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P_6- and Triangle-Free Graphs Revisited: Structure and Bounded Clique-Width

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The Maximum Weight Stable Set (MWS) Problem is one of the fundamental problems on graphs. It is well-known to be NP-complete for triangle-free graphs, and Mosca has shown that it is solvable in polynomial time when restricted to P_6- and triangle-free graphs. We give a complete structure analysis of (nonbipartite) P_6- and triangle-free graphs which are prime in the sense of modular decomposition. It turns out that the structure of these graphs is simple implying bounded clique-width and thus, efficient algorithms exist for all problems expressible in terms of Monadic Second Order Logic with quantification only over vertex predicates. The problems Vertex Cover, MWS, Maximum Clique, Minimum Dominating Set, Steiner Tree, and Maximum Induced Matching are among them.

Our results improve the previous one on the MWS problem by Mosca with respect to structure and time bound but also extends a previous result by Fouquet, Giakoumakis, and Vanherpe which have shown that bipartite P_6-free graphs have bounded clique-width. Moreover, it covers a result by Randerath, Schiermeyer, and Tewes on polynomial time 3-colorability of P_6- and triangle-free graphs.

Keywords: Maximum Weight Stable Set Problem; clique-width of graphs; efficient graph algorithms.

1 Introduction

Basic problems on graphs such as Vertex Cover and Maximum Weight Stable Set (MWS) which are NP-complete in general can be solved efficiently for various graph classes. Thus, for example, the problems Vertex Cover and MWS are NP-complete even for triangle-free graphs [24] but can be solved in polynomial time for bipartite graphs. Mosca [21] has shown that MWS can be solved in time $\mathcal{O}(n^{4.5})$ for any P_6- and triangle-free graph. Let P_k denote the induced path of k vertices and let $S_{i,j,k}$ denote the tree with exactly one vertex r of degree 3 and three leaves which have distance i, j, k from r, respectively.

In this paper, we give a complete structure analysis of P_6- and triangle-free graphs by showing that such graphs which are not bipartite but prime in the sense of modular decomposition have simple structure which implies bounded clique-width for these graph classes. This leads to the following improvements over previous results:

- it improves the time bound of the MWS algorithm in [21] from $\mathcal{O}(n^{4.5})$ to $\mathcal{O}(n^2)$ by using the clique-width approach;

- it leads to more efficient algorithms not only for the MWS problem but also for problems such as Maximum Clique, Minimum Dominating Set, Steiner Tree, and Maximum Induced Matching and in general for all problems expressible in terms of Monadic Second Order Logic with quantification only over vertex predicates - see [10] - based on a so-called k-expression of the input graph;

- it extends the previous result by Fouquet, Giakoumakis and Vanherpe [12] on bounded clique-width from bipartite P_6-free graphs to (P_6,K_3)-free graphs;

- it extends efficient algorithms to larger classes: the input graph is not necessarily assumed to be (P_6,K_3)-free. Our algorithm for constructing a k-expression of the input graph either results in such an expression or proves that the input graph contains a P_6 or K_3. This approach is called *robust* in [27].

Remark. There is great and constant interest in triangle-free graphs and in particular, P_6- and triangle-free graphs and variants; many papers are dealing with such graphs. Some examples are:

1. Prömel, Schickinger and Steger [25] have shown that with "high probability", a triangle-free graph can be made bipartite by removing a single vertex (which extends the famous result by Erdös, Kleitman and Rothschild saying that almost all triangle-free graphs are bipartite).

2. Liu and Zhou [16] showed that a triangle-free graph G is P_6-free if and only if every connected induced subgraph of G has a dominating complete bipartite subgraph or a dominating C_6.

3. Brandt [5] also studied the structure of P_6- and triangle-free graphs but neither the results of Liu and Zhou nor the results of Brandt lead to a complete structure analysis and bounded clique-width for this graph class.

4. Sumner [28] proved that P_6-, C_6- and triangle-free graphs are 3-colorable. This result was extended by Randerath, Schiermeyer and Tewes [26] to 4-colorability of P_6- and triangle-free graphs and a polynomial time algorithm for 3-colorability on P_6-free graphs.

5. In [17], Lozin shows that bipartite $S_{1,2,3}$-free graphs (an extension of P_6-free bipartite graphs) have bounded clique-width.

Recently, there is increasing interest in classes of bipartite graphs of bounded clique-width. The motivation partially comes from applications in Model Checking which is crucial in the theory of Database Systems as well as Constraint Satisfaction of Artificial Intelligence - see [14]. The model checking problem can be formulated in terms of hypergraphs, and the corresponding bipartite vertex-hyperedge incidence graph of a hypergraph H uniquely determines H. It turns out that bounded clique-width of this bipartite incidence graph is one of the most general conditions under which the model checking problem can be solved efficiently.

Subsequently, we always focus on the nonbipartite case since bipartite P_6-free graphs were studied in [12, 17]. The main result of this paper, namely Theorem 1, gives a complete structure analysis of P_6- and triangle-free graphs and implies bounded clique-width of these graphs.

2 Basic Notions and Tools

Throughout this paper, let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges and let $|V| = n$, $|E| = m$. For a vertex $v \in V$, let $N(v) = \{u \mid uv \in E\}$ denote the *neighborhood* of v in G, and, more generally, let $N^i(v)$ denote the set of vertices with distance i to v, $i \geq 1$.

Disjoint vertex sets X, Y form a *join*, denoted by $X \textcircled{1} Y$ (*co-join*, denoted by $X \textcircled{0} Y$) if for all pairs $x \in X$, $y \in Y$, $xy \in E$ ($xy \notin E$) holds.

Subsequently, we will consider join and co-join also as operations, i.e., the join operation between disjoint vertex sets X, Y adds all edges between them, whereas the co-join operation for X and Y is the disjoint union of the subgraphs induced by X and Y (without edges between them).

Let $u \sim v$ if $uv \in E$ and $u \not\sim v$ otherwise. We will call $u \not\sim v$ a *coedge*. A vertex $z \in V$ *distinguishes* vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$.

A vertex set $M \subseteq V$ is a *module* if no vertex from $V \setminus M$ distinguishes two vertices from M, i.e., every vertex $v \in V \setminus M$ has either a join or a co-join to M. A vertex set is *trivial* if it is empty, one-elementary or the entire vertex set. Note that trivial vertex sets are modules, the so-called *trivial modules*. Nontrivial modules are called *homogeneous sets*.

A graph is *prime* if it contains only trivial modules. The notion of module plays a crucial role in the *modular* (or *substitution*) *decomposition* of graphs (and other discrete structures) which is of basic importance for the design of efficient algorithms - see e.g. [20] for modular decomposition of discrete structures and its algorithmic use and [19] for a linear-time algorithm constructing the modular decomposition tree of a given graph.

