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On the number of spanning trees of $K_n^m \pm G$ graphs

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The K_n-complement of a graph G, denoted by $K_n - G$, is defined as the graph obtained from the complete graph K_n by removing a set of edges that span G; if G has n vertices, then $K_n - G$ coincides with the complement $\overline{G}$ of the graph G. In this paper we extend the previous notion and derive determinant based formulas for the number of spanning trees of graphs of the form $K_n^m \pm G$, where K_n^m is the complete multigraph on n vertices with exactly m edges joining every pair of vertices and G is a multigraph spanned by a set of edges of K_n^m; the graph $K_n^m + G$ (resp. $K_n^m - G$) is obtained from K_n^m by adding (resp. removing) the edges of G. Moreover, we derive determinant based formulas for graphs that result from K_n^m by adding and removing edges of multigraphs spanned by sets of edges of the graph K_n^m. We also prove closed formulas for the number of spanning tree of graphs of the form $K_n^m \pm G$, where G is (i) a complete multipartite graph, and (ii) a multi-star graph. Our results generalize previous results and extend the family of graphs admitting formulas for the number of their spanning trees.

Keywords: Kirchhoff matrix tree theorem, complement spanning tree matrix, spanning trees, K_n-complements, multigraphs.

1 Introduction

The number of spanning trees of a graph G, denoted by $\tau(G)$, is an important, well-studied quantity in graph theory, and appears in a number of applications. Most notable application fields are network reliability [9, 17, 22], enumerating certain chemical isomers [5], and counting the number of Eulerian circuits in a graph [15].

Thus, both for theoretical and for practical purposes, we are interested in deriving formulas for the number of spanning trees of a graph G, and also of the K_n-complement of G; the *K_n-complement* of a graph G, denoted by $K_n - G$, is defined as the graph obtained from the complete graph K_n by removing a set of edges (of the graph K_n) that span G; if G has n vertices, then $K_n - G$ coincides with the complement $\overline{G}$ of the graph G. Many cases have been examined depending on the choice of G. For example, there exist closed formulas for the cases where G is is a pairwise disjoint set of edges [24], a chain of edges [16], a cycle [11], a star [20], a multi-star [19, 25], a multi-complete/star graph [7],

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a labelled molecular graph [5], and more recent results when G is a circulant graph [12, 26], a quasi-threshold graph [18], and so on (see Berge [2] for an exposition of the main results).

In this paper, we extend the previous notion and consider graphs that result from the complete multigraph K_n^m by removing multiple edges; we denote by K_n^m the complete multigraph on n vertices with exactly m edges joining every pair of vertices. Based on the properties of the Kirchhoff matrix, which permits the calculation of the number of spanning trees of any given graph, we derive a determinant based formula for the number of spanning trees of the graph $K_n^m - G$, where G is a subgraph of K_n^m, and, thus, it is a multigraph. Note that, if $m = 1$ then $K_n^m - G$ coincides with the graph $K_n - G$.

We also consider graphs that result from the complete multigraph K_n^m by adding multiple edges. More precisely, we consider multigraphs of the form $K_n^m + G$ that result from the complete multigraph K_n^m by adding a set of edges (of the graph K_n^m) that span G. Again, based on the properties of the Kirchhoff matrix, we derive a determinant based formula for the number of spanning trees of the graph $K_n^m + G$. To the best of our knowledge, not as much seems to be known about the number $\tau(K_n^m + G)$. Bedrosian in [1] considered the number $\tau(K_n + G)$ for some simple configurations of G, i.e., when G forms a cycle, a complete graph, or when its vertex set is quite small. More recently, Golin et al. in [12] derive closed formula for the number $\tau(K_n + G)$ using Chebyshev polynomials, introduced in [4], for the case where G forms a circulant graph.

We denote $K_n^m \pm G$ the family of graphs of the forms $K_n^m + G$ and $K_n^m - G$, and derive a determinant based formula for the number $\tau(K_n^m \pm G)$. Moreover, based on these results, we generalize our formulas and extend the family $K_n^m \pm G$ to the more general family of graphs $F_n^m \pm G$, where F_n^m is the complete multigraph on n vertices with at least $m \geq 1$ edges joining every pair of vertices.

Based on our results–that is, the determinant based formulas for the number of spanning trees of the family of graphs $\tau(K_n^m \pm G)$, and using standard algebraic techniques, we generalize known closed formulas for the number of spanning trees of simple graphs of the form $K_n - G$. In particular, we derive closed formulas for the number of spanning trees $\tau(K_n^m \pm G)$, in the case where G forms (i) a complete multipartite graph, and (ii) a multi-star graph.

We point out that our proposed formulas express the number of spanning trees $\tau(K_n^m \pm G)$ as a function of the determinant of a matrix that can be easily constructed from the adjacency relation of the graph G. Our results generalize previous results and extend the family of graphs of the form $K_n^m \pm G$ admitting formulas for the number of their spanning trees.

2 Preliminaries

We consider finite undirected simple graphs and multigraphs with no loops; the term *multigraph* is used when multiple edges are allowed in a graph. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively.

The *multiplicity* of a vertex-pair (v, u) of a graph G, denoted by $\ell_G(vu)$, is the number of edges joining the vertices v and u in G. The minimum multiplicity among all the vertex-pairs of G is denoted $\lambda(G)$ while $\Lambda(G)$ is the largest such number. Thus, if $\lambda(G) > 0$, then every pair of vertices in G is connected with at least $\lambda(G)$ edges; if $\Lambda(G) = 1$, then G contains no multiple edges, that is, G is a *simple* graph (note that a simple graph or a multigraph contains no loops). The *degree* of a vertex v of a graph G, denoted by $d_G(v)$, is the number of edges incident with v in G. The minimum degree among the vertices of G is denoted $\delta(G)$ while $\Delta(G)$ is the largest such number.

