characterization of efficient open domination graphs, for the case of disjunctive product, which has quite similar behavior like lexicographic product. Thus, we have the following result, which proof is quite similar to the proof of Theorem 2.1.

Theorem 2.3 *The disjunctive product of two graphs is an efficient open domination graph if and only if one graph is an efficient open domination graph and the other one contains an isolated vertex.*

Proof. Let G and H be any graphs. Assume that G is an efficient open domination graph with an efficient open dominating set D_G and let h_0 be an isolated vertex of H. Proceeding like in the proof of Theorem

2.1 we show that $D_G \times \{h_0\}$ is an efficient open dominating set of $G \oplus H$. Therefore, $G \oplus H$ is an efficient open domination graph.

On the contrary, we consider $G \oplus H$ is an efficient open domination graph with an efficient open dominating set D and we also proceed as in the proof of Theorem 2.1. Let $(g,h),(g',h') \in D$ be adjacent vertices. If $g \neq g'$, then h (and by symmetry also h') is an isolated vertex of H. Since H contains an isolated vertex, it follows that G has no isolated vertices, otherwise $G \oplus H$ would contain isolated vertices, which is impossible for an efficient open domination graph. Thus, if $(g_1,h_1),(g_2,h_2) \in D$, then h_1 and h_2 are isolated vertices of H. If $g_1 \in N_G(g) \cap N_G(g')$ for some $g,g' \in p_G(D)$, then ${}^{g_1}H \subseteq N_{G\oplus H}(g,h) \cap N_{G\oplus H}(g',h')$ for $(g,h),(g',h') \in D$, which is a contradiction. Also $\bigcup_{g_1 \in p_G(D)} N_G(g_1) = V(G)$, since $\bigcup_{(g_1,h_1)\in D} N_{G\oplus H}(g_1,h_1) = V(G \oplus H)$ and D is an efficient open dominating set. Thus, G is an efficient open domination graph (with an efficient open dominating set $p_G(D)$).

Similarly, if $h \neq h'$, then we obtain that g_1 and g_2 are isolated vertices of G. An analogous procedure shows that, in this case, H is an efficient open domination graph. Therefore, the proof is complete. $\square$

For the case of direct product graphs, the efficient open domination graphs among them were already characterized in [1]. By completeness we include their result in this section. The *direct product* $G \times H$ of graphs G and H is a graph with vertex set $V(G \times H) = V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent in $G \times H$ whenever $gg' \in E(G)$ and $hh' \in E(H)$. The open neighborhoods of vertices in direct product graphs are nicely connected to open neighborhoods of projections to the factors. Namely, $N_{G\times H}(g,h) = N_G(g) \times N_H(h)$ for every vertex $(g,h) \in V(G \times H)$ and this is the main reason for nice behavior of efficient open domination graphs among direct product with respect to its factors.

Theorem 2.4 [1] *Let G and H be two graphs. The direct product $G \times H$ is an efficient open domination graph if and only if G and H are efficient open domination graphs.*

By Observation, 1.1 the following corollary is obtained.

Corollary 2.5 *If G and H are efficient open domination graphs, then $\gamma_t(G \times H) = \gamma_t(G)\gamma_t(H)$.*

3 The zig-zag graphs

This section deals with a family of graphs which is the heart of the characterization of the efficient open domination graphs among the so called prism graphs $G\Box K_2$. In this sense, we give some deep description of such a family. To do so, we need to introduce some notation. By $d_G(u,v)$ we mean the *geodesic* or *shortest path distance*, which is the number of edges on a shortest u,v-path in G. Distance $d_G(e,v)$ between edge e and a vertex v in G is the shortest distance between end vertices of e and v, while the distance $d_G(e_1,e_2)$ between edges e_1 and e_2 is the shortest distance between end vertices of e_1 and end vertices of e_2. In general, for $P,Q \subseteq V(G)$, the distance $d_G(P,Q)$ between them is the shortest distance between a vertex from P and a vertex from Q. Two different edges are *incident* if they have a common end vertex.

Let G be a graph on at least three vertices and $E' = \{e_1,\dots,e_k\}$ be a subset of $E(G)$, where $e_i = u_iv_i$, with the following properties:

(i) $N_G(u_i) \cap N_G(v_i) = \emptyset$;

(ii) $d_G(e_i, e_j) \geq 2$ for every different pair $i, j \in \{1, \ldots, k\}$;

(iii) for every $w \in V(G) - \{u_i, v_i : i \in \{1, \ldots, k\}\}$ there exist unique j and ℓ, $j \neq \ell$, such that $d(w, e_j) = d(w, e_\ell) = 1$;

(iv) for every sequence of distinct edges $e_{i_1}, e_{i_2}, \ldots, e_{i_j}$, $j > 2$, with $d_G(e_{i_\ell}, e_{i_{\ell+1 (\mathrm{mod}\, j)}}) = 2$ for $\ell \in \{1, \ldots, j\}$, j must be an even number.

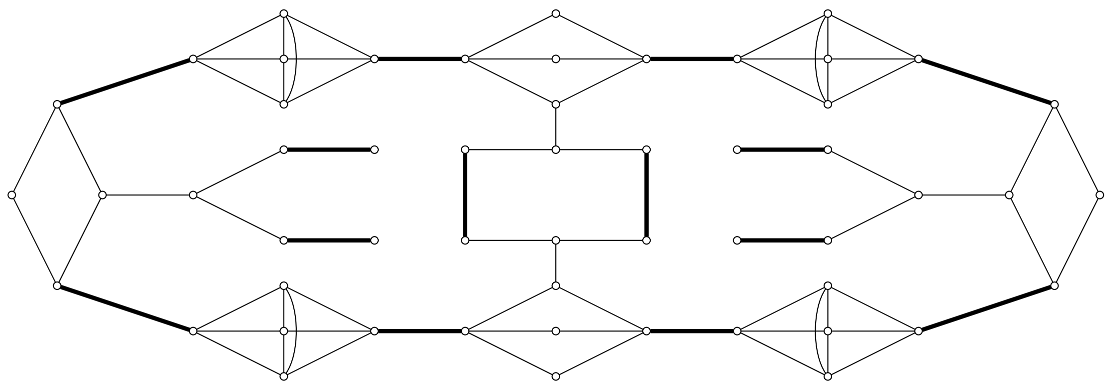

Fig. 1: A zig-zag graph where the zig-zag set is formed by the edges in bold.