For $U \subseteq V$, let $G[U]$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood to be induced subgraphs. Let $\mathcal{F}$ denote a set of graphs. A graph is *$\mathcal{F}$-free* if none of its induced subgraphs is in $\mathcal{F}$.

A vertex set $U \subseteq V$ is *stable* (or *independent*) in graph G if the vertices in U are pairwise nonadjacent. For a given graph with vertex weights, the *Maximum Weight Stable Set* (*MWS*) Problem asks for a stable set of maximum vertex weight.

Let co-$G = \overline{G} = (V, \overline{E})$ denote the *complement graph* of G. A vertex set $U \subseteq V$ is a *clique* in G if U is a stable set in $\overline{G}$. Let K_ℓ denote the clique with ℓ vertices, and let ℓK_1 denote the stable set with ℓ vertices. K_3 is called *triangle*.

Recall that for $k \geq 1$, P_k denotes a chordless path with k vertices and $k-1$ edges, and for $k \geq 3$, C_k denotes a chordless cycle with k vertices and k edges.

Moreover, recall that $S_{i,j,k}$ denotes the tree with exactly one vertex r of degree 3 and three leaves which have distance i, j, k from r, respectively. Thus, the $S_{1,2,3}$ has vertices a, b, c, d, e, f, g and edges ab, bc, cd, de, ef, cg.

The *paw* has vertices a, b, c, d and edges ab, ac, bc, cd. The *house* is the co-P_5. The *bull* has vertices a, b, c, d, e and edges ab, bc, cd, be, ce. The *double-gem* has vertices a, b, c, d, e, f and edges ab, ac, bc, bd, cd, ce, de, df, ef.

For a subgraph H of G, a vertex not in H is a *k-vertex of H* (or *for H*) if it has exactly k neighbors in H. We say that H *has no k-vertex* if there is no k-vertex for H. The subgraph H *dominates* the graph G if there is no 0-vertex for H in G.

In what follows, we need the following classes of bipartite and co-bipartite graphs:

- G is *matched co-bipartite* if its vertex set is partitionable into two cliques C_1, C_2 with $|C_1| = |C_2|$ or $|C_1| = |C_2| - 1$ such that the edges between C_1 and C_2 are a matching and at most one vertex in C_1 and C_2 is not covered by the matching.

- G is *co-matched bipartite* if G is the complement graph of a matched co-bipartite graph.

If a graph is triangle-free but not bipartite, it must contain an odd cycle of length at least 5. For P_6-free graphs, this must be a C_5, say, C with vertices $v_1, \ldots, v_5$ and edges $\{v_i, v_{i+1}\}$, $i \in \{1, \ldots, 5\}$ (throughout this paper, all index arithmetic with respect to a C_5 is done modulo 5). Obviously, in a triangle-free graph, a C_5 C has no 3-, 4- and 5-vertex, and 2-vertices of C have nonconsecutive neighbors in C. Let X denote the set of 0-vertices of C, and for $i \in \{1, \ldots, 5\}$, let Y_i denote the set of 1-vertices of C being adjacent to v_i, and let $Z_{i,i+2}$ denote the set of 2-vertices of C being adjacent to v_i and v_{i+2}.

Moreover, let $Y = Y_1 \cup \ldots \cup Y_5$ and $Z = Z_{1,3} \cup Z_{2,4} \cup Z_{3,5} \cup Z_{4,1} \cup Z_{5,2}$. Obviously, $\{v_1, \ldots, v_5\} \cup X \cup Y \cup Z$ is a partition of V.

Lemma 1 *Let G be a triangle-free graph containing a C_5 C with vertices $v_1, \ldots, v_5$ and edges $\{v_i, v_{i+1}\}$, $i \in \{1, \ldots, 5\}$. Then the following properties hold for all $i \in \{1, \ldots, 5\}$:*

(i) Y_i *and* $Z_{i,i+2}$ *are stable sets;*

(ii) $Y_i \circledS (Z_{i,i+2} \cup Z_{i-2,i})$;

(iii) $Z_{i,i+2} ⓪ Z_{i+2,i+4}$;

(iv) if G is connected P_6-free then the set X of 0-vertices of the C_5 C is a stable set, $Y_i ⓪ X$, and $Y_i ① Y_{i+2}$ as well as $Y_i ⓪ Y_{i+1}$.

Lemma 2 *Let G be a prime (P_6, K_3)-free graph containing a C_5 C with vertices $v_1, \ldots, v_5$ and edges $\{v_i, v_{i+1}\}$, $i \in \{1, \ldots, 5\}$. Then the following properties hold for all $i \in \{1, \ldots, 5\}$:*

(i) every vertex in $Z_{i-1,i+1}$ with a neighbor in Y_i has a join to $Z_{i-2,i}$ and to $Z_{i,i+2}$;

(ii) vertices in Y_i can only be distinguished by vertices in $Z_{i-1,i+1}$;

(iii) coedges between consecutive 2-vertex sets cannot be distinguished by 0-vertices;

(iv) there are no two 0-vertices $x, y \in X$ with x adjacent to $Z_{i,i+2}$ and y adjacent to $Z_{i+2,i+4}$;

(v) if $|Y_i| \geq 2$ then $Y_{i-2} = Y_{i+2} = \emptyset$;

(vi) if $x \not\sim y$ is a coedge with $x \in Z_{i,i+2}$ and $y \in Z_{i+1,i+3}$ then $x ① Y_{i+3}$ and $y ① Y_i$;

(vii) vertices $x \in Z_{i,i+2}$, $y \in Z_{i+1,i+3}$ with $x \not\sim y$ cannot be distinguished by vertices in Y_{i+4};

Proof: In this proof and subsequent ones, without loss of generality, we choose some fixed values for $i \in \{1, \ldots, 5\}$.

(i) If $x \in Y_3$, $y \in Z_{2,4}$ and $z \in Z_{3,5}$ with $x \sim y$ and $y \not\sim z$ then x, y, v_2, v_1, v_5, z induce a P_6 in G.

(ii) By Lemma 1, $Z_{i-2,i}$ and $Z_{i,i+2}$ have a co-join to Y_i. Assume that $x, y \in Y_3$ and $z \in Z_{5,2}$ such that $x \sim z$ and $y \not\sim z$. Then v_1, v_5, z, x, v_3, y induce a P_6 in G.

(iii) If $x \in Z_{1,3}$, $y \in Z_{2,4}$ and $z \in X$ such that $x \not\sim y$ and $z \sim x$, $z \not\sim y$ then z, x, v_1, v_5, v_4, y induce a P_6 in G.