It is worth noting that a multigraph can also be viewed as a simple graph whose edges are labeled with non-zero integers (weights) which correspond to the multiplicity of the edges. Thus, a multigraph can be thought as a weighted graph. In this paper we adopt the standard approach.

Let $\mathcal{F}$ be the family of complete multigraphs on n vertices with multiplicity at least $m \geq 1$, and let $F_n^m \in \mathcal{F}$. Then, $\lambda(F_n^m) = m$, since F_n^m has at least $m \geq 1$ edges joining every pair of its vertices. A complete multigraph on n vertices with exactly m edges joining every pair of its vertices is called *m-complete multigraph* and denoted by K_n^m. Thus, for the m-complete multigraph K_n^m we have that $\lambda(K_n^m) = \Lambda(K_n^m) = m$ and $\delta(K_n^m) = \Delta(K_n^m) = (n-1)m$. Note that, the 1-complete multigraph is the graph K_n. By definition, every complete multigraph F_n^m contains a subgraph isomorphic to an m-complete multigraph K_n^m.

Let F_n^m be a complete multigraph and let $\mathcal{C}$ be a set of edges of F_n^m such that the graph which is obtained from F_n^m by removing the edges of $\mathcal{C}$ is an m-complete multigraph K_n^m; the graph spanned by the edges of $\mathcal{C}$ is called a *characteristic* graph of F_n^m and denoted by $\mathcal{H}(F_n^m)$. By definition, a characteristic graph $\mathcal{H}(F_n^m)$ contains no isolated vertex.

Let G and H be two multigraphs. The graph $G + H$ is defined as follows:

$$V(G+H) = V(G) \cup V(H)$$

and

$$vu \in E(G+H) \Longleftrightarrow vu \in E(G) \text{ or } vu \in E(H).$$

By definition, both graphs G and H are subgraphs of $G + H$. Moreover, if $v, u \in V(G) \cap V(H)$, then $\ell_{G+H}(vu) = \ell_G(vu) + \ell_H(vu)$.

Let G and H be two multigraphs such that $E(H) \subseteq E(G)$. The graph $G - H$ is defined as the graph obtained from G by removing the edges of H.

Having defined the graphs $G + H$ and $G - H$, it is easy to see that $F_n^m = K_n^m + \mathcal{H}(F_n^m)$ and $K_n^m = F_n^m - \mathcal{H}(F_n^m)$. In general, $\mathcal{H}(F_n^m) \neq F_n^m - K_n^m$; the equality holds if $\mathcal{H}(F_n^m)$ has n vertices.

The *adjacency matrix* of a multigraph G on n vertices, denoted by $A(G)$, is an $n \times n$ matrix with diagonal elements $A(G)[i,i] = 0$ and off-diagonal elements $A(G)[i,j] = \ell_G(v_i v_j)$. The *degree matrix* of the multigraph G, denoted by $D(G)$, is an $n \times n$ matrix with diagonal elements $D(G)[i,i] = d_G(v_i)$ and off-diagonal elements $D(G)[i,j] = 0$. Throughout the paper empty entries in matrices represent 0s.

For an $n \times n$ matrix M, the $(n-1)$-st order *minor* μ_j^i is the determinant of the $(n-1) \times (n-1)$ matrix obtained from M after having deleted row i and column j; the i-th *cofactor* equals μ_i^i. The *Kirchhoff matrix* $L(G)$ (also known as the *Laplacian matrix*) for a multigraph G on n vertices is an $n \times n$ matrix with elements

$$L(G)[i,j] = \begin{cases} d_G(v_i) & \text{if } i = j, \\ -\ell_G(v_i v_j) & \text{otherwise} \end{cases}$$

where $d_G(v_i)$ is the degree of vertex v_i in the graph G and $\ell_G(v_i v_j)$ is the number of edges joining the vertices v_i and v_j in G. The matrix $L(G)$ is symmetric, has nonnegative real eigenvalues and its determinant is equal to zero. Note that, $L(G) = D(G) - A(G)$.

The *Kirchhoff matrix tree theorem* [3] is one of the most famous results in graph theory. It provides a formula for the number of spanning trees of a graph G in terms of the cofactors of G's Kirchhoff matrix; it is stated as follows:

Theorem 2.1 (Kirchhoff Matrix Tree Theorem [3]) *For any multigraph G with $L(G)$ defined as above, the cofactors of $L(G)$ have the same value, and this value equals the number of spanning trees $\tau(G)$ of the multigraph G.*

The Kirchhoff matrix tree theorem provides a powerful tool for computing the number $\tau(G)$ of spanning trees of a graph G. For this computation, we first form the Kirchhoff matrix $L(G)$ of the graph G and obtain the $(n-1)\times(n-1)$ matrix $L_i(G)$ from $L(G)$ by removing its i-th row and column (arbitrary), and then compute the determinant of the matrix $L_i(G)$. We note that the determinant of $L_i(G)$ simply counts the number of spanning trees rooted at vertex $v_i \in V(G)$, which justifies the reason of removing the i-th row and the i-th column from $L(G)$. A combinatorial proof of the Kirchhoff matrix tree theorem can be found in [6].