We call E' a *zig-zag set* of G and, if there exists a zig-zag set in G, we call G as a *zig-zag graph*. (The motivation for this name follows from the property (iv) and the zig-zag role of edges at distance 2 in efficient open domination set of $G \Box K_2$ as can be seen in the following section.) In Figure 1 appears an example of a zig-zag graph, where the zig-zag set is given by the edges in bold. Notice that zig-zag graphs among cycles are exactly C_{6k} for a positive integer k (we need six to fulfill the property (iv) of the definition). Also, if a zig-zag graph G contains a vertex v of degree one, then the edge $vu \in E(G)$ must be in any zig-zag set of G. This follows from property (iii), since any vertex of degree one has not two different non incident edges at distance one. Thus, for instance, a path P_n is a zig-zag graph whenever $n = 2 + 3k$ for some positive integer k. Moreover, observe that if a zig-zag graph G contains no cycles, then property (iv) is fulfilled. Hence, we need a cycle C for property (iv) to be nontrivial. If edges from a zig-zag set E' which have at least one end vertex on C are successively at distance two, then its number must be even. If this holds for all such cycles of a graph G, then property (iv) is fulfilled for G.

Since zig-zag graphs play an important role in the efficient open domination graphs among prisms, we give somewhat deeper insight into their structure. Next we consider a construction that describes the family $\mathcal{G}$ of all graphs that satisfy properties (i), (ii), and (iii) of the definition. We begin with an arbitrary graph H on n vertices, and let $E' = \{e_i : e_i = u_i v_i, i \in \{1, \ldots, r\}\}$ be a set of r disjoint pairs of edges, with $r \geq 2$. Now, to obtain a graph $G \in \mathcal{G}$, for every vertex $h \in V(H)$ choose two edges $e_j, e_\ell \in E'$ and connect h by an edge with exactly one end vertex of e_j and one of e_ℓ. Notice that we can choose the same edges for different vertices. It is straightforward to observe that G is a graph satisfying properties (i), (ii), and (iii) for the set E'.

On the other hand, if G is a graph carrying out the properties (i), (ii), and (iii) for the set of edges $F \subseteq E(G)$, then F represents E' and the subgraph induced by $V(G) - V(F)$ represents H in the above description. Thus, this construction gives a characterization of all graphs for which properties (i), (ii), and (iii) hold. However, property (iv) does not always hold in this construction.

We will incorporate property (iv) to the above construction in two special cases, which also reveals richness of the structure of zig-zag graphs. The first case is related to the following. If $G \in \mathcal{G}$ was constructed in such a way that every $v \in V(G)$ has two private edges at distance one, then such edges do not lie in any cycle, and, as mentioned before, property (iv) is fulfilled. As a consequence, G is a zig-zag graph. Notice that in this construction we need $2n$ disjoint edges in E', where n is the order of the graph H used to generate G.

For the second construction of zig-zag graphs, we do the following. Let H be an arbitrary graph of order n and let $E = \{e_i : e_i = u_i v_i, i \in \{1, \dots, r\}\}$ be a set of disjoint edges not in H where $r = n$ if n is even, or $r = n + 1$ if n is odd. We generate a zig-zag graph G as follows.

- We partition the set $V(H)$ into the sets $S_1, \dots, S_t$ such that at most one set, say S_t, has cardinality one and other ones have even cardinality.
- We partition the set E into the sets $E_1, \dots, E_t$ such that, $|E_i| = |S_i|$ for every $i \in \{1, \dots, t-1\}$ and, if n is odd, then $|E_t| = 2$ or, if n is even, then $|E_t| = |S_t|$.
- Let $S_i = \{x_{i1}, x_{i2}, \dots, x_{i|S_i|}\}$ and let $E_i = \{e_{i_1}, e_{i_2}, \dots, e_{i_{|S_i|}}\}$.
- Now, for every $i \in \{1, \dots, t-1\}$, we add edges between S_i and the vertices of E_i in such a way that we form the cycle $x_{i1} u_{i_1} v_{i_1} x_{i2} u_{i_2} v_{i_2} \dots x_{i|S_i|} u_{i_{|S_i|}} v_{i_{|S_i|}} x_{i1}$.
- Now, if n is even, then we proceed with S_t as above. On the contrary, if n is odd, then $S_t = \{x_{t1}\}$ and $E_t = \{e_{t_1}, e_{t_2}\}$. Hence, we form the path $u_{t_1} v_{t_1} x_{t1} u_{t_2} v_{t_2}$.

Notice that a graph G obtained by such a construction is a zig-zag graph with zig-zag set E. It is also easy to see that there are more zig-zag graphs (with less edges in zig-zag set). Take for instance, the complete graph K_{2n} and choose for a set E' to be any perfect matching of K_{2n}. In addition subdivide by one vertex every remaining edge of K_{2n}. It is straightforward to observer that the obtained graph, K_{2n}^+ is a zig-zag graph with zig-zag set E'.

We end the discussion on zig-zag graphs by a recursive description of the family $\mathcal{T}$ of all zig-zag trees. Since trees have no cycles, it follows that any tree T is a zig-zag graph if and only it satisfies properties (i), (ii), and (iii). Recall that a path P_n is a zig-zag graph whenever $n = 2 + 3k$ for some positive integer k. Let T_1 and T_2 be zig-zag trees with corresponding zig-zag sets E_1' and E_2', respectively. The following two rules can be used to obtain a tree T by a *zig-zag tree construction*.

(a) choose $e_1 = u_1 v_1 \in E_1'$ and $e_2 = u_2 v_2 \in E_2'$ and then identify the vertices u_1 with u_2 (hence vertex u) and v_1 with v_2 (hence vertex v) to maintainin the edge uv in T;

(b) choose any $x \in V(T_1) - V(E_1')$ and any $y \in V(T_2) - V(E_2')$ and connect x with y by an edge to obtain T.