(iv) Assume that there are vertices $x, y \in X$, $x \neq y$ and $u \in Z_{1,3}$, $v \in Z_{3,5}$ such that $x \sim u$, $y \sim v$. Since x, u, v_1, v_5, v, y induce no P_6, either $x \sim v$ or $y \sim u$ but if $x \sim v$ and $y \not\sim u$ then v_2, v_1, u, x, v, y induce a P_6. Thus, x and y have the same neighborhood in $Z_{1,3}$ and $Z_{3,5}$. Since G is prime, there must be a vertex z distinguishing x and y, say $z \sim x$ and $z \not\sim y$. Recall that by Lemma 1 (iv), no 0- and no 1-vertex can distinguish x and y. Thus, z must be a 2-vertex. If $z \in Z_{5,2}$ then y, v, x, z, v_2, v_1 induce a P_6, and similarly for $z \in Z_{4,1}$. Finally, assume that $z \in Z_{2,4}$. Note that $z \not\sim u$ and $z \not\sim v$ since $z \sim x$, $x \sim u$, $x \sim v$ and G is triangle-free but now v, y, u, v_1, v_2, z induce a P_6 - contradiction.

(v) If $|Y_3| \geq 2$ then, since Y_3 is no module, there are $s, s' \in Y_3$, $s \neq s'$, and $x \in Z_{2,4}$ such that $x \sim s$ and $x \not\sim s'$. If $Y_1 \neq \emptyset$ and $r \in Y_1$ then s', r, s, x, v_4, v_5 induce a P_6.

(vi) If $x \in Z_{1,3}$, $y \in Z_{2,4}$ and $u \in Y_4$ ($v \in Y_1$, respectively) with $x \not\sim y$ and $x \not\sim u$ ($y \not\sim v$, respectively) then u, v_4, y, v_2, v_1, x (v, v_1, x, v_3, v_4, y, respectively) induce a P_6 in G.

(vii) If $x \in Z_{1,3}$, $y \in Z_{2,4}$ and $z \in Y_5$ such that $x \not\sim y$ and $x \sim z$, $y \not\sim z$ then z, x, v_1, v_2, y, v_4 induce a P_6 in G.

$\square$

Lemma iv can also be expressed in the following way:

If the set of 0-vertices adjacent to $Z_{i,i+2}$ is nontrivial then the set of 0-vertices adjacent to $Z_{i+2,i+4}$ is empty.

An immediate consequence of Lemma v is

Corollary 1 *At most two of the 1-vertex sets are nontrivial, namely consecutive ones.*

For the next section we need the following notions:

$Z^0_{i,i+2} := \{x \mid x \in Z_{i,i+2}$ and x has a nonneighbor in $Z_{i-1,i+1}$ or in $Z_{i+1,i+3}\}$ for $i \in \{1, \dots, 5\}$, and let

$Z_0 := \bigcup_{i=1}^{5} Z^0_{i,i+2}$. We say that

2-vertices $v \in Z_0$ are of *type 0* and 2-vertices $v \in Z \setminus Z_0$ are of *type 1*.

Let X_0 denote the set of 0-vertices being adjacent to a vertex in Z_0, and let

$G_0 := G[X_0 \cup Z_0]$.

3 Structure of the Subgraph $G_0 := G[X_0 \cup Z_0]$

Throughout this section, let G be a P_6- and triangle-free graph containing C_5 C with vertices $v_1, \dots, v_5$ and edges $\{v_i, v_{i+1}\}$, $i \in \{1, \dots, 5\}$. The aim of this section is to describe the structure of G_0 as a first step of the complete structure description of G. We first focus on the structure of the subgraph $G[Z_0]$. In the subsequent Lemmas 3, 4, 5 and 6, primality of the graph is not required.

Lemma 3 *(i) There are no four vertices $x, u \in Z_{i,i+2}$, $y, v \in Z_{i+1,i+3}$ such that $x \sim y$, $x \not\sim v$, $u \not\sim y$, and $u \not\sim v$.*

(ii) There are no four vertices $x \in Z_{i-1,i+1}$, $y, y' \in Z_{i,i+2}$ and $z \in Z_{i+1,i+3}$ such that $x \sim y$, $x \not\sim y'$, $y \not\sim z$, and $y' \sim z$.

(iii) There are no four vertices $u \in Z_{i-1,i+1}$, $v \in Z_{i,i+2}$, $x \in Z_{i+1,i+3}$, $y \in Z_{i+2,i+4}$ such that $u \not\sim v$, $v \sim x$ and $x \not\sim y$.

Proof:

(i) If there are four vertices $x, u \in Z_{1,3}$, $y, v \in Z_{2,4}$ such that $x \sim y$, $x \not\sim v$, $u \not\sim y$, and $u \not\sim v$ then u, v_1, x, y, v_4, v induce a P_6 in G.

(ii) If there are four vertices $x \in Z_{1,3}$, $y, y' \in Z_{2,4}$ and $z \in Z_{3,5}$ such that $x \sim y$, $x \not\sim y'$, $y \not\sim z$, and $y' \sim z$ then y, x, v_1, v_5, z, y' induce a P_6 in G.

(iii) If there are four vertices $u \in Z_{5,2}$, $v \in Z_{1,3}$, $x \in Z_{2,4}$, $y \in Z_{3,5}$ such that $u \not\sim v$, $v \sim x$ and $x \not\sim y$ then v, x, v_2, u, v_5, y induce a P_6 in G.

□

Figure 1 shows the three forbidden configurations of Lemma 3 (boldface edges indicate P_6).

Fig. 1: Forbidden configurations of Lemma 3

A simple consequence of Lemma 3 is the following:

Lemma 4 *If there are $x \in Z^0_{i,i+2}$, $y \in Z^0_{i-1,i+1}$, $z \in Z^0_{i-2,i}$ such that $x \sim y$ and $y \not\sim z$ then $Z^0_{i+1,i+3} = \emptyset$. Analogously, if there are $x \in Z^0_{i,i+2}$, $y \in Z^0_{i-1,i+1}$, $z \in Z^0_{i-2,i}$ such that $x \not\sim y$ and $y \sim z$ then $Z^0_{i-3,i-1} = \emptyset$.*

Proof: Let $x \in Z^0_{5,2}$, $y \in Z^0_{4,1}$ and $z \in Z^0_{3,5}$ such that $x \sim y$ and $y \not\sim z$. Then $x \textcircled{1} Z^0_{1,3}$, since otherwise there is a $u \in Z^0_{1,3}$ with $x \not\sim u$, and then z, y, x, u contradict to Lemma 3 (iii). Since $x \in Z_0$, x must have a nonneighbor $y' \in Z^0_{4,1}$. Assume that there is a vertex $u \in Z^0_{1,3}$. Then $u①Z^0_{2,4}$, since otherwise there is a $w \in Z^0_{2,4}$ with $u \not\sim w$, and then y', x, u, w contradict to Lemma 3 (iii). Thus, since $u \in Z_0$, u must have a nonneighbor $x' \in Z^0_{5,2}$. Now $x' \not\sim y'$, otherwise y', x', x, u contradict to Lemma 3 (ii), and $x' \sim y$, otherwise y, y', x, x' contradict to Lemma 3 (i), but now z, y, x', u contradict to Lemma 3 (iii) Thus, $Z^0_{1,3} = \emptyset$. The second claim follows by symmetry. □