The number of spanning trees of a graph G can be computed directly (without removing any row or column) in terms of a matrix $L'(G)$ similar to the Kirchhoff matrix $L(G)$, which is associated with the graph G [23], or, alternatively, it can be computed by defining a characteristic polynomial $\det(L(G)-xI)$ on $L(G)$; the latter approach takes into account the computation of the eigenvalues of the matrix $L(G)$ (see [4, 5, 10, 12, 22, 26]).

In our work, we express the number of spanning trees of a graph of the form $K_n^m \pm G$, where G is a subgraph of K_n^m on p vertices, in terms of a $p\times p$ matrix $B(G)$ associated with the graph G, and not in terms of an $n\times n$ matrix $L(K_n^m \pm G)$ associated with the whole graph $K_n^m \pm G$.

3 The $K_n^m \pm G$ graphs

In this section, we consider graphs that result from the m-complete multigraph K_n^m by removing or/and adding multiple edges. We are interested in deriving determinant based formulas for the number of spanning trees of the graphs $K_n^m - G$ and $K_n^m + G$, where G is a multigraph spanned by a set of edges $S \subseteq E(K_n^m)$. To this end, we define a parameter α as follows: (i) $\alpha = 1$, for the case $K_n^m + G$, and (ii) $\alpha = -1$, for the case $K_n^m - G$. In other words, $\alpha = \pm 1$, according to $K_n^m \pm G$. Based on the value of α we conclude with the following result.

Let G be a multigraph spanned by a set of edges of the graph K_n^m. We derive formulas for the number of spanning trees of the graph $K_n^m \pm G$; the graph G has $p \le n$ vertices and $\Lambda(G) \le m$.

In order to compute the number $\tau(K_n^m - G)$ we will make use of Theorem 2.1. Thus, we consider the $n\times n$ Kirchhoff matrix $L = L(K_n^m \pm G)$, which has the form:

$$L = \left[\begin{array}{ccc|cccc}
m(n-1) & \cdots & -m & -m & -m & \cdots & -m \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-m & \cdots & m(n-1) & -m & -m & \cdots & -m \\
\hline
-m & \cdots & -m & m(n-1)+\alpha\cdot d_G(v_1) & & & -m-\alpha\cdot\ell_G(v_jv_i) \\
-m & \cdots & -m & & m(n-1)+\alpha\cdot d_G(v_2) & & \\
\vdots & \cdots & \vdots & & & \ddots & \\
-m & \cdots & -m & -m-\alpha\cdot\ell_G(v_iv_j) & & & m(n-1)+\alpha\cdot d_G(v_p)
\end{array}\right] \qquad (1)$$

where $d_G(v_i)$ is the degree of the vertex $v_i \in G$ and $\ell_G(v_iv_j)$ is the multiplicity of the vertices v_i and v_j in G. The entries of the off-diagonal positions $(n-p+i, n-p+j)$ of the matrix L are equal to

$-m-\alpha \cdot \ell_G(v_iv_j)$, $1 \le i,j \le p$. Note that, the first $n-p$ rows and columns of L correspond to the $n-p$ vertices of the set $V(K_n^m) - V(G)$ and, thus, they have degree $m(n-1)$ in $K_n^m \pm G$.

Let L_1 be the $(n-1)\times(n-1)$ matrix obtained from L by removing its first row and column. Then, from Theorem 2.1 we have that

$$\tau(K_n^m - G) = \det(L_1).$$

In order to compute the determinant of the matrix L_1, we add one row and one column to the matrix L_1; the resulting $n \times n$ matrix L_1' has 1 in position $(1,1)$, $-m$ in positions $(1,j)$, $2 \le j \le n$, and 0 in positions $(i,1)$, $2 \le i \le n$. It is easy to see that, $\det(L_1') = \det(L_1)$. Thus, the $n \times n$ matrix L_1' has the following form:

$$L_1' = \left[\begin{array}{cccc:cccc}
1 & -m & \cdots & -m & -m & -m & \cdots & -m \\
0 & m(n-1) & \cdots & -m & -m & -m & \cdots & -m \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -m & \cdots & m(n-1) & -m & -m & \cdots & -m \\ \hdashline
0 & -m & \cdots & -m & m(n-1)+\alpha\cdot d_G(v_1) & & & -m-\alpha\cdot\ell_G(v_jv_i) \\
0 & -m & \cdots & -m & & m(n-1)+\alpha\cdot d_G(v_2) & & \\
\vdots & \vdots & \cdots & \vdots & & & \ddots & \\
0 & -m & \cdots & -m & -m-\alpha\cdot\ell_G(v_iv_j) & & & m(n-1)+\alpha\cdot d_G(v_p)
\end{array}\right].$$

We denote by L_1'' the matrix obtained from L_1' after multiplying the first row of L_1' by -1 and adding it to the next $n-1$ rows. Thus, the determinant of L_1'' is equal to the determinant of L_1'. Moreover it becomes:

$$\det(L_1'') = \left|\begin{array}{cccc:cccc}
1 & -m & \cdots & -m & -m & -m & \cdots & -m \\
-1 & mn & & & & & & \\
\vdots & & \ddots & & & & & \\
-1 & & & mn & & & & \\ \hdashline
-1 & & & & mn+\alpha\cdot d_G(v_1) & & & -\alpha\cdot\ell_G(v_jv_i) \\
-1 & & & & & mn+\alpha\cdot d_G(v_2) & & \\
\vdots & & & & & & \ddots & \\
-1 & & & & -\alpha\cdot\ell_G(v_iv_j) & & & mn+\alpha\cdot d_G(v_p)
\end{array}\right|,$$

where the entries of the off-diagonal positions $(n-p+i, n-p+j)$ of the matrix L_1'' are equal to $-\alpha \cdot \ell_G(v_iv_j)$, $1 \le i,j \le p$. Note that, the first $n-p$ rows of the matrix L_1'' have non-zero elements in positions $(1,i)$ and (i,i), $2 \le i \le n-p$. We observe that the sum of all the elements on each row of L_1'', except of the first row, is equal to $mn-1$; recall that, $d_G(v_i) = \sum_{1\le j\le p} \ell_G(v_iv_j)$, for every $v_i \in V(G)$.