We say that a tree $T \in \mathcal{T}$ if it is obtained from zig-zag trees T_1 and T_2 by one of the above rules (a zig-zag tree construction) or T is P_5.

Theorem 3.1 *A tree T is a zig-zag graph if and only if $T \in \mathcal{T}$.*

Proof. If T is isomorphic to P_5, then it is clearly a zig-zag tree. So, suppose that $T \in \mathcal{T}$ obtained from zig-zag trees T_1 and T_2 with corresponding zig-zag sets E_1' and E_2', respectively. Let $E' = E_1' \cup E_2'$. Notice that properties (i),(ii), and (iii) of the definition of zig-zag set are clearly satisfied for E' in T after applying rule (a) or (b) to T_1 and T_2, since E_1' and E_2' are zig-zag sets. Since trees have no cycles, we do not need to check property (iv). Therefore, T is a zig-zag tree.

Conversely, let T be a zig-zag tree with corresponding zig-zag set E'. If T is isomorphic to P_5, then $T \in \mathcal{T}$. If T is isomorphic to P_{2+3k} for $k \geq 2$, then it can be obtained from P_5 and $P_{2+3(k-1)}$ by using rule (a) of the zig-zag tree construction. Therefore, $T \in \mathcal{T}$.

Assume now that there exists a vertex v of degree at least three in T. If v is not an end vertex of any edge of E', then exactly two vertices $x, y \in N(v)$ are end vertices of an edge in E' by property (iii). Choose any neighbor $u \in N(v) - \{x, y\}$. By property (iii) there exists two edges $e_u, e_u' \in E'$ that are at distance one to u. Now, delete edge uv from T. Clearly T splits into two trees T_u and T_v, where T_u contains u and T_v contains v. Consider the sets of edges $E_u = E' \cap E(T_u)$ and $E_v = E' \cap E(T_v)$. Notice that properties (i) and (ii) hold for E_u and E_v, since E' is a zig-zag set. Property (iii) must be checked only for u. But both e_u and e_u' must be in E_u, since v is not an end vertex of any edge of E'. So, property (iii) is clear for E_u and E_v in T_u and T_v, respectively. Thus, T_u and T_v are zig-zag graphs with zig-zag sets E_u and E_v, respectively. Moreover, T is obtained from T_u and T_v by using rule (b). Therefore, $T \in \mathcal{T}$.

Now let v be an end vertex of the edge $e \in E'$ and let $e = uv$. We shall "divide" the tree T using a kind of "reverse-like process" of rule (a) into two trees T_u and T_v, where T_u contains u and T_v contains v. Let $E_u = E' \cap E(T_u)$ and $E_v = E' \cap E(T_v)$. We notice that properties (i), (ii), and (iii) hold for E_u and E_v by the same reason as above. Thus, again T_u and T_v are zig-zag graphs with zig-zag sets E_u and E_v, respectively, and T is obtained from T_u and T_v by using rule (a). Therefore, $T \in \mathcal{T}$ and the proof is complete. $\square$

The Figure 2 represents an example of a tree belonging to the family $\mathcal{T}$.

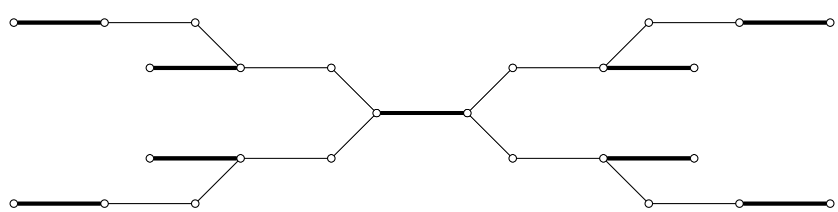

Fig. 2: A tree of the family $\mathcal{T}$ obtained only by rule (a).

4 Efficient open domination graphs $G \Box K_2$

The *Cartesian product* $G \Box H$ of graphs G and H is a graph with $V(G \Box H) = V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \Box H$ whenever $(gg' \in E(G)$ and $h = h')$ or $(g = g'$ and $hh' \in E(H))$. Hence $E(G \boxtimes H) = E(G \Box H) \cup E(G \times H)$. The Cartesian product is commutative and associative (see [10]). Layers and projections are defined identically as for the lexicographic product. The subgraph of $G \Box H$ induced by G^h or ${}^gH$ is isomorphic to G or H, respectively.

As usual in domination related problems, it seems that the Cartesian product of graphs is the most problematic of all four standard products for the efficient open domination problem. For instance, until now only the Cartesian product of two paths (grid graphs) [4, 14] and some particular case of the Cartesian product of two cycles (torus graphs) [5] have been studied. In this section we characterize all efficient open domination graphs among $G\Box K_2$.

Cartesian product graphs $G\Box K_2$ are often called *prisms* and can be described as two copies of G with a matching between the corresponding vertices of each copy of G. We denote $V(K_2) = \{1, 2\}$ and for a vertex $v \in V(G)$ we denote by v^i, $i \in \{1, 2\}$, the copy of v in the G^i-layer.

With the knowledge about zig-zag graphs we show now the relationship between them and efficient open domination prisms.