Another consequence of Lemma 3 (i) and Lemma 2 is the following property:

Lemma 5 *Vertices $x, x' \in Z^0_{i,i+2}$ having a common nonneighbor $y \in Z^0_{i-1,i+1}$ ($y \in Z^0_{i+1,i+3}$, respectively) can only be distinguished by a vertex $y' \in Z^0_{i+1,i+3}$ ($y' \in Z^0_{i-1,i+1}$, respectively).*

Proof: Let $x, x' \in Z^0_{5,2}$ and let $y \in Z^0_{4,1}$ be a common nonneighbor of x and x'. By Lemma 2 (iii) x and x' cannot be distinguished by 0-vertices of C since a 0-vertex z being adjacent to x is also adjacent to y and thus to x'. Since G is K_3-free, x and x' do not have edges to Y_5 and Y_2. By Lemma 2 (i), x and x' do not have edges to Y_1. By Lemma 2 (vi, x and x' have a join to Y_4. By Lemma 2 (vii), x and x' cannot be distinguished by Y_3 vertices. By Lemma 3 (i), x and x' cannot be distinguished by a vertex from $Z_{4,1}$. Thus, if y' distinguishes x and x' then necessarily $y' \in Z^0_{1,3}$. □

Lemma 6 *If there is an edge between a vertex in $Z^0_{i,i+2}$ and $Z^0_{i+1,i+3}$ then the following conditions are fulfilled:*

(i) $Z^0_{i+2,i+4} = \emptyset$ *or* $Z^0_{i-1,i+1} = \emptyset$.

(ii) If $Z^0_{i-1,i+1} \neq \emptyset$ *then* $Z^0_{i-1,i+1}$ ⓪ $Z^0_{i,i+2}$.

Proof: Let $xy \in E$ be an edge with $x \in Z^0_{5,2}$ and $y \in Z^0_{4,1}$

(i) Assume that $Z^0_{1,3} \neq \emptyset$ and $Z^0_{3,5} \neq \emptyset$. If y has a nonneighbor in $Z^0_{3,5}$ then by Lemma 4, $Z^0_{1,3} = \emptyset$, and analogously, if x has a nonneighbor in $Z^0_{1,3}$ then by Lemma 4, $Z^0_{3,5} = \emptyset$. Thus, for every edge $xy \in E$, $x \in Z^0_{5,2}$, $y \in Z^0_{4,1}$, $x①Z^0_{1,3}$ and $y①Z^0_{3,5}$ holds.

Since $x \in Z^0_{5,2}$, x has a nonneighbor $y' \in Z^0_{4,1}$, and analogously, y has a nonneighbor $x' \in Z^0_{5,2}$. By Lemma 3 (i), $x' \sim y'$. Now let $z \in Z^0_{1,3}$. By Lemma 4, applied to y', x, z, $Z^0_{2,4} = \emptyset$ follows. Thus, z must have a nonneighbor $x'' \in Z^0_{5,2}$, $x'' \neq x$, $x'' \neq x'$ (note that by Lemma 3 (ii), applied to z, x, x', y', also $x' \sim z$). If x'' has a neighbor $y'' \in Z^0_{4,1}$ then by Lemma 4, z, x'', y'' imply $Z^0_{3,5} = \emptyset$ - contradiction. Thus, x'' has a co-join to $Z^0_{4,1}$, but now x, y, x'', y' contradict to Lemma 3 (i). Thus (i) holds.

(ii) Let $Z^0_{3,5} \neq \emptyset$ and assume that there is an edge between $Z^0_{3,5}$ and $Z^0_{4,1}$. Then by (i), $Z^0_{1,3} = Z^0_{2,4} = \emptyset$. Let $z \in Z^0_{3,5}$ have a neighbor $y' \in Z^0_{4,1}$, and recall that $x \in Z^0_{5,2}$ has neighbor $y \in Z^0_{4,1}$. If x and z have no common neighbor in $Z^0_{4,1}$ (i.e., $x \not\sim y'$ and $z \not\sim y$) then x, y, y', z contradicts to

Lemma 3 (ii). Thus x and z have a common neighbor, say $w \in Z^0_{4,1}$.

Since $Z^0_{1,3} = Z^0_{2,4} = \emptyset$, x has a nonneighbor $u \in Z^0_{4,1}$ and z has a nonneighbor $u' \in Z^0_{4,1}$. If x and z have no common nonneighbor in $Z^0_{4,1}$ then x, u, u', z contradict to Lemma 3 (ii). Thus let $u \in Z^0_{4,1}$ be a common nonneighbor of x and z. Then w has a nonneighbor $x' \in Z^0_{5,2}$ or $z' \in Z^0_{3,5}$; without loss of generality, say $x'w \notin E$. By Lemma 3 (i), $x' \sim u$ but now x', u, w, z contradict to Lemma 3 (ii). Thus, (ii) holds.

□

Lemma 7 *Let G be a* prime *P_6- and triangle-free graph containing C_5 as above. If there is an edge between a vertex in $Z^0_{i,i+2}$ and a vertex in $Z^0_{i+1,i+3}$ then $G[Z^0_{i,i+2} \cup Z^0_{i+1,i+3}]$ is a co-matched bipartite graph.*

Proof: Let $ab \in E$ with $a \in Z^0_{4,1}$ and $b \in Z^0_{5,2}$. By Lemma 6 (i), $Z^0_{1,3} = \emptyset$ or $Z^0_{3,5} = \emptyset$, say $Z^0_{1,3} = \emptyset$. Assume that $Z^0_{4,1} \cup Z^0_{5,2}$ do not induce a co-matched bipartite graph. Then two vertices in one of the two sets have a common nonneighbor in the other. First assume that $x, x' \in Z^0_{5,2}$ have a common nonneighbor $y \in Z^0_{4,1}$. Then, by Lemma 5, only vertices in $Z^0_{1,3}$ can distinguish x and x' but $Z^0_{1,3} = \emptyset$. Now assume that $x, x' \in Z^0_{4,1}$ have a common nonneighbor $y \in Z^0_{5,2}$. Then, again by Lemma 5, only vertices in $Z^0_{3,5}$ can distinguish x and x' but by Lemma 6 (ii), there are no edges between $Z^0_{3,5}$ and $Z^0_{4,1}$. Since G is prime, $G[Z^0_{4,1} \cup Z^0_{5,2}]$ is co-matched bipartite. □

Corollary 2 *$G[Z_0]$ is either the disjoint union of two co-matched bipartite graphs (which are possibly empty or one of their color classes is empty and the other is trivial) or for each $i \in \{1, \dots, 5\}$, $|Z^0_{i,i+2}| = 1$.*

Proof: If there are no edges between consecutive sets $Z^0_{i,i+2}$, $i \in \{1, \dots, 5\}$, then by Lemma 5, all $Z^0_{i,i+2}$ are modules, and since G is prime, $|Z^0_{i,i+2}| \leq 1$. If in addition at least one of them is empty, $G[Z_0]$ is the disjoint union of two (trivial) co-matched bipartite graphs.