Thus, we multiply each column of matrix L_1'' by $\frac{1}{mn}$ and add it to the first column, and we obtain:

$$\det(L_1'') = \begin{vmatrix} \frac{1}{n} & -m & \cdots & -m & -m & -m & \cdots & -m \\ 0 & mn & & & & & & \\ \vdots & & \ddots & & & & & \\ 0 & & & mn & & & & \\ 0 & & & & mn+\alpha\cdot d_G(v_1) & & & -\alpha\cdot \ell_G(v_jv_i) \\ 0 & & & & & mn+\alpha\cdot d_G(v_2) & & \\ \vdots & & & & & & \ddots & \\ 0 & & & & -\alpha\cdot\ell_G(v_iv_j) & & & mn+\alpha\cdot d_G(v_p) \end{vmatrix} \quad (2)$$
$$= m\cdot(mn)^{n-p-2}\cdot\det(B),$$

where $B = mnI_p + \alpha L(G)$ is a $p\times p$ matrix; recall that, $L(G)$ is the Kirchhoff matrix of the multigraph G and thus, $L(G) = D(G) \pm A(G)$, where $D(G)$ and $A(G)$ are the degree matrix and the adjacency matrix of G, respectively. Concluding, we obtain the following result.

Theorem 3.1 *Let K_n^m be the m-complete multigraph on n vertices, and let G be a multigraph on p vertices such that $V(G)\subseteq V(K_n^m)$ and $E(G)\subseteq E(K_n^m)$. Then,*

$$\tau(K_n^m \pm G) = m\cdot(mn)^{n-p-2}\det(B),$$

where $B = mnI_p + \alpha\cdot L(G)$ is a $p\times p$ matrix, $\alpha = \pm 1$ according to $K_n^m \pm G$, and $L(G)$ is the Kirchhoff matrix of G.

We note that, for simple graphs K_n and G in case $K_n - G$, Theorem 3.1 has been stated first by Moon in [16] and numerous authors in various guises used it as a constructive tool to obtain formulas for the number of spanning trees of graphs of the type $K_n - G$.

In the previous theorem the graph G is a subgraph of K_n^m, and, thus, it has multiplicity $\Lambda(G)\leq m$. It follows that the graph $K_n^m + G$ has multiplicity $\Lambda(K_n^m + G)\leq 2m$. However, we can relax the previous restriction in the case of the graph $K_n^m + G$. It is not difficult to see that for the matrix L of Equation (1) we have $\lambda(G)\geq 0$. Thus, we can define the graph G to be a multigraph on p vertices such that $V(G)\subseteq V(K_n^m)$. The following theorem holds.

Theorem 3.2 *Let K_n^m be the m-complete multigraph on n vertices, and let G be a multigraph on p vertices such that $V(G)\subseteq V(K_n^m)$. Then,*

$$\tau(K_n^m + G) = m\cdot(mn)^{n-p-2}\cdot\det(B),$$

where $B = mnI_p + L(G)$ is a $p\times p$ matrix, and $L(G)$ is the Kirchhoff matrix of G.

4 The $F_n^m \pm G$ graphs

In this section we derive determinant based formulas for the number $\tau(F_n^m \pm G)$, where F_n^m is a complete multigraph and G is a subgraph of F_n^m. We first take into consideration the graph $K_n^m + G_1 - G_2$ and derive a determinant based formula for the number $\tau(K_n^m + G_1 - G_2)$, and, then, we derive a formula for the number $\tau(F_n^m \pm G)$ using the graph $K_n^m + G_1 - G_2$ and a characteristic graph $\mathcal{H}(F_n^m)$.

4.1 The case $K_n^m + G_1 - G_2$

Here, we consider graphs that result from the m-complete multigraph K_n^m by adding multiple edges of a graph G_1 and removing multiple edges from a graph G_2. Let G_1 be a multigraph on p_1 vertices, such that $V(G_1) \subseteq V(K_n^m)$, and let G_2 be a multigraph on p_2 vertices, such that $V(G_2) \subseteq V(K_n^m)$ and $E(G_2) \subseteq E(K_n^m + G_1)$. We next focus on the graph $K_n^m + G_1 - G_2$, which is obtained from the m-complete multigraph K_n^m by first adding the edges of the graph G_1 and then removing from the resulting graph $K_n^m + G_1$ the edges of G_2; that is, $K_n^m + G_1 - G_2 = (K_n^m + G_1) - G_2$.

It is worth noting that $(K_n^m + G_1) - G_2 \neq K_n^m + (G_1 - G_2)$, since the graph G_2 is not necessarily a subgraph of G_1. Moreover, $V(G_1) \neq V(G_2)$.