Theorem 4.1 *If G is a zig-zag graph, then $G\Box K_2$ is an efficient open domination graph.*

Proof. Let G be a zig-zag graph and $E' = \{e_1, \ldots, e_k\}$, $e_i = u_iv_i$, its zig-zag set. In addition we may assume that E' is ordered such that for every e_i, $i > 1$, there exists e_j, $j < i$, with $d_G(e_i, e_j) = 2$. We call such an edge e_j an *ancestor* of e_i. We define a subset D of $V(G\Box K_2)$ inductively as follows. Let $u_1^1, v_1^1 \in D$. For an edge e_i, $i > 1$, with ancestor e_j we have either ($u_i^1, v_i^1 \in D$ if $u_j^2, v_j^2 \in D$) or ($u_i^2, v_i^2 \in D$ if $u_j^1, v_j^1 \in D$). Since an edge e_i, $i > 1$, can have many ancestors, we need to show that D is well defined. Suppose not, and let e_i be the first edge which has two ancestor e_j and e_ℓ with $u_j^1, v_j^1 \in D$ and $u_\ell^2, v_\ell^2 \in D$ (without loss of generality). By the choice of order of E' there must exist a sequence $e_j, e_{j_1}, \ldots, e_{j_t}, e_1$ with $d(e_j, e_{j_1}) = d(e_{j_t}, e_1) = d(e_{j_p}, e_{j_{p+1}}) = 2$ for $p \in \{1, \ldots, t-1\}$. Notice that every edge of this sequence has index smaller than i. Similarly, there exists a sequence $e_\ell, e_{\ell_1}, \ldots, e_{\ell_s}, e_1$ with $d(e_\ell, e_{\ell_1}) = d(e_{\ell_s}, e_1) = d(e_{\ell_p}, e_{\ell_{p+1}}) = 2$ for $p \in \{1, \ldots, s-1\}$, where index of every edge is smaller than i. Let $e_k = e_{j_k} = e_{\ell_p}$ be the first common edge of these two sequences. For the sequence $e_j, e_{j_1}, \ldots, e_{j_k}, e_{\ell_{p-1}}, e_{\ell_{p-2}}, \ldots, e_\ell, e_i$ holds that any two consecutive edges are at distance two, and also the first and the last edge are at this distance. By the construction of D and since all indices are smaller than i (with the exception of e_i), we have that consecutive edges lie zig zag in $G\Box K_2$: one edge is in G^1 and the other in G^2. Hence, from e_j to e_ℓ we have an even number of edges. Together with e_i we have an odd number of edges, which contradicts to the property (iv) of the definition of zig-zag sets. Hence, D is well defined and for every pair of edges $e_i, e_j \in E'$ with $d_G(e_i, e_j) = 2$ it follows u_i^p, v_i^p, $u_j^q, v_j^q \in D$, where $\{p, q\} = \{1, 2\}$.

Next we show that D is an efficient open dominating set of $G\Box K_2$. Notice that if $u_i^j, v_i^j \in D$, $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$, then they are dominated by v_i^j and u_i^j, respectively. On the contrary, if $u_i^j, v_i^j \notin D$, then u_i^ℓ and v_i^ℓ dominate u_i^j and v_i^j, respectively, where $\ell \in \{1, 2\} - \{j\}$. Let now $w \in V(G) - \{u_i, v_i : i \in \{1, \ldots, k\}\}$. By the property (iii) of the definition of zig-zag sets, there exist unique j and ℓ with $d(w, e_j) = d(w, e_\ell) = 1$. Hence, either e_j is an ancestor of e_ℓ or vice versa. In each case we may assume without loss of generality that $u_j^1, v_j^1 \in D$ and $u_\ell^2, v_\ell^2 \in D$. Now w^1 is dominated by u_j^1 or v_j^1 and w^2 is dominated by u_ℓ^2 or v_ℓ^2. Hence $\bigcup_{v\in D} N_{G\Box K_2}(v) = V(G\Box K_2)$. By the property (i) of the definition of zig-zag sets we have that $N_{G\Box K_2}(u_i^j) \cap N_{G\Box K_2}(v_i^j) = \emptyset$ for $u_i^j, v_i^j \in D$ and $j \in \{1, 2\}$. If $d_G(e_i, e_j) = 2$, then $N_{G\Box K_2}(x_i^p) \cap N_{G\Box K_2}(y_j^q) = \emptyset$ for $x, y \in \{u, v\}$ since $\{p, q\} = \{1, 2\}$ by the construction of D. Finally, if $d_G(e_i, e_j) > 2$, then clearly $N_{G\Box K_2}(x_i^p) \cap N_{G\Box K_2}(y_j^q) = \emptyset$ for $x, y \in \{u, v\}$ and $p, q \in \{1, 2\}$. Therefore, the neighborhoods of D form a partition of $G\Box K_2$ and, as a consequence, $G\Box K_2$ is an efficient open domination graph. □

Notice that in above proof all edges induced by D project to edges of G. However, if we wish that all edges induced by an efficient open domination set D of $G\Box K_2$ projet into K_2, then it is not hard to see which property is needed for G. Namely, if G is 1-perfect graph with 1-perfect code P, then D contains exactly P in both copies of G. In particular notice that a path graph P_k is a 1-perfect graph for every integer $k \geq 1$, and hence $P_k\Box K_2$ is an efficient open domination graph. The following result is clear.

Remark 4.2 *If G is a 1-perfect graph, then $G\Box K_2$ is an efficient open domination graph.*

To describe all efficient open domination graphs among Cartesian products of graphs with K_2 we need a combination of both: 1-perfect graphs and zig-zag graphs. Let H_1 be 1-perfect graph with 1-perfect code P and let H_2 be a zig-zag graph with a zig-zag set $E' = \{e_1, \ldots, e_k\}$, $e_i = u_iv_i$. (Notice that H_1 and H_2 do not need to be connected.) A graph G is called a *1-perfect zig-zag graph* if $V(G) = V(H_1) \cup V(H_2)$, $E(G) = E(H_1) \cup E(H_2) \cup E_3$ where E_3 is an arbitrary subset of the set $\{xw : x \in V(H_1) - P, w \in V(H_2) - \{u_i, v_i : i \in \{1, \ldots, k\}\}\}$. In other words, $E(G)$ contains all edges of H_1, all edges of H_2 and for every $x \in V(H_1) - P$ and every $w \in V(H_2) - \{u_i, v_i : i \in \{1, \ldots, k\}\}$ we may insert an edge xw to $E(G)$ or not. In particular, G is isomorphic to the disjoint union of H_1 and H_2 if no edges of the type xw are added.

Notice that a 1-perfect zig-zag graph could be formed only by a 1-perfect graph or only by a zig-zag graph. This means that, for instance, all paths (which are always 1-perfect graphs, but not always zig-zag graphs) are included into the family of 1-perfect zig-zag graphs. While it is easy to see that there exists 1-perfect graphs which are not zig-zag graphs, one needs a bit more effort to find a graph that is a zig-zag, but not a 1-perfect graph. An example for this is K_{2n}^+. Recall that we obtain K_{2n}^+ by subdividing with one vertex every edge which does not belong to a fixed perfect matching of K_{2n}.