Now assume without loss of generality that there is an edge between $Z^0_{4,1}$ and $Z^0_{5,2}$. Then by Lemma 7, $Z^0_{4,1}$ and $Z^0_{5,2}$ induce a co-matched bipartite graph, and by Lemma 6 (i), $Z^0_{3,5} = \emptyset$ or $Z^0_{1,3} = \emptyset$, say $Z^0_{1,3} = \emptyset$, and if also $Z^0_{3,5} = \emptyset$ then $Z^0_{2,4} = \emptyset$ by definition of type 0 2-vertices. If $Z^0_{3,5} \neq \emptyset$ then $Z^0_{3,5}$ ⓪ $Z^0_{4,1}$ by Lemma 6 (ii). Now, if there is an edge between $Z^0_{2,4}$ and $Z^0_{3,5}$ then again by Lemma 7, $Z^0_{2,4}$ and $Z^0_{3,5}$ induce a co-matched bipartite graph, and if there is no edge between $Z^0_{2,4}$ and $Z^0_{3,5}$ then by Lemma 5, both sets have at most one vertex since G is prime In either case, $G[Z_0]$ is the disjoint union of two co-matched bipartite graphs. □

Now, we add the 0-vertices X_0 being adjacent to Z_0 to $G[Z_0]$. A *copath in $G[Z_0]$* is a sequence of coedges $x_i x_{i+1}$, $i \in \{1, \dots, k\}$, such that $x_i \in Z^0_{j,j+2}$, $x_{i+1} \in Z^0_{j+1,j+3}$ or $x_i \in Z^0_{j,j+2}$, $x_{i+1} \in Z^0_{j-1,j+1}$ for some $j \in \{1, \dots, 5\}$. A *cocomponent in $G[Z_0]$* is a maximal vertex subset $U \subseteq Z_0$ such that for every x and y in U, there is a copath connecting x and y.

Lemma 8 *For each cocomponent Q in $G[Z_0]$, the set X_Q of 0-vertices being adjacent to Q is a module in G.*

Proof: Let Q be a cocomponent in $G[Z_0]$, and let X_Q denote the set of 0-vertices being adjacent to Q. Assume that X_Q is no module in G. Then there are $x, y \in X_Q$ and $z \notin X_Q$ such that $x \sim z$ and $y \not\sim z$. By Lemma 2 (iii), every 0-vertex adjacent to Q has a join to Q, i.e., x①Q and y①Q. Let $u \in Z^0_{1,3}$, $v \in Z^0_{2,4}$, $u \not\sim v$, be neighbors of x, y: $\{x, y\}$①$\{u, v\}$.

By Lemma 1 (iv), the distinguishing vertex z is no 0-vertex since X is a stable set, and z is no 1-vertex since X⓪Y. Thus, z must be a 2-vertex.

By Lemma 2 (iv), $z \notin (Z_{3,5} \cup Z_{4,1} \cup Z_{5,2})$. Thus, let $z \in Z_{1,3}$ ($z \in Z_{2,4}$, respectively). Then $z \not\sim v$ ($z \not\sim u$, respectively), since otherwise x, z, v (x, z, u, respectively) induce a triangle but now z and v (z and u, respectively) form a coedge distinguished by y, a contradiction to Lemma 2 (iii). □

Since G is prime, $|X_Q| \leq 1$ for all cocomponents Q of $G[Z_0]$. Thus, if $G[Z_0]$ has only one cocomponent then there is at most one 0-vertex being adjacent to Z_0. Note that by Lemma 6 (ii) and by Corollary 2, if $G[Z_0]$ consists of two nonempty co-matched bipartite graphs then it has only one cocomponent. In the case in which $G[Z_0]$ consists of only one co-matched bipartite graph, it may have arbitrarily many cocomponents (which are just single coedges), and then by Lemma 8, every coedge of $G[Z_0]$ can have exactly one neighbor in X. Since G is K_3-free, every 0-vertex is adjacent to at most one coedge (note that in a co-matched bipartite graph every two coedges are connected by two edges).

4 Structure of Nonbipartite Prime (P_6,K_3)-free Graphs

The aim of this section is to give a complete structure description of nonbipartite prime (P_6,K_3)-free graphs G which also will lead to bounded clique-width. For this purpose, we subdivide G into the subgraph G_0 and into five bipartite subgraphs based on the other 2-vertex sets, the 1-vertex sets and the other 0-vertices. It is already clear that $\{v_1, \dots, v_5\}$, X, Y and Z define a partition of the vertex set V of G.

Let $Z^1_{i,i+2} := Z_{i,i+2} \setminus Z^0_{i,i+2}$ and $Z_1 := Z \setminus Z_0$. For $i \in \{1, \dots, 5\}$, let X_i denote the set of 0-vertices being adjacent to $Z^1_{i-1,i+1}$. Now, if for $i \in \{0, 1, \dots, 5\}$, X_i is trivial, we will omit the single vertex in X_i, i.e., let

$$X'_i = \begin{cases} X_i & \text{if } X_i \text{ is nontrivial} \\ \emptyset & \text{otherwise} \end{cases}$$

For $i \in \{1, \dots, 5\}$, let $B_i := G[X'_i \cup Y_i \cup Z^1_{i-1,i+1}]$. By Lemma 1 (iv), $X \cup Y_i$ is a stable set, and thus, B_i is bipartite. Let X_T denote the union of trivial X_i, $i \in \{0, 1, \dots, 5\}$.

The *basic subgraphs* in G are the subgraphs G_0 and B_i, $i \in \{1, \dots, 5\}$.

Lemma 9 *The vertex sets X'_0, Z_0 of G_0 and the vertex sets X'_i, Y_i, $Z^1_{i-1,i+1}$ of B_i, $i \in \{1, \dots, 5\}$, define a partition of $V \setminus (\{v_1, \dots, v_5\} \cup X_T)$.*

Proof: Obviously, X, Y and Z define a partition of $V \setminus \{v_1, \dots, v_5\}$, Z_0 and Z_1 define a partition of Z, and $Z^1_{i,i+2}$, $i \in \{1, \dots, 5\}$, define a partition of Z_1. Moreover, G_0 contains no 1-vertices, and Y_i, $i \in \{1, \dots, 5\}$, define a partition of Y.