For the pair of multigraphs (G_1, G_2) we define the *union-stable* graphs G_1^* and G_2^* of (G_1, G_2) as follows: G_1^* is the multigraph that results from G_1 by adding in $V(G_1)$ the vertices of the set $V(G_2) - V(G_1)$ and G_2^* is the multigraph that results from G_2 by adding in $V(G_2)$ the vertices of the set $V(G_1) - V(G_2)$. Thus, by definition $V(G_1^*) = V(G_2^*)$.

By definition, both G_1^* and G_2^* are multigraphs on $p = |V(G_1) \cup V(G_2)|$ vertices, with at least $p - p_1$ and $p - p_2$ isolated vertices, respectively. Since $K_n^m + G_1^* - G_2^* = K_n^m + G_1 - G_2$, we focus on the graph $K_n^m + G_1^* - G_2^*$.

Based on Theorem 2.1, we construct the $n \times n$ Kirchhoff matrix $L = L(K_n^m + G_1^* - G_2^*)$ associated with the graph $K_n^m + G_1^* - G_2^*$; it is similar to that of the case of $K_n^m \pm G$. The difference here is the $p \times p$ submatrix which is formed by the last p rows and columns of L, where $p = |V(G_1) \cup V(G_2)|$. More precisely, the matrix L has the following form:

$$L = \left[\begin{array}{cccc:cccc}
m(n-1) & -m & \cdots & -m & -m & -m & \cdots & -m \\
-m & m(n-1) & \cdots & -m & -m & -m & \cdots & -m \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-m & -m & \cdots & m(n-1) & -m & -m & \cdots & -m \\
\hdashline
-m & -m & \cdots & -m & B'(G_1^*, G_2^*)[1,1] & & & B'(G_1^*, G_2^*)[j,i] \\
-m & -m & \cdots & -m & & B'(G_1^*, G_2^*)[2,2] & & \\
\vdots & \vdots & \cdots & \vdots & & & \ddots & \\
-m & -m & \cdots & -m & B'(G_1^*, G_2^*)[i,j] & & & B'(G_1^*, G_2^*)[p,p]
\end{array}\right]$$

where the $p \times p$ submatrix $B'(G_1^*, G_2^*)$ has elements

$$B'(G_1^*, G_2^*)[i,j] = \begin{cases} m(n-1) + d_{G_1^*}(v_i) - d_{G_2^*}(v_i) & \text{if } i = j, \\ -m - \ell_{G_1^*}(v_i v_j) + \ell_{G_2^*}(v_i v_j) & \text{otherwise.} \end{cases}$$

We note that $\ell_{G_1^*}(v_i v_j)$ and $\ell_{G_2^*}(v_i v_j)$ are the number of edges of the vertices v_i and v_j in G_1^* and G_2^*, respectively. The entries $d_{G_1^*}(v_i)$ and $d_{G_2^*}(v_i)$, $1 \leq i \leq p$, are the degrees of vertex v_i of G_1^* and G_2^*, respectively. Note that, $V(G_1^*) = V(G_2^*)$.

It is straightforward to apply a technique similar to that we have applied for the computation of the determinant of the matrix L_1' in the case of $K_n^m - G$. Thus, in the case of $K_n^m + G_1^* - G_2^*$, the determinant of the matrix L'_1 of Equation (2) becomes

$$\det(L'_1) = \begin{vmatrix} \frac{1}{n} & -m & \cdots & -m & -m & & -m & \cdots & -m \\ 0 & mn & & & & & & & \\ \vdots & & \ddots & & & & & & \\ 0 & & & mn & & & & & \\ 0 & & & & B(G_1^*, G_2^*)[1,1] & & & & B(G_1^*, G_2^*)[j,i] \\ 0 & & & & & B(G_1^*, G_2^*)[2,2] & & & \\ \vdots & & & & & & \ddots & & \\ 0 & & & & B(G_1^*, G_2^*)[i,j] & & & & B(G_1^*, G_2^*)[p,p] \end{vmatrix}$$
$$= m \cdot (mn)^{n-p-2} \cdot \det(B(G_1^*, G_2^*)),$$

where the $p \times p$ submatrix $B(G_1^*, G_2^*)$ has elements

$$B(G_1^*, G_2^*)[i,j] = \begin{cases} mn + d_{G_1^*}(v_i) - d_{G_2^*}(v_i) & \text{if } i = j, \\ -\ell_{G_1^*}(v_i v_j) + \ell_{G_2^*}(v_i v_j) & \text{otherwise.} \end{cases}$$

Since $B(G_1^*, G_2^*) = mnI_p + L(G_1^*) - L(G_2^*)$, we set $B = B(G_1^*, G_2^*)$ and obtain the following result.

Theorem 4.1 *Let K_n^m be the m-complete multigraph on n vertices, and let G_1, G_2 be two multigraphs such that $V(G_1) \subseteq V(K_n^m)$ and $E(G_2) \subseteq E(K_n^m + G_1)$. Then,*

$$\tau(K_n^m + G_1 - G_2) = m \cdot (mn)^{n-p-2} \det(B),$$

where $p = |V(G_1) \cup V(G_2)|$, $B = mnI_p + L(G_1^*) - L(G_2^*)$ is a $p \times p$ matrix, $L(G_1^*)$ and $L(G_2^*)$ are the Kirchhoff matrices of the union-stable graphs G_1^* and G_2^* of (G_1, G_2), respectively.

