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Recurrence among trees with most numerous efficient dominating sets

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A dominating set D of vertices in a graph G is called an efficient dominating set if the distance between any two vertices in D is at least three. A tree T of order n is called maximum if T has the largest number of efficient dominating sets among all n-vertex trees. A constructive characterization of all maximum trees is given. Their structure has recurring aspects with period 7. Moreover, the number of efficient dominating sets in maximum n-vertex trees is determined and is exponential. Also the number of maximum n-vertex trees is shown to be bounded below by an increasing exponential function in n.
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1 Introduction

Our aim is to maximize the number of efficient dominating sets among n-vertex trees. This is a sequel to our article [4]. Paper [4] presents characterizations of trees with largest or smallest (and still large) numbers of dominating sets. Trees without any efficient dominating set exist. Determining whether or not a graph has an efficient dominating set is an NP-complete problem, see [2, 8]. A constructive characterization of trees which have an efficient dominating set can be found in [2]. That is why we concentrate on characterizing trees with the largest number of efficient dominating sets. There are several papers devoted to characterizing graphs with maximal numbers of specified substructures. Erdős is credited for raising a question which is answered in the pioneering paper [9] of 1965 by Moon and Moser on maximizing the number of cliques among n-vertex graphs. There is a series of publications on the related problem of maximizing the number of maximal independent sets among different graphs, and trees also are considered [16, 10, 17]. At the beginning of 1980s Tomescu, when dealing with cliques in specialized hypergraphs, studied [14] number-theoretical problem of maximizing a constrained product of binomial coefficients, the simplest possible subcase of the problem being solved by Moon and Moser [9]. The second author, when dealing with maximizing the number of maximum path-factors among trees, arrived independently [12] at another subcase of Tomescu's problem. A solution to Tomescu's problem was found by Schinzel and Skupień, cf. [13].

Given a graph G, a vertex set $S \subseteq V(G)$ is called a *dominating set* of G if any vertex of G either is in S or has a neighbor in S. Following Bange et al. [1], a dominating set D of G is called an EDS (*efficient dominating set*) if the distance between any two members of D is at least three in G. Bange, Barkauskas and Slater [2] proved the following result.

Proposition 1 *If G has an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number of G. In particular, all efficient dominating sets have the same cardinality.*

The number of efficient dominating sets in oriented graphs, and in particular, oriented trees was studied [3, 11].

Observation 1 *Given any dominating set D of a graph G, if $v \in V(G)$ then either $v \in D$ or a neighbor of v is in D. If D is any* EDS *and $v \notin D$ then exactly one neighbor of v is in D.* $\square$

Let $\mathcal{T}_n$ be the class of unlabeled trees of order n, where isomorphic trees are considered identical. In what follows T stands for a tree with vertex set $V(T)$, $|T|$ denotes the cardinality of $V(T)$. A *leaf* (or *pendant vertex*) and a B*-vertex* (or *branch vertex*) are vertices of degree at most one and at least three, respectively. Let $B(T)$ and $b(T)$ denote respectively the *set* and *number of B-vertices* in T. A *twig* in T is a nontrivial path joining a leaf of T to the closest vertex in T which is not of degree two. Hence the *number of twigs*, say $\theta(T)$, either is the number of leaves if $b(T) > 0$ or is one less than the number of leaves if T is a path (whether trivial or not). Given an $i \in \{0, 1, 2\}$ and $x \in B(T)$, let $\tau_i(x)$ be the *number of twigs* attached to x with lengths congruent to i modulo 3.

Let T have a B-vertex. Consider the subtree obtained by ungluing all twigs from T (i.e., by deleting all non-B-vertices of twigs together with all incident edges). Then every pendant vertex x of the subtree is called a PB*-vertex* (or a *pendant B-vertex*) of T. Let $PB(T)$ stand for the *set of pendant branch vertices* in T.