Claim 4.1 *If a 0-vertex is adjacent to some $Z^1_{i,i+2}$ then it is not adjacent to $Z^0_{i,i+2}$.*

Proof. Without loss of generality, let $x \in Z^1_{1,3}$ and $y \in Z^0_{1,3}$ with a nonneighbor $z \in Z^0_{2,4}$, $z \not\sim y$. If for a 0-vertex u, $u \sim x$ and $u \sim y$ then by Lemma 2 (iii), $u \sim z$ but now x, u, z induce a triangle - contradiction. This shows Claim 4.1.

Claim 4.2 *If a 0-vertex from X'_{i+1} is adjacent to some $Z^1_{i,i+2}$ then it is not adjacent to $Z^0_{j,j+2}$ and not adjacent to $Z^1_{j,j+2}$ for $j \neq i$.*

Proof. Since G is K_3-free, a 0-vertex being adjacent to $Z^1_{i,i+2}$ is nonadjacent to $Z_{i-1,i+1}$ and $Z_{i+1,i+3}$, and by Lemma 2 (iv), if X_{i+1} is nontrivial then no vertex of X_{i+1} is adjacent to $Z_{i-2,i}$ or $Z_{i+2,i+4}$. This shows Claim 4.2.

By Claims 4.1 and 4.2, $X_0 \cap (X_1 \cup \ldots \cup X_5 \setminus X_T) = \emptyset$. Now by Claim 4.2 the sets $X_i, X_j, i \neq j$, are disjoint Thus, $X_0, X_1, \dots, X_5$ form a partition of $X \setminus X_T$. This completes the proof of Lemma 9. □

Recall that by Lemma 2 (ii), vertices in Y_i can only be distinguished by vertices in $Z_{i-1,i+1}$. Thus, every vertex in $Z_{i-1,i+1}$ has either a join or a co-join to Y_{i+2} (Y_{i+3}, respectively).

Let $Z_{i-1,i+1;00}$ ($Z_{i-1,i+1;01}$, $Z_{i-1,i+1;10}$, $Z_{i-1,i+1;11}$, respectively) be the set of 2-vertices in $Z_{i-1,i+1}$ having a co-join to Y_{i+2} and Y_{i+3} (having a co-join to Y_{i+2} and a join to Y_{i+3}, having a join to Y_{i+2} and a co-join to Y_{i+3}, having a join to Y_{i+2} and Y_{i+3}, respectively). Moreover, let $Z^a_{i-1,i+1;bc} = Z^a_{i-1,i+1} \cap Z_{i-1,i+1;bc}$, $a \in \{0,1\}$, $bc \in \{00, 01, 10, 11\}$.

The *basic vertex subsets* of G are $X'_0, X'_1, \ldots, X'_5$, $Y_1, \ldots, Y_5$, and $Z^a_{i-1,i+1;bc}$, $i \in \{1, \ldots, 5\}$, $a \in \{0,1\}$, $bc \in \{00, 01, 10, 11\}$.

Lemma 10 *For all pairs of basic vertex subsets U, W from different basic subgraphs, either $U①W$ or $U⓪W$.*

Proof: First assume that $U \in \{X'_0, X'_1, \ldots, X'_5\}$. If also $W \in \{X'_0, X'_1, \ldots, X'_5\}$ then $U⓪W$ since by Lemma 1 (iv), X is a stable set.

If $W \in \{Y_1, \ldots, Y_5\}$ then $U⓪W$ since by Lemma 1 (iv), there are no edges between 0-vertices and 1-vertices.

If W is a basic subset of 2-vertices, we have the following cases:

First assume that $U = X'_0$. Let $W = Z^1_{i-1,i+1;bc}$ (note that $W = Z^0_{i-1,i+1;bc}$ is impossible since W belongs to another basic subgraph). Assume that $u \in U$ with u being adjacent to some $x \in Z^0_{4,1}$ and $y \in Z^0_{5,2}$, $x \not\sim y$. Then $u⓪W$ for $W \in \{Z^1_{3,5}, Z^1_{4,1}, Z^1_{5,2}, Z^1_{1,3}\}$ since G is K_3-free, and $u⓪W$ for $W = Z^1_{2,4}$ by Lemma 2 (iv) since X'_0 is assumed to be nontrivial or empty.

Now assume that $U \in \{X'_1, \ldots, X'_5\}$, say $U = X'_1$. Then, by Lemma 9, $U⓪Z_0$ since $X_0 \cap X'_1 = \emptyset$. This completes the case analysis when U is a basic set of 0-vertices.

Next assume that $U \in \{Y_1, \ldots, Y_5\}$. If also $W \in \{Y_1, \ldots, Y_5\}$ then $U①W$ or $U⓪W$ by Lemma 1 (iv).

Now, without loss of generality, let $U = Y_1$, and assume that W is a basic subset from another basic subgraph. Since G is K_3-free, $U⓪Z_{4,1} \cup Z_{1,3}$. By Lemma 2 (i), $U⓪Z^0_{5,2}$. By definition of the basic subsets $Z^a_{2,4;bc}$ and $Z^a_{3,5;bc}$, U has join or co-join to all these basic subsets.

Finally, the connections between basic 2-vertex subsets from different basic subgraphs are join or co-join by Lemma 1 (iii), by definition of $Z^1_{i,i+2}$ and by the definition of B_i and G_0. Thus, $Z^1_{i,i+2}$ has a join to $Z_{i-1,i+1}$ and $Z_{i+1,i+3}$ and a co-join to $Z_{i-2,i}$ and $Z_{i+2,i+4}$, $i \in \{1, \ldots, 5\}$.

This shows Lemma 10. □

An immediate consequence of Lemma 9 and Lemma 10 is the following decomposition of G which is the main result of this paper (notation of basic subsets and basic subgraphs as above):

Theorem 1 (Structure Theorem) *Let G be a prime (P_6,K_3)-free graph which is not bipartite, and let C be a C_5 in G. Then the vertex set of $G[V \setminus (V(C) \cup X_T)]$ can be partitioned into the (possibly empty) basic subgraph $G_0 = G[Z_0 \cup X_0]$ and into the five (possibly empty) basic bipartite subgraphs $B_1, \ldots, B_5$ such that the connections between the basic vertex subsets of different basic subgraphs are only join or co-join.*

Recall that G_0 consists of the bipartite subgraph $G[Z_0]$ which, according to Corollary 2, is either the disjoint union of two co-matched bipartite graphs or fulfills $|Z^0_{i,i+2}| = 1$ for $i \in \{1, \ldots, 5\}$, and the 0-vertices in X_0 being adjacent to Z^0.

5 Bounded Clique-Width of (P_6,K_3)-free Graphs

The P_4-free graphs (also called *cographs*) play a fundamental role in graph decomposition; see [8] for linear time recognition of cographs, [6, 7, 8] for more information on P_4-free graphs and [4] for a survey on this graph class and related ones.