4.2 The general case $F_n^m \pm G$

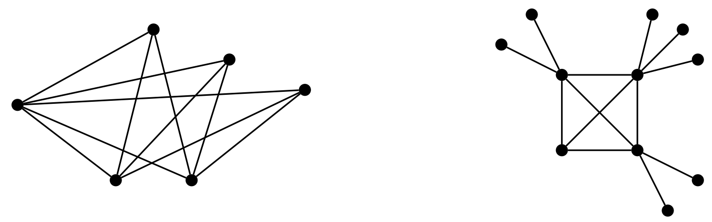
Let F_n^m be a complete multigraph on n vertices and let G be a subgraph of F_n^m. We will show that the previous theorem provides the key idea for computing the number $\tau(F_n^m \pm G)$, where $F_n^m \pm G$ is the graph that results from F_n^m by adding or removing the edges of G. Since $\lambda(F_n^m) > 0$, we have that $F_n^m = K_n^m + \mathcal{H}(F_n^m)$, where $\mathcal{H}(F_n^m)$ is a characteristic graph of F_n^m. Then we have that,

$$F_n^m \pm G = K_n^m + \mathcal{H}(F_n^m) \pm G.$$

The addition of the edges of G in the graph F_n^m, implies that $F_n^m + G = K_n^m + G'$, where the graph $G' = \mathcal{H}(F_n^m) + G$. Thus, for the computation of the number $\tau(F_n^m + G)$ we can apply Theorem 3.2. On the other hand, in the case of removal the edges of G from the graph F_n^m, for the computation of the number $\tau(F_n^m - G)$ we can apply Theorem 4.1, since $F_n^m - G = K_n^m + \mathcal{H}(F_n^m) - G$. Concluding we have the following result.

Lemma 4.1 *Let F_n^m be a complete multigraph on n vertices and $\mathcal{H}(F_n^m)$ be a characteristic graph of F_n^m, and let G be a subgraph of F_n^m. Then,*

$$\tau(F_n^m \pm G) = m \cdot (mn)^{n-p-2} \det(B),$$



(a) (b)

Fig. 1: (a) A complete multipartite graph $K_{1,2,3}$ and (b) a multi-star graph $K_4(0, 2, 2, 3)$.

where $p = |V(\mathcal{H}(F_n^m)) \cup V(G)|$, $B = mnI_p + L(\mathcal{H}(F_n^m)^*) \pm L(G^*)$ is a $p \times p$ matrix and $L(\mathcal{H}(F_n^m)^*)$ and $L(G^*)$ are the Kirchhoff matrices of the union-stable graphs $\mathcal{H}(F_n^m)^*$ and G^* of $(\mathcal{H}(F_n^m), G)$, respectively.

Note that, we consider the graph $F_n^m \pm G$ and therefore G must be a subgraph of F_n^m. However in the case of the $F_n^m + G$ graph, similar to Theorem 3.2, it is obvious that G can be a graph spanned by any set of edges joining the vertices of F_n^m.

5 Classes of graphs

In this section, we generalize known closed formulas for the number of spanning trees of families of graphs of the form $K_n - G$. As already mentioned in the introduction there exist many cases for the $\tau(K_n - G)$, depending on the choice of G. The purpose of this section is to prove closed formulas for $\tau(K_n^m \pm G)$, by applying similar techniques to that of the case of $K_n - G$. Thus we derive closed formulas for the number of spanning trees $\tau(K_n^m \pm G)$, in the cases where G forms (i) a complete multipartite graph, and (ii) a multi-star graph.

5.1 Complete multipartite graphs

A graph is defined to be a *complete multipartite* (or complete k-partite) if there is a partition of its vertex set into k disjoint sets such that no two vertices of the same set are adjacent and every pair of vertices of different sets are adjacent. We denote a complete multipartite graph on p vertices by $K_{m_1,m_2,\ldots,m_k}$, where $p = m_1 + m_2 + \cdots + m_k$; see Figure 1(a). We note that the number of spanning trees of a complete multipartite graph has been considered by several authors in the past [8, 14].

Let $G = K_{m_1,m_2,\ldots,m_k}$ be a complete multipartite graph on p vertices. In [21] it has been proved that the number of spanning trees of $K_n - G$ is given by the following formula:

$$\tau(K_n - G) = n^{n-p-1}(n-p)^{k-1} \prod_{i=1}^{k} (n - (p - m_i))^{m_i - 1},$$

where p is the number of vertices of G.

In this section, we extend the previous result by deriving a closed formula for the number of their spanning trees for the graphs $K_n^m \pm G$, where G a complete multipartite graph on $p \leq n$ vertices. From Theorem 3.1, we construct the $p \times p$ matrix $B(G)$ and add one row and column to the matrix $B(G)$; the resulting $(p+1)\times(p+1)$ matrix $B'(G)$ has 1 in position $(1,1)$, α in positions $(1,j)$, $2\le j\le p+1$, and 0 in positions $(i,1)$, $2\le i\le p+1$; recall that, $\alpha=\pm 1$. Thus, the resulting matrix $B'(G)$ has the following form:

$$B'(G)=\begin{bmatrix} 1 & \alpha & \alpha & \cdots & \alpha \\ & M_1 & -\alpha & \cdots & -\alpha \\ & -\alpha & M_2 & \cdots & -\alpha \\ & \vdots & \vdots & \ddots & \vdots \\ & -\alpha & -\alpha & \cdots & M_k \end{bmatrix},$$

where the diagonal $m_i\times m_i$ submatrices M_i have diagonal elements $mn+\alpha\cdot(p-m_i)$, $1\le i\le k$. Note that, $\det(B(G))=\det(B'(G))$.