Theorem 4.3 *Let G be a graph. The Cartesian product $G\Box K_2$ is an efficient open domination graph if and only if G is a 1-perfect zig-zag graph.*

Proof. First, if G is formed either by a zig-zag graph or by a 1-perfect graph, then by Theorem 4.1 or by Remark 4.2, respectively, we obtain that $G\Box K_2$ is an efficient open domination graph.

Now, let G be a 1-perfect zig-zag graph built from 1-perfect graph H_1 with 1-perfect code P and zig-zag graph H_2 with zig-zag set $E' = \{e_1, \ldots, e_k\}$, $e_i = u_iv_i$. Let P^i, $i \in \{1, 2\}$, be a copy of P in the G^i-layer of $G\Box K_2$ and let D be a set of vertices obtained from E' as in the proof of Theorem 4.1. The set $D' = P^1 \cup P^2 \cup D$ dominates $G\Box K_2$ since $P^1 \cup P^2$ dominates $H_1\Box K_2$ and D dominates $H_2\Box K_2$. In addition, $N_{G\Box K_2}(x) \cap N_{G\Box K_2}(y) = \emptyset$ for any $x, y \in D'$. Finally, since all additional edges in G between H_1 and H_2 are connecting vertices which neither belong to P nor to $\{u_i, v_i : i \in \{1, \ldots, k\}\}$, we obtain that $G\Box K_2$ is an efficient open domination graph.

For the other direction, let $G\Box K_2$ be an efficient open domination graph with an efficient open dominating set D. Clearly D contains adjacent pairs of vertices and we split them into two subsets as follows. In D_1 we put pairs of adjacent vertices from D for which its edge projects to the edge of K_2 and in D_2 are all the remaining vertices (those whose edge projects to an edge of G). First we define a graph H_1 as follows. Let $V(H_1\Box K_2) = \bigcup_{v\in D_1} N_{G\Box K_2}(v)$ and $H_1\Box K_2$ is an induced subgraph of $G\Box K_2$ on $V(H_1\Box K_2)$. It is clear that H_1 which is induced by vertices of layer H_1^1 (and also by vertices of layer H_1^2) is a 1-perfect graph with 1-perfect code $p_G(D_1)$.

Next, let $H_2\Box K_2 = G\Box K_2 - V(H_1\Box K_2)$. The projection of all edges induced by D_2 projects to edges of the first factor H_2. We will show that these edges, namely $p_{H_2}(D_2)$, form a zig-zag set of H_2. Let $|D_2| = 2k$ and denote adjacent vertices of D_2 by u_i^j and v_i^j for $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$. Since D_2

is a subset of an efficient open dominating set D, it follows $N_{H_2\Box K_2}(u_i^j) \cap N_{H_2\Box K_2}(v_i^j) = \emptyset$ for every $i \in \{1,\ldots,k\}$ and $j \in \{1,2\}$, consequently $N_{H_2}(u_i) \cap N_{H_2}(v_i) = \emptyset$ and the property (i) holds.

For the property (ii), let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be two different edges induced by $p_{H_2}(D_2)$ and let $e_1', e_2' \in E(H_2\Box K_2)$ be edges which project to e_1 and e_2, respectively. If $d_{H_2}(e_1,e_2) = 0$, then two end vertices coincide and the other two differ. Without loss of generality, let $u_1 = u_2$ and $v_1 \neq v_2$. If both e_1' and e_2' lie in the same H_2-layer, say H_2^1, then $u_1^1 \in N_{H_2\Box K_2}(v_1^1) \cap N_{H_2\Box K_2}(v_2^1)$ which is not possible, since D is an efficient open dominating set. If e_1' and e_2' lie in different H_2-layers, say e_1' in H_2^1 and e_2' in H_2^2, then $u_1^1 \in N_{H_2\Box K_2}(v_1^1) \cap N_{H_2\Box K_2}(u_2^1)$, which again yields to the same contradiction. If $d_{H_2}(e_1,e_2) = 1$, then there are two end vertices, say u_1 and u_2, adjacent in H_2. If both e_1' and e_2' lie in the same H_2-layer, say H_2^1, then $u_1^1 \in N_{H_2\Box K_2}(v_1^1) \cap N_{H_2\Box K_2}(u_2^1)$, which is not possible by the same reason. If e_1' and e_2' lie in different H_2-layers, say e_1' in H_2^1 and e_2' in H_2^2, then $u_1^2 \in N_{H_2\Box K_2}(u_1^1) \cap N_{H_2\Box K_2}(u_2^2)$, which is not possible. Hence, $d_{H_2}(e_1,e_2) \geq 2$ and the property (ii) holds for $p_{H_2}(D_2)$.

Let $w \in V(H_2) - p_{H_2}(D_2)$. Hence both w^1 and w^2 are not in D_2. Suppose that they are dominated by u^1 and u^2, respectively. Both u^1 and u^2 have a neighbor in D_2 in the same layer: v^1 and v^2, respectively. Let $e = u^1v^1$ and $e' = u^2v^2$. If $p_{H_2}(e) = p_{H_2}(e')$, then $u^1 \in N_{H_2\Box K_2}(v^1) \cap N_{H_2\Box K_2}(u^2)$, which is not possible. Moreover, by the property (ii) we have $d_{H_2}(p_{H_2}(e), p_{H_2}(e')) = 2$. Clearly $d_{H_2}(w, p_{H_2}(e)) = d_{H_2}(w, p_{H_2}(e')) = 1$. If there exists a third edge $p_{H_2}(e'')$ induced by $p_{H_2}(D_2)$ with $d_{H_2}(w, p_{H_2}(e'')) = 1$, then either w^1 or w^2 is dominated by two vertices of $D_2 \subseteq D$, which is not possible, and the property (iii) is satisfied by $p_{H_2}(D_2)$.