For a cograph G, either G or its complement is disconnected, and the *cotree* of G expresses how the graph is recursively generated from single vertices by repeatedly applying join and co-join operations. Note that the cographs are those graphs whose modular decomposition tree contains only join and co-join nodes as internal nodes.

Based on the following operations on vertex-labeled graphs, namely

(i) create a vertex u labeled by integer ℓ, denoted by $\ell(u)$,

(ii) disjoint union (i.e., co-join), denoted by $\oplus$,

(iii) join between all vertices with label i and all vertices with label j for $i \neq j$, denoted by $\eta_{i,j}$, and

(iv) relabeling all vertices of label i by label j, denoted by $\rho_{i \to j}$,

the notion of *clique-width* $cwd(G)$ of a graph G is defined in [9] as the minimum number of labels which are necessary to generate G by using these operations. It is easy to see that cographs are exactly the graphs whose clique-width is at most two.

A k-*expression* for a graph G of clique-width k describes the recursive generation of G by repeatedly applying these operations using at most k different labels.

Proposition 1 ([10, 11])

(i) *The clique-width cwd*(G) *of a graph G is the maximum of the clique-width of its prime induced subgraphs if G has nontrivial prime subgraphs.*

(ii) $cwd(\overline{G}) \leq 2 \cdot cwd(G)$.

In [10], it is shown that every problem expressible in a certain kind of Monadic Second Order Logic, called *LinEMSOL*($\tau_{1,L}$), is linear-time solvable on any graph class with bounded clique-width for which a k-expression can be constructed in linear time.

Roughly speaking, MSOL(τ_1) is Monadic Second Order Logic with quantification over subsets of vertices but not of edges; MSOL($\tau_{1,L}$) is the extension of MSOL(τ_1) with the addition of labels added to the vertices, and LinEMSOL($\tau_{1,L}$) is the extension of MSOL($\tau_{1,L}$) which allows to search for sets of vertices which are optimal with respect to some linear evaluation functions. The problems Vertex Cover, Maximum Weight Stable Set, Maximum Weight Clique, Steiner Tree, Domination and Maximum Induced Matching are examples of LinEMSOL($\tau_{1,L}$) expressible problems.

Theorem 2 ([10]) *Let $\mathcal{C}$ be a class of graphs of clique-width at most k such that there is an $\mathcal{O}(f(|E|,|V|))$ algorithm, which for each graph G in $\mathcal{C}$, constructs a k-expression defining it. Then for every LinEMSOL*($\tau_{1,L}$) *problem on $\mathcal{C}$, there is an algorithm solving this problem in time $\mathcal{O}(f(|E|,|V|))$.*

Observe that, trivially, the clique-width of a graph with n vertices is at most n. The following result by Johansson gives a slightly sharper bound.

Lemma 11 ([15]) *If G has n vertices then $cwd(G) \leq n - k$ as long as $2^k + 2k \leq n$.*

Thus, for instance, the clique-width of a graph with nine vertices is at most seven. Another helpful tool is

Lemma 12 ([1]) *Let $G = (V, E)$ be a graph and $V = F_1 \cup F_2$ be a partition of V with $|F_2| \leq s$ for some s. If there is a t-expression for $G[F_1]$ then there is a $(2^s \cdot (t + 1))$-expression for G.*

This lemma means that adding a constant number s of vertices to a graph H from a class of bounded clique-width maintains bounded clique-width. This allows to disregard certain specific vertices and thus to reduce graph G to its *essential part* G'.

We also need the following principle:

Principle 1 ([3]) *Let $G = (V, E)$ be a graph and $V = V_1 \cup \ldots \cup V_p$ be a partition of V. If there is a t-expression for G then there is a $p \cdot t$-expression for G such that finally for each $i \in \{1, 2, \ldots, p\}$, vertices in V_i get the same label and $l(u) \neq l(v)$ for each pair $u \in V_i$, $v \in V_j$, $i \neq j$.*

Our clique-width analysis of (P_6,K_3)-free graphs is based on the following results by Lozin [17] and, slightly earlier, by Fouquet, Giakoumakis and Vanherpe [12].

Theorem 3 ([17]) *The clique-width of bipartite $S_{1,2,3}$-free graphs is at most 5.*

Theorem 4 ([12]) *The clique-width of bipartite P_6-free graphs is at most 4, and given such a graph, a 4-expression can be constructed in linear time.*

We also need:

Proposition 2 *The clique-width is at most 4 for matched co-bipartite as well as for co-matched bipartite graphs, and corresponding k-expressions, $k \leq 4$, can be obtained in linear time.*

The proof of Proposition 2 is straightforward.

To give an example, we describe how a graph $G = (X \cup Y \cup Z, E)$ with pairwise disjoint vertex sets X, Y, Z, each of size n, consisting of a prime co-matched bipartite graph $B = (X, Y, E)$ with coedges $x_i y_i$, $i \in \{1, \ldots, n\}$, and with an additional neighbor $z_i \in Z$, $i \in \{1, \ldots, n\}$, to each coedge $x_i y_i$, can be constructed with 6 labels:

$\alpha_1 := \rho_{5\to6}(\eta_{5,2}(\eta_{5,1}(5(z_1) \oplus (1(x_1) \oplus 2(y_1)))))$

For $i := 2$ to n let

$\alpha_i := \rho_{4\to2}(\rho_{3\to1}(\rho_{5\to6}(\eta_{3,2}(\eta_{4,1}(\eta_{5,4}(\eta_{5,3}(5(z_i) \oplus (3(x_i) \oplus (4(y_i) \oplus \alpha_{i-1})))))))))$.

Now, Theorem 1 implies:

Corollary 3 *The clique-width of (P_6,K_3)-free graphs is bounded.*

Proof: If G is bipartite then by Theorem 4, its clique-width is at most 4. Now assume that G is prime and not bipartite. Then Theorem 1 describes its structure. If all sets of 0-, 1- and 2-vertices are trivial then G has at most 16 vertices and, by Lemma 11, its clique-width is at most 13.

The clique-width of G_0 is at most 6 as the example above shows, and for every $i \in \{1, \ldots, 5\}$, B_i is bipartite and thus, by Theorem 4, its clique-width is at most 4.

Now, applying Principle 1, the 2-vertex subsets in the basic subgraphs G_0 and B_i are subdivided into the four basic subsets. It is clear that there are k-expressions for G_0 and B_i where the basic subsets finally get different labels. By Theorem 1, the edge sets between basic subsets from different basic subgraphs are only join or co-join. Thus, finally these edges can easily be generated. This gives bounded clique-width for G. □

A more detailed analysis shows that the clique-width bound for (P_6,K_3)-free graphs can be improved to 36 since in all cases, some of the subsets are trivial.

In [18], it is shown that ($S_{1,1,3}$,K_3)-free graphs as well as ($S_{1,2,2}$,K_3)-free graphs have bounded clique-width.