In order to compute the determinant of the matrix $B'(G)$ we add the first row to the next p rows. Let $B''(G)$ be the resulting matrix. It follows that the determinant of $B'(G)$ is equal to the determinant of $B''(G)$. We multiply the $2,3,\ldots,p+1$ columns of the matrix $B''(G)$ by $-1/(mn+\alpha\cdot p)$ and add them to the first column; note that, the sum of each row of the matrix $B''(G)$ is equal to $mn+\alpha\cdot p$. Thus, the determinant of matrix $B''(G)$ becomes:

$$\begin{aligned}\det(B''(G)) &= \begin{vmatrix} 1-\frac{\alpha\cdot p}{mn+\alpha\cdot p} & \alpha & \alpha & \cdots & \alpha \\ & M'_1 & & & \\ & & M'_2 & & \\ & & & \ddots & \\ & & & & M'_k \end{vmatrix} \\ &= \frac{mn}{mn+\alpha\cdot p}\cdot\det(M'_1)\cdot\det(M'_2)\cdot\cdots\cdot\det(M'_k), \end{aligned} \tag{3}$$

where the $m_i\times m_i$ submatrices M'_i, $1\le i\le k$, have the following form:

$$M'_i=\begin{vmatrix} mn+\alpha\cdot(p-m_i)+\alpha & \alpha & \cdots & \alpha \\ \alpha & mn+\alpha\cdot(p-m_i)+\alpha & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & mn+\alpha\cdot(p-m_i)+\alpha \end{vmatrix}$$

For the determinant of matrix M'_i we multiply the first row by -1 and add it to the next m_i-1 rows. Then, we add the columns of matrix M'_i to the first column. Observing that $mn+\alpha\cdot(p-m_i)+\alpha\cdot m_i=mn+\alpha\cdot p$, we obtain

$$\det(M'_i)=(mn+\alpha\cdot p)\cdot(mn+\alpha\cdot(p-m_i))^{m_i-1}.$$

Thus, from Equation (3) we have the following result.

Theorem 5.1 *Let $G=K_{m_1,m_2,\ldots,m_k}$ be a complete multipartite graph on $p=m_1+m_2+\cdots+m_k$ vertices. Then,*

$$\tau(K_n^m\pm G)=m\cdot(mn)^{n-p-1}(mn+\alpha\cdot p)^{k-1}\prod_{i=1}^{k}(mn+\alpha\cdot(p-m_i))^{m_i-1},$$

where $p \leq n$ and $\alpha = \pm 1$ according to $K_n^m \pm G$.

Remark 5.1. The class of complete multipartite graphs contains the class of c-split graphs (complete split graphs); a graph is defined to be a c-split graph if there is a partition of its vertex set into a stable set S and a complete set K and every vertex in S is adjacent to all the vertices in K [13].

Thus, a c-split graph G on p vertices and $V(G) = K + S$ is a complete multipartite graph $K_{m_1,m_2,\ldots,m_k}$ with $m_1 = |S|$, $m_2 = m_3 = \cdots = m_k = 1$ and $k = |K| + 1$. A closed formula for the number of spanning trees of the graph $K_n - G$ was proposed in [18], where G is a c-split graph.

Let G be a c-split graph on p vertices and let $V(G) = K + S$. Then, from Theorem 5.1 we obtain that the number of spanning trees of the graphs $K_n^m \pm G$ is given by the following closed formula:

$$\tau(K_n^m \pm G) = m \cdot (mn)^{n-p-1}(mn + \alpha \cdot |K|)^{|S|-1}(mn + \alpha \cdot p)^{|K|},$$

where $p = |K| + |S|$ and $p \leq n$. $\square$

5.2 Multi-star graphs

A *multi-star* graph, denoted by $K_r(b_1, b_2, \ldots, b_r)$, consists of a complete graph K_r with vertices labelled $v_1, v_2, \ldots, v_r$, and b_i vertices of degree one, which are incident with vertex v_i, $1 \leq i \leq r$ [7, 19, 25]; see Figure 1(b).

Let $G = K_r(b_1, b_2, \ldots, b_r)$ be a multi-star graph on $p = r + b_1 + b_2 + \cdots + b_r$ vertices. It has been proved [7, 19, 25] that the number of spanning trees of the graph $K_n - G$ is given by the following closed formula:

$$\tau(K_n - G) = n^{n-p-2}(n-1)^{p-r}\left(1 + \sum_{i=1}^{r} \frac{1}{q_i - 1}\right) \cdot \prod_{i=1}^{r} (q_i - 1)\,,$$

where $q_i = n - (r - 1 + b_i) - \frac{b_i}{n-1}$.

In this section, based on Theorem 3.1, we generalize the previous result by deriving a closed formula for the number of spanning trees of the graphs $K_n^m \pm G$, where G is a multi-star on $p \leq n$ vertices. Let K_r be the complete graph of the multi-star graph G and let $v_1, v_2, \ldots, v_r$ be its vertices. The vertex set consisting of the vertex v_i and the b_i vertices of degree one which are incident with vertex v_i induces a star on $b_i + 1$ vertices, $1 \leq i \leq r$. We construct a $(b_i + 1) \times (b_i + 1)$ matrix M_i which corresponds to the star with center vertex v_i; it has the following form:

$$M_i = \begin{bmatrix} mn + \alpha & & & -\alpha \\ & mn + \alpha & & -\alpha \\ & & & \vdots \\ -\alpha & -\alpha & \cdots & mn + \alpha \cdot (r - 1 + b_i) \end{bmatrix},$$

where $\alpha = \pm 1$.