If the property (iv) does not hold, then there exists a sequence of distinct edges $e_{i_1}, e_{i_2}, \ldots, e_{i_j}$, $j > 2$, with $d(e_{i_\ell}, e_{i_{\ell+1(\mathrm{mod}\, j)}}) = 2$ for $\ell \in \{1,\ldots,j\}$ and j is an odd number. Since $d(e_{i_\ell}, e_{i_{\ell+1(\mathrm{mod}\, j)}}) = 2$ for $\ell \in \{1,\ldots,j\}$, there exists a vertex w_ℓ for which $d(w_\ell, e_{i_\ell}) = 1 = d(w_\ell, e_{i_{\ell+1(\mathrm{mod}\, j)}})$. By e_{i_ℓ}' we denote the edge which projects to e_{i_j} for every $\ell \in \{1,\ldots,j\}$. Two consecutive edges e_{i_p}' and $e_{i_{p+1(\mathrm{mod} j)}}'$ must be in the same H_2-layer, say H_2^1, since j is odd. Without loss of generality, we may assume that w_p^1 is a common neighbor of $u_{i_p}^1$ and $v_{i_{p+1(\mathrm{mod} j)}}^1$. Thus $w_p^1 \in N_{H_2\Box K_2}(u_{i_p}^1) \cap N_{H_2\Box K_2}(v_{i_{p+1(\mathrm{mod} j)}}^1)$, a final contradiction. Hence, the property (iv) also holds, $p_{H_2}(D_2)$ is a zig-zag set of H_2 and H_2 is a zig-zag graph. Therefore, G is 1-perfect zig-zag graph, which ends the proof. □

To generalize this results from K_2 to K_p, $p > 2$, it is easy to see that no edge induced by an efficient open dominating set D of $G\Box K_p$ can project to K_p. Hence 1-perfect graphs have no analogue for $p > 2$. However, it seems that zig-zag graphs could be generalized to higher orders of K_p, where property (iv) represents the greatest problem.

5 Grid-like graphs

The study of efficient open domination graphs among Cartesian product of path and/or cycles has attracted some researchers in the last few years. For instance, [4, 14] are dedicated to investigate the efficient open domination graphs among the grid graphs $P_r\Box P_t$. Between both works all the efficient open domination grid graphs were characterized. Notice that, in [4], efficient open dominating sets are referred as exact transversals, while in [14], are called total perfect codes. The union of these two works produced the following characterization.

Theorem 5.1 [4, 14] *A grid graph $P_r\Box P_t$, $t \geq r \geq 3$, is an efficient open domination graph if and only if r is an even number and $t \equiv x \pmod{r+1}$ for $x \in \{1, r-2, r\}$.*

Some partial results on the efficient open domination graphs among torus graphs $C_r \Box C_t$ were presented in [5]. There was given the following characterization of all the torus graphs $C_r \Box C_t$ having an efficient open dominating set where all its edges are parallel (a parallel total perfect code).

Theorem 5.2 [5] *A parallel total perfect code in a Cartesian product graph $C_r \Box C_t$ of two cycles C_r and C_t exists if and only if r and t are multiples of four.*

From now on we give some partial results on the torus $C_r \Box C_t$ and the cylinder $P_r \Box C_t$. To do so, in this subsection we use the following notation: $U = \{u_0, \ldots, u_{r-1}\}$ and $V = \{v_0, \ldots, v_{t-1}\}$ are the vertex sets of G and H, where G and H are isomorphic to a path or a cycle of order r and t, respectively. Operations with the subindexes of vertices of U and V are done modulo r and t, respectively. With respect to the previous section we may assume that $r, t \geq 3$. The adjacency in G and H is defined as $u_0 \sim u_1 \sim \ldots \sim u_{r-1} (\sim u_0)$ and $v_0 \sim v_1 \sim \ldots \sim v_{t-1} (\sim v_0)$, respectively.

5.1 The torus $C_r \Box C_t$

By Theorem 5.2 we know that $C_{4r} \Box C_{4t}$ are efficient open domination graphs. Motivated by this we state the following conjecture.

Conjecture 5.3 *The torus $C_r \Box C_t$, $r, t \geq 3$, is an efficient open domination graph if and only if $r, t \equiv 0\ (mod\ 4)$.*

Till the end of this subsection we settle this conjecture for $r \in \{3, 4, 5, 6, 7\}$. We start with $r = 4$ and continue with all other values.

Proposition 5.4 *Torus $C_4 \Box C_t$, $t \geq 4$, is an efficient open domination graph if and only if $t \equiv 0 \pmod 4$.*

Proof. If $t \equiv 0 \pmod 4$, then it follows by Theorem 5.2 that $C_4 \Box C_t$ is an efficient open domination graph.

Now suppose that the torus graph $C_4 \Box C_t$, $t \geq 4$, is an efficient open domination graph and let F be an efficient open dominating set in $C_4 \Box C_t$. By Observation 1.2 we have that t is even. So either $t \equiv 0 \pmod 4$ or $t \equiv 2 \pmod 4$. Suppose $t \equiv 2 \pmod 4$. According to the symmetry of $C_4 \Box C_t$, we can suppose, without loss of generality, that $(u_0, v_0) \in F$ and $(u_0, v_{t-1}) \notin F$.