Let $q(T)$ denote the *number of all efficient dominating sets* of T. Given a positive integer n, let $\hat{q}(n)$ denote the *maximum* of $q(T)$ over all trees T on n vertices, $n = |T|$. Then each n-vertex tree such that $q(T) = \hat{q}(n)$ is called a *maximum tree.*

We shall characterize all maximum trees T on n vertices. If $n \geq 9$ then the structure of T is quite easy to describe. Namely, each maximum tree T has uniquely defined number b of branch vertices, $b = b(n)$. Branch vertices induce a subtree, and all remaining vertices of T are on twigs of length two only. To define $b = b(n)$ for any $n \geq 39$ or odd $n \geq 33$, let $\tilde{b} = \lfloor \frac{n+2}{7} \rfloor$. Then b is an integer such that $b \equiv n \pmod 2$ and $b \in \{\tilde{b}, \tilde{b} - 1\}$. Note that if $7|n$ and $n \geq 35$ then $b = n/7$. Moreover, $b(n) = b$ for all $n = 7b + 2s$ with $s = -1, 0, 1, \ldots, 5$ provided that $b \geq 5$. Thus s, which determines some structural aspects, is a periodic function of n with period 7. Our main result is that any tree on b vertices is a subtree induced by branch vertices of an n-vertex maximum tree with n such that $b = b(n)$. Additionally we give a detailed specification of the following estimation. The number of maximum trees on n vertices is exponential in n because, for $n \geq 39$, it is at least the number $|\mathcal{T}_b|$ of all trees on b vertices with $b = b(n) \sim n/7$ as $n \to \infty$, the lower bound being attained if $7|n$ and $n \geq 35$. We determine $\hat{q}(n)$, and $\hat{q}(n)$ is exponential in n, too.

EDS' and maximum trees can possibly be used to design computer or communication networks, facility and guard locations, surveillance systems, interference-free transmission and/or cooperation, cf. [7, p. 891].

2 Preliminaries

Using Proposition 1 we can see that in case of a nontrivial path P_n ($n \geq 2$) each efficient dominating set of P_n contains no endvertex, both endvertices and exactly one endvertex depending on whether n is congruent modulo 3 to $0, 1, 2$, respectively. Hence the number of efficient dominating sets in P_n is

$$q(P_n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod 3, \\ 2 & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{1}$$

Assume that T is a tree with a twig P of length 3 or more. Let z be a leaf of T in P and $wxyz$ a subpath in P. Then $wxyz$ is called a *terminal subtwig* in T. The transformation $T \mapsto T - \{x, y, z\}$ is called a 3-*reduction* of T. Recursive repetition of a 3-reduction eventually gives a subtree of T, say $\bar{T}$, which does not have any twig of length 3 or more. Call $\bar{T}$ to be the 3-*trim* of T.

Proposition 2 *Any* 3*-reduction $T \mapsto T'$ does not change the number of EDS', i.e., $q(T') = q(T)$.*

Proof: Given a leaf z of T such that $wxyz$ is a terminal subtwig in T, let $T' = T - \{x, y, z\}$. Due to Observation 1 and the definition of EDS, if D is an EDS of T then $\{z, w\} \subseteq D$ or, if $T \neq P_4$, $\{y, u\} \subseteq D$ where u is a neighbor of w in T'. Hence, for each D, $D - \{z, y\}$ is an EDS of T'. Conversely, if D' is any EDS of T' then, due to Observation 1 (with $v = w$ in T'), D' is uniquely augmentable to an EDS of T. Therefore $D \mapsto D - \{z, y\}$ is a bijection from EDS' in T onto those in T'. □

Corollary 1 *$q(\bar{T}) = q(T)$ if $\bar{T}$ is the* 3*-trim of T.* □

Recall that for $x \in B(T)$, $\tau_i(x)$ is the number of twigs of length congruent to i modulo 3, $i = 0, 1, 2$, twigs being attached to x.

Proposition 3 *Let T be a tree with exactly one branch vertex x. Let $\tau_i = \tau_i(x)$. Then*

$$q(T) = \begin{cases} 0 & \text{if} \quad \tau_2 \geq 1 \quad \text{and} \quad \tau_1 \geq 2, \\ 1 & \text{if} \quad \tau_2 \geq 1 \quad \text{and} \quad \tau_1 = 1 \quad \text{or} \ \ \tau_2 = 0 \ \ \text{and} \ \ \tau_1 \geq 2, \\ & \qquad\qquad\qquad\qquad\qquad\quad\ \text{or} \ \ \tau_2 = \tau_1 = 0, \\ 2 & \text{if} \quad \tau_2 = 0 \quad \text{and} \quad \tau_1 = 1, \\ \tau_2 & \text{if} \quad \tau_2 \geq 1 \quad \text{and} \quad \tau_1 = 0. \end{cases}$$