In order to compute the determinant of the matrix M_i we first multiply the first row by -1 and add it to the next $b_i - 1$ rows. We then add the b_i columns to the first column. Finally, we multiply the first column by $\frac{\alpha}{mn+\alpha}$ and add it to the last column. Thus, by observing that $\alpha^2 = 1$, we obtain:

$$\begin{aligned}
\det(M_i) &= (mn+\alpha)^{b_i} \cdot \left(mn + \alpha \cdot (r-1+b_i) - \frac{\alpha^2 b_i}{mn+\alpha} \right) \\
&= (mn+\alpha)^{b_i} \cdot \left(mn + \alpha \cdot (r-1+b_i) - \frac{b_i}{mn+\alpha} \right) \\
&= (mn+\alpha)^{b_i} \cdot q_i,
\end{aligned}$$

where

$$q_i = mn + \alpha \cdot (r-1+b_i) - \frac{b_i}{mn+\alpha}, 1 \le i \le r. \tag{4}$$

We are now in a position to compute the number of spanning trees $\tau(K_n^m \pm G)$ using Theorem 3.1. Thus, we have

$$\tau(K_n^m \pm G) = m \cdot (mn)^{n-p-2} \cdot \det(B(G)) \tag{5}$$

where

$$B(G) = \begin{bmatrix}
M_{1,1} & & & & -\alpha & & & \\
& M_{2,2} & & & & -\alpha & & \\
& & \ddots & & & & \ddots & \\
& & & M_{r,r} & & & & -\alpha \\
-\alpha & & & & mn+\alpha \cdot d_G(v_1) & -\alpha & \cdots & -\alpha \\
& -\alpha & & & -\alpha & mn+\alpha \cdot d_G(v_2) & \cdots & -\alpha \\
& & \ddots & & \vdots & \vdots & \ddots & \vdots \\
& & & -\alpha & -\alpha & -\alpha & \cdots & mn+\alpha \cdot d_G(v_r)
\end{bmatrix}$$

is a $p \times p$ matrix and $M_{i,i}$ is a submatrix which is obtained from M_i by deleting its last row and its last column, $1 \le i \le r$. The degrees of the vertex v_i of K_r is equal to $d_G(v_i) = r - 1 + b_i$, $1 \le i \le r$. It now suffices to compute the determinant of the matrix $B(G)$. Following a procedure similar to that we applied to the matrix M_i, we obtain:

$$\begin{aligned}
\det(B(G)) &= (mn+\alpha)^{p-r} \cdot \begin{vmatrix}
q_1 & -\alpha & \cdots & -\alpha \\
-\alpha & q_2 & \cdots & -\alpha \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha & -\alpha & \cdots & q_r
\end{vmatrix} \\
&= (mn+\alpha)^{p-r} \cdot \det(D).
\end{aligned}$$

Recall that, $q_i = mn + \alpha \cdot (r-1+b_i) - \frac{b_i}{mn+\alpha}$; see Equation (4). In order to compute the determinant of the $r \times r$ matrix D we first multiply the first row of D by -1 and add it to the $r-1$ rows. Then, we multiply column i by $\frac{q_1+\alpha}{q_i+\alpha}$, $2 \le i \le r$, and add it to the first column. Expanding in terms of the rows of matrix D, we have that

$$\det(D) = \left(1 - \alpha \sum_{i=1}^{r} \frac{1}{q_i+\alpha} \right) \cdot \prod_{i=1}^{r} (q_i + \alpha).$$

Thus, substituting the value of $\det(D)$ into Equation (5), we obtain the following theorem.

Theorem 5.2 *Let $G = K_r(b_1, b_2, \ldots, b_r)$ be a multi-star graph on $p = r + b_1 + b_2 + \cdots + b_r$ vertices. Then,*

$$\tau(K_n^m \pm G) = m \cdot (mn)^{n-p-2}(mn+\alpha)^{p-r} \left(1 - \alpha \cdot \sum_{i=1}^{r} \frac{1}{q_i + \alpha}\right) \cdot \prod_{i=1}^{r} (q_i + \alpha)\,,$$

where $p \leq n$, $q_i = mn + \alpha \cdot (r - 1 + b_i) - \frac{b_i}{mn+\alpha}$ and $\alpha = \pm 1$ according to $K_n^m \pm G$.

6 Concluding remarks

In this paper we derived determinant based formulas for the number of spanning trees of the family of graphs of the form $K_n^m \pm G$, and also for the more general family of graphs $F_n^m \pm G$, where K_n^m (resp. F_n^m) is the complete multigraph on n vertices with exactly (resp. at least) m edges joining every pair of vertices and G is a multigraph spanned by a set of edges of K_n^m (resp. K_n^m). Based on these determinant based formulas, we prove closed formulas for the number of spanning trees $\tau(K_n^m \pm G)$, in the case where G is (i) a complete multipartite graph, and (ii) a multi-star graph.

In light of our results, it would be interesting to consider the problem of proving closed formulas for the number of spanning tree $\tau(K_n^m \pm G)$ in the cases where G belongs to other classes of simple graphs or multigraphs. Moreover, instead of a multigraph G that we have taken into account, we do not know whether a generalazisation for an arbitrarily weighted graph G holds.

The problem of maximizing the number of spanning trees was solved for some families of graphs of the form $K_n - G$, where G is a multi-star graph, a union of paths and cycles, etc. (see [7, 11, 19, 22]). Thus, an interesting open problem is that of maximizing the number of spanning trees of graphs of the form $K_n^m \pm G$.

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