If $(u_0, v_1) \in F$, then we have that $(u_2, v_2), (u_2, v_3) \in F$ and consequently, the vertices (u_0, v_4), $(u_0, v_5), (u_2, v_6), (u_2, v_7), \ldots, (u_0, v_{t-2}), (u_0, v_{t-1})$ also belong to F, which is a contradiction since (u_0, v_{t-1}) is dominated by (u_0, v_0) and (u_0, v_{t-2}). Analogously, we obtain a contradiction if $(u_1, v_0) \in F$. Therefore $t \equiv 0 \pmod 4$. $\Box$

Theorem 5.5 *Torus $C_r \Box C_t$, $r \in \{3, 5, 6, 7\}$ and $t \geq r$, is not an efficient open domination graph.*

Proof. If $r = 3$, then it is straightforward to observe that $C_3 \Box C_t$ is not an efficient open domination graph for every $t \geq 3$. Now suppose $r = 5$ and let F_5 be an efficient open dominating set in $C_5 \Box C_t$. According to the symmetry of $C_5 \Box C_t$, we consider without loss of generality that $(u_0, v_0) \in F_5$ and $(u_0, v_{t-1}) \notin F_5$. If $(u_1, v_0) \in F_5$, then we have that $(u_3, v_1), (u_3, v_2) \in F_5$ and consequently, $(u_0, v_3), (u_1, v_3) \in F_5$. Thus, we have that at least one of the vertices of the set $\{(u_2, v_4), (u_3, v_4), (u_4, v_4)\}$ cannot be efficiently open dominated by F_5, a contradiction. On the other hand, if $(u_0, v_1) \in F_5$, then $(u_2, v_2), (u_3, v_2) \in F_5$ and at least one of the vertices of the set $\{(u_0, v_3), (u_1, v_3), (u_4, v_3)\}$ cannot be efficiently open dominated by F_5, a contradiction again.

Suppose $r = 6$ and let F_6 be an efficient open dominating set in $C_6 \Box C_t$. We proceed similarly to the above case. We may assume that $(u_0, v_0) \in F_6$ and $(u_0, v_{t-1}) \notin F_6$. If $(u_0, v_1) \in F_6$, then we have the following cases.
Case 1: $(u_3, v_0), (u_3, v_1) \in F_6$. As a consequence, we have that two vertices of the set $\{(u_1, v_2), (u_2, v_2), (u_4, v_2), (u_5, v_2)\}$ cannot be efficiently open dominated by F_6, a contradiction.
Case 2: $(u_3, v_1), (u_3, v_2) \in F_6$. Consequently, either (u_2, v_0) or (u_4, v_0) cannot be efficiently open dominated by F_6, a contradiction.
Case 3: Either $(u_2, v_2), (u_2, v_3) \in F_6$ or $(u_4, v_2), (u_4, v_3) \in F_6$. Consequently, either (u_4, v_1) or (u_2, v_1), respectively, cannot be efficiently open dominated by F_6, a contradiction.
Case 4: $(u_3, v_2), (u_3, v_3) \in F_6$. Consequently, (u_2, v_1) and (u_4, v_1) cannot be efficiently open dominated by F_6, a contradiction.
Case 5: Either $(u_2, v_2), (u_3, v_2) \in F_6$ or $(u_3, v_2), (u_4, v_2) \in F_6$. Consequently, either (u_4, v_1) or (u_2, v_1), respectively, cannot be efficiently open dominated by F_6, a contradiction.

On the other hand, if $(u_1, v_0) \in F_6$, then we have the following cases.
Case 6: $(u_3, v_1), (u_4, v_1) \in F_6$. As a consequence, we have that two vertices of the set $\{(u_0, v_2), (u_1, v_2), (u_2, v_2)(u_5, v_2)\}$ cannot be efficiently open dominated by F_6, a contradiction.
Case 7: Either $(u_3, v_1), (u_3, v_2) \in F_6$ or $(u_4, v_1), (u_4, v_2) \in F_6$. Consequently, either (u_5, v_1) or (u_2, v_1), respectively, cannot be efficiently open dominated by F_6, a contradiction.

Now, if $r = 7$, then by using a similar cases analysis like in the constructive procedure of the set F_6 for $r = 6$, we obtain contradictions which lead to that $C_7 \Box C_t$ is not an efficient open domination graph. We leave the details to the reader. $\Box$

5.2 The cylinder $P_r \Box C_t$

As we will see, the situation for cylinders is a little bit more complicated. The first indicator for this is that $P_2 \Box C_t \cong C_t \Box K_2$ and, by Theorem 4.3, we know that this is an efficient open domination graph whenever $t \equiv 0\ (mod\ 3)$. Also other cases are more complicated. We start with cycles on $4t$ vertices.

Proposition 5.6 *Cylinder $P_{2r+1} \Box C_{4t}$ is an efficient open domination graph for every $r, t \geq 1$.*

Proof. The result follows immediately from the fact that the set F of vertices of $P_{2r+1} \Box C_{4t}$ given in the following way is an efficient open dominating set for $P_{2r+1} \Box C_{4t}$.

If $2r + 1 \equiv 1 \pmod 4$, then the set F is the union of sets $\{u_0, u_4, \ldots, u_{2r}\} \times \{v_0, v_1, v_4, v_5, \ldots, v_{4t-4}, v_{4t-3}\}$ and $\{u_2, u_6, \ldots, u_{2r-2}\} \times \{v_2, v_3, v_6, v_7, \ldots, v_{4t-2}, v_{4t-1}\}$.

If $2r + 1 \equiv 3 \pmod 4$, then the set F is the union of sets $\{u_0, u_4, \ldots, u_{2r-2}\} \times \{v_0, v_1, v_4, v_5, \ldots, v_{4t-4}, v_{4t-3}\}$ and $\{u_2, u_6, \ldots, u_{2r}\} \times \{v_2, v_3, v_6, v_7, \ldots, v_{4t-2}, v_{4t-1}\}$. $\Box$

Next we see characterizations for $P_r \Box C_t$ to be efficient open domination graphs for $t \in \{3, 4, 5, 6, 7\}$.

Theorem 5.7

(i) *$P_r \Box C_3$ is an efficient open domination graph if and only if $r = 2$.*

(ii) *$P_r \Box C_4$ is an efficient open domination graph if and only if r is an odd integer.*

(iii) *$P_r \Box C_5$ is an efficient open domination graph if and only if $r = 4$.*

(iv) $P_r \Box C_6$ *is an efficient open domination graph if and only if* $r = 2$.

(v) $P_r \Box C_7$ *is an efficient open domination graph if and only if* $r = 6$.

Proof. It is easy to observe that $P_2 \Box C_3$, $P_4 \Box C_5$, $P_2 \Box C_6$ and $P_6 \Box C_7$ are efficient open domination graphs, while $P_{2t+1} \Box C_4$ is efficient open domination graph by Proposition 5.6. Also, it is straightforward to check that if $P_r \Box C_3$ is an efficient open domination graph, then $r = 2$. So (i) is proved.