The clique-width of (P_6,K_4)-free graphs is unbounded since in [2], it is shown that the clique-width for the smaller class of ($2K_2$,K_4)-free graphs is unbounded (the $2K_2$ is the complement of C_4). In the same paper, it is mentioned that for P_7-free bipartite graphs, it is unknown whether the clique-width of these graphs is bounded or unbounded. It also seems to be unknown whether the clique-width of (P_7,K_3)-free graphs is bounded or unbounded.

6 Time Bound for Robustly Constructing a k-Expression for G

By a result of Giakoumakis and Vanherpe [13], P_6-free bipartite graphs can be recognized in linear time. By Theorem 4, the clique-width of these graphs is at most 4, and given such a graph, a 4-expression can be constructed in linear time. From now on, we focus on nonbipartite graphs.

The aim of this section is to give an efficient algorithm which for arbitrary prime nonbipartite input graph G either determines a k-expression of G or finds out that G is not (P_6,K_3)-free. The time bound of our algorithm is $\mathcal{O}(n^2)$. Algorithm 1 does not check whether the input graph is indeed (P_6,K_3)-free; instead, it tries to find an induced C_5 or K_3 or P_6 in G, and if a C_5 is found, it checks whether G fulfills the conditions of the Structure Theorem. Thus, for constructing a k-expression for G, it is of crucial importance to find a C_5 in G.

Algorithm 1:
Input: An arbitrary prime nonbipartite graph $G = (V, E)$.
Output: An induced C_5 or K_3 or P_6 in G.

(1) Pick a vertex $v \in V$ and determine the levels $N^i(v)$, $i \geq 1$, by applying Breadth-First Search to G with start vertex v.

(2) Check whether $N^5(v) \neq \emptyset$. If yes then G contains a P_6 - STOP. {Otherwise, from now on, $N^k(v) = \emptyset$ for $k \geq 5$.}

(3) Check whether $N(v)$ is a stable set. If not then G contains a K_3 - STOP.

(4) Check whether $N^2(v)$ is a stable set. If not then let $x \sim y$ for some $x, y \in N^2(v)$.

(4.1) Check whether x and y have a common neighbor in $N(v)$. If yes then G contains a K_3 - STOP.

(4.2) Otherwise, let u_x (u_y) be a neighbor of x (y) in $N(v)$. Then v, u_x, x, y, u_y is a C_5 - STOP.

(5) Check whether $N^3(v)$ is a stable set. If not then let $x \sim y$ for some $x, y \in N^3(v)$.

(5.1) Check whether x and y have a common neighbor in $N^2(v)$. If yes then G contains a K_3 - STOP.

(5.2) Otherwise, let u_x (u_y) be a neighbor of x (y) in $N^2(v)$. Check whether u_x and u_y have a common neighbor w in $N(v)$. If yes then w, u_x, x, y, u_y is a C_5 - STOP. Otherwise, let w_x (w_y) be a neighbor of u_x (u_y) in $N(v)$. Then w_x, u_x, x, y, u_y, w_y is a P_6 - STOP.

(6) $\{$Now, since G is not bipartite, $N^4(v)$ is not a stable set.$\}$ Determine an edge $x \sim y$ for some $x, y \in N^4(v)$.

(6.1) Check whether x and y have a common neighbor in $N^3(v)$. If yes then G contains a K_3 - STOP.

(62) Otherwise, let u_x (u_y) be a neighbor of x (y) in $N^3(v)$. Check whether u_x and u_y have a common neighbor w in $N^2(v)$. If yes then w, u_x, x, y, u_y is a C_5 - STOP. Otherwise, let w_x (w_y) be a neighbor of u_x (u_y) in $N^2(v)$. Then w_x, u_x, x, y, u_y, w_y is a P_6 - STOP.

Theorem 5 *Algorithm 1 is correct and works in time $\mathcal{O}(n^2)$.*

Proof: *Correctness*: Assume that G is a nonbipartite graph and $N^k(v)$, $k \geq 1$, is a hanging of G with start vertex v. Obviously, if $N^5(v) \neq \emptyset$ then G contains a P_6. Otherwise, since G is not bipartite, one of the levels $N^k(v)$, $1 \leq k \leq 4$, contains an edge $x \sim y$. Let k_0 be the smallest index k such that $N^k(v)$ contains an egde. If $k_0 = 1$ then G contains a K_3; if $k = 2$ then G contains a K_3 or C_5 depending on the question whether x and y have a common neighbor in $N(v)$; if $k = 3$ then G contains a K_3, C_5 or P_6 depending on the criteria given in steps (5.1), (5.2) of the algorithm. Finally, if $k = 4$ then again G contains a K_3, C_5 or P_6 depending on the criteria given in steps (6.1), (6.2) of the algorithm. This shows the correctness of the algorithm.

Time bound: Step (1): Breadth-First Search for one start vertex v can be done in linear time $\mathcal{O}(n + m)$.
Steps (2), (3), (4), (5) and (6): can obviously be done in linear time.
Steps (4.1) and (4.2), (5.1) and (5.2), (6.1) and (6.2): can obviously be done in time $\mathcal{O}(n^2)$. $\square$

If Algorithm 1 ends with a C_5 C then the next task is to classify the vertices not in C as k-vertices, $0 \leq k \leq 5$. If there is a k-vertex for $k \in \{3, 4, 5\}$ then G contains a K_3. Otherwise, check whether C together with its 0-, 1- and 2-vertices fulfill all the conditions of the Structure Theorem. If not then G is not (P_6,K_3)-free. Otherwise, a k-expression for G can be constructed by Corollary 3.

7 Conclusion

In this paper, we give a complete structure description of (prime) (P_6,K_3)-free graphs. Moreover, we show that the clique-width of these graphs is bounded, and we give a robust algorithm which, for an arbitrary nonbipartite input graph G, either constructs a corresponding k-expression of G if Algorithm 1 returns a C_5 and G fulfills the conditions of the Structure Theorem or states that G does not fulfill these conditions (in which case it cannot be (P_6,K_3)-free) or finds an induced P_6 or K_3 in G. The running time of this algorithm is at most $\mathcal{O}(n^2)$.

The fact that (P_6,K_3)-free graphs have bounded clique-width can be extended to (P_6,paw)-free graphs by the following observation by Olariu [22]: A graph G is paw-free if and only if each component of G is either triangle-free or complete multipartite.

Moreover, in [23], Olariu has observed that if a prime graph contains a triangle then it contains a house, bull or double-gem (these are the minimal prime extensions of the non-prime paw graph). Thus also (P_6,house,bull,double-gem)-free graphs have bounded clique-width.

It remains a challenging open problem whether there is a linear time algorithm for constructing k-expressions of (P_6,K_3)-free graphs (k-expressions for (P_6,house,bull,double-gem)-free graphs, respectively) and whether the class of ($S_{1,2,3}$,K_3)-free graphs has bounded clique-width.

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