For (ii) suppose that cylinder $P_r \Box C_4$, $r \geq 4$, is an efficient open domination graph and let F be an efficient open dominating set in $P_r \Box C_4$. Clearly, at least one vertex of ${}^{u_0}C_4$ must be in F and according to the symmetry of $P_r \Box C_4$, we can suppose, without loss of generality, that $(u_0, v_0) \in F$. If $(u_1, v_0) \in F$, then we have that the vertex (u_0, v_2) cannot be efficiently open dominated by F, a contradiction. Now, if $(u_0, v_1) \in F$, then $(u_2, v_2), (u_2, v_3) \in F$, and consequently only vertices of the type $(u_{2k}, v_2), (u_{2k}, v_3), (u_{4\ell}, v_0), (u_{4\ell}, v_1)$ belong to F, where k, ℓ are integers and k is odd. Thus, if r is even, then there exist two vertices $(u_{r-1}, v_i), (u_{r-1}, v_j)$, $i \neq j$, and $i, j \in \{0, 1, 2, 3\}$, which cannot be efficiently open dominated by F, a contradiction. Therefore, r is odd and (ii) is clear.

Now, suppose $P_r \Box C_5$ is an efficient open domination graph and let F_5 be an efficient open dominating set in $P_r \Box C_5$. Clearly, at least one vertex of ${}^{u_0}C_5$ must be in F_5 and according to the symmetry of $P_r \Box C_5$, we consider without loss of generality that $(u_0, v_0) \in F_5$ and $(u_0, v_4) \notin F_5$. If $(u_1, v_0) \in F_5$, then we vertices (u_0, v_2) and (u_0, v_3) cannot be efficiently open dominated by F_5, a contradiction. On the other hand, let $(u_0, v_1) \in F_5$. If $r = 2$, then (u_0, v_3) is not efficiently dominated. If $r > 2$, then $(u_1, v_3), (u_2, v_3) \in F_5$. If $r = 3$, then (u_2, v_0) and (u_2, v_1) are not efficiently open dominated by F_5. Hence $r > 3$ and also $(u_3, v_0), (u_3, v_1) \in F_5$. Now, if $r > 4$, then one vertex of the set $\{(u_4, v_2), (u_4, v_3), (u_4, v_4)\}$ cannot be efficiently open dominated by F_5, a contradiction again. Thus, the only possible choice is $r = 4$ and (ii) is proved.

Assume $P_r \Box C_6$ is an efficient open domination graph and let F_6 be an efficient open dominating set in $P_r \Box C_6$. We proceed similarly to the above case. Consider $(u_0, v_0) \in F_6$ and $(u_0, v_5) \notin F_6$. If $(u_1, v_0) \in F_6$, then $(u_0, v_3), (u_1, v_3) \in F_6$. If $r > 2$, then two vertices of the set $\{(u_2, v_1), (u_2, v_2),$ $(u_2, v_4), (u_2, v_5)\}$ cannot be efficiently open dominated by F_6, a contradiction. On the other hand, if $(u_0, v_1) \in F_6$, then $(u_1, v_3), (u_1, v_4) \in F_6$. Analogously, if $r > 2$, then two vertices of the set $\{(u_2, v_0), (u_2, v_1), (u_2, v_2),$
$(u_2, v_5)\}$ cannot be efficiently open dominated by F_6, a contradiction. So, the only possible choice is $r = 2$ and (iii) is proved.

Finally, suppose $P_r \Box C_7$ is an efficient open domination graph and let F_7 be an efficient open dominating set in $P_r \Box C_7$. As above we consider $(u_0, v_0) \in F_7$ and $(u_0, v_6) \notin F_7$. If $(u_1, v_0) \in F_7$, then (u_0, v_2) and (u_0, v_5) can be efficiently open dominated only by (u_0, v_3) and (u_0, v_4), respectively. Also $r > 2$, since (u_1, v_2) and (u_1, v_5) are not efficiently open dominated yet. Now (u_1, v_2) and (u_1, v_5) can be efficiently open dominated only by (u_2, v_2) and (u_2, v_5), respectively, and these two with (u_3, v_2) and (u_3, v_5), respectively. Hence $r > 3$ and $r \neq 4$ since (u_3, v_0) is not efficiently dominated. The vertex (u_3, v_0) yields that $(u_4, v_0), (u_5, v_0) \in F_7$ and consequently $r \geq 6$. To continue, $(u_5, v_3), (u_5, v_4) \in F_7$ to efficiently open dominate (u_4, v_3) and (u_4, v_4). If $r = 6$, then we are done. If $r > 6$, then at least one vertex out of $\{(u_6, v_1), (u_6, v_2), (u_6, v_5), (u_6, v_6)\}$ cannot be efficiently open dominated by F_7, a contradiction. On the other hand, if $(u_0, v_1) \in F_7$, then $(u_0, v_4), (u_1, v_4) \in F_7$. If $r = 2$, then (u_1, v_2) and (u_1, v_6) are not dominated by F_7, a contradiction. Hence $r > 2$ and to efficiently open dominate these two vertices, (u_2, v_2) and (u_2, v_6), respectively, must be in F_7. Furthermore, to efficiently open dominate (u_2, v_2) and (u_2, v_6), (u_3, v_2) and (u_3, v_6), respectively, must be in F_7. Hence $r > 3$ and if

$r = 4$, then (u_3, v_4) is not dominated by F_7. Thus, $r > 4$ and to efficiently open dominate (u_3, v_4) we need $(u_4, v_4), (u_5, v_4) \in F_7$. Moreover, the vertices $(u_5, v_0), (u_5, v_1) \in F_7$ to efficiently open dominate (u_4, v_0) and (u_4, v_1). If $r = 6$, then we are done and if $r > 6$, then at least one vertex of the set $\{(u_6, v_2), (u_6, v_3), (u_6, v_5), (u_6, v_6)\}$ cannot be efficiently open dominated by F_7, a contradiction. Therefore, the only possible choice is $r = 6$ and (iv) is proved. $\square$

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