□

Corollary 2 *Let T be an n-vertex tree with $b(T) = 1$, $n \geq 7$, $n \neq 8$, and with $q(T)$ being the largest possible number of EDS'. Then $\tau_1 = 0$ and $q(T) = \tau_2 \geq 3$ where*

$$\tau_2 = \begin{cases} \frac{n-1}{2} & \text{for odd } n, \\ \frac{n-4}{2} & \text{for even } n, \end{cases}$$

whence all twigs are of length 2 *but one of length* 3 *for even n only.* □

3 Useful recursive relations

We are going to present a recurrence relations for the number $q(T)$. Assume that $b(T) \geq 2$ and x is a PB-vertex. Then there exists a unique vertex $a(x)$ which is the neighbor of x on a path from x to another B-vertex in T. The vertex $a(x)$ is said to be the *attachment vertex* of x. Let $T - *x$ be the tree obtained from T by deleting the PB-vertex x together with all twigs attached to x. Call $T - *x$ to be tree T *minus star at* x. Let $T_{a(x)+i}$ stand for a tree obtained from $T - *x$ by adding a new y–$a(x)$ path of length $i = 0, 1, 2$, where $y = a(x)$ if $i = 0$. Let $q(T_{a(x)+i}, y)$ be the count of all efficient dominating sets that contain y. The following result is easily seen due to Observation 1.

Lemma 1 *Let $b(T) \geq 2$ and $\tau_0 = 0$ at all B-vertices of T. Assume that $x \in PB(T)$, $a(x)$ is the attachment vertex of x, and $\tau_i = \tau_i(x)$ for $i = 1, 2$. Let y be the endvertex of path y–$a(x)$ of length $j = 0, 1, 2$ in the tree $T_{a(x)+j}$. Then*

$$q(T) = \begin{cases} 0 & \textit{if} \quad \tau_1 \geq 2 \;\; \textit{and} \;\; \tau_2 \geq 1, \\ q(T_{a(x)+1}, y) & \textit{if} \quad \tau_1 \geq 2 \;\; \textit{and} \;\; \tau_2 = 0, \\ q(T_{a(x)+2}, y) & \textit{if} \quad \tau_1 = 1 \;\; \textit{and} \;\; \tau_2 \geq 1, \\ \tau_2(x) q(T_{a(x)+2}, y) + q(T_{a(x)+0}, y) & \textit{if} \quad \tau_1 = 0 \;\; \textit{and} \;\; \tau_2 \geq 2. \end{cases}$$

□

One can easily check that orders of trees $T_{a(x)+1}$ and $T_{a(x)+2}$ involved in Lemma 1 are at most $|T| - 2$. Using some of the above results and inspecting the list of small trees in Harary [5] gives the following.

Proposition 4 *All trees T on n vertices with $n \leq 10$ which are maximum, i.e., with $q(T) = \hat{q}(|T|)$, are listed in Fig. 1, T being unique if $n \neq 4, 6, 8$.* □

The symbol $S(G)$ (which appears in Fig. 1) stands for subdivision graph of G. Recall that $S(G)$ results in inserting a new vertex of degree 2 into each edge of G.

A vertex x of a tree T is called *EDS-avoided* if x does not belong to any efficient dominating set of T.

Proposition 5 *Each tree of order $n \geq 3$ has an EDS-avoided vertex. In fact, each vertex at distance 2 from a leaf is EDS-avoided.* □

Lemma 2 *Let $b(T) \geq 2$, $x \in PB(T)$, $\tau_1(x) = \tau_0(x) = 0$, and $\tau_2(x) \geq 2$. Then*

$$q(T) \leq \tau_2(x)\, q(T - *x),$$

where T minus star at x is involved, and equality is true if and only if $a(x)$, the attachment vertex of x, is EDS-avoided in $T - *x$ (i.e., $q(T_{a(x)+0}, y) = 0$).

Proof: On applying Lemma 1 we have

$$q(T) = q(T_{a(x)+0}, y) + \tau_2(x)\, q(T_{a(x)+2}, y). \tag{2}$$

Due to Observation 1, one can see that $q(T - *x) = q(T_{a(x)+0}, y) + q(T_{a(x)+2}, y)$. Hence, by (2),

$$\begin{aligned} q(T) &= q(T_{a(x)+0}, y) + \tau_2(x)\,[q(T - *x) - q(T_{a(x)+0}, y)] \\ &= \tau_2(x)\, q(T - *x) + [1 - \tau_2(x)]\, q(T_{a(x)+0}, y) \\ &\leq \tau_2(x)\, q(T - *x), \end{aligned}$$

where equality holds whenever $q(T_{a(x)+0}, y) = 0$, which ends the proof. □

For $n = 1, 2, 3, 5,\ \ T = P_n$ (uniquely), $\hat{q} \in \{1, 2\}$,

$n = 4, \hat{q} = 1, \quad T = P_4$ or $K_{1,3}$

$n = 6, \hat{q} = 1, \ T = P_6, K_{1,5},$

$n = 7, \hat{q} = 3, \qquad S(K_{1,3})$:

$n = 8, \hat{q} = 2, \ \ T = P_8,$

$n = 9, \hat{q} = 4, \qquad S(K_{1,4})$:

$n = 10, \hat{q} = 4, \qquad \tilde{H}$:

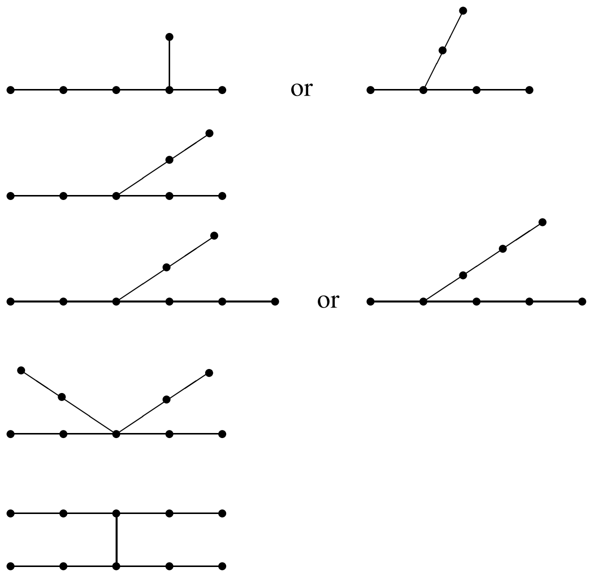

Fig. 1: All maximum n-vertex trees T with $n \leq 10$

4 Preliminary maximization

Let $\mathcal{T}^2$ be the family of unlabeled trees T such that $b(T) \geq 1$ with both $\tau_1 = \tau_0 = 0$ and $\tau_2 \geq 2$ at each B-vertex. Moreover, all $\theta(T)$ twigs are of length two and contain all degree-2 vertices of T. Let $\mathcal{T}_n^2$ be the subclass comprising n-vertex elements of $\mathcal{T}^2$. Let $T \in \mathcal{T}^2$ and let $b = b(T)$, $\theta = \theta(T)$. It follows that T includes a subtree, say T_b, on b vertices. If $b = 1$ then $\theta \geq 3$, $|T| = 1 + 2\theta \geq 7$ whence $|T|$ is odd and $T = S(K_{1,\theta})$, the unique subdivision of the star $K_{1,\theta}$. If $b \geq 2$ then $\theta \geq 2b$ and $|T| = b + 2\theta \geq 5b$. Hence

$$n_b := \min\{|T| \colon T \in \mathcal{T}^2, b(T) = b\} = \begin{cases} 7 & \text{for } \ b = 1, \\ 5b & \text{for } \ b \geq 2. \end{cases} \tag{3}$$

Proposition 6 *Let $T \in \mathcal{T}_n^2$. Then branch vertices induce a subtree $\langle B(T) \rangle$ of T and*

$$\begin{aligned} n &= |T| = b(T) + 2\,\theta(T), && (4)\\ b(T) &\equiv |T| \pmod 2, && (5)\\ q(T) &= \theta(T) \text{ if } b(T) = 1, && (6)\\ 7 &\leq |T| \neq 8, && (7)\\ \theta(T) &= \sum_{x \in B(T)} \tau_2(x) \geq \max\{2b(T), 3\} \geq 3. && (8) \end{aligned}$$

□

The following observation complements Proposition 5.

Proposition 7 *If $T \in \mathcal{T}^2$ then EDS-avoided vertices in T coincide with B-vertices.* □

From (6), Lemma 2 and Proposition 7, for $T \in \mathcal{T}^2$, we get the following.

$$q(T) = \prod_{x \in B(T)} \tau_2(x). \tag{9}$$

Observation 2 *The* RHS (*right-hand side*) *of* (9) *determines the corresponding tree T up to the distribution of twigs in the following sense.*

Consider a finite product $\prod_{i \in I} t_i$ of integers $t_i \geq 2$ where $|I| \geq 1$. Let $b = |I|$, the number of factors in the product, $\theta = \sum_{i \in I} t_i$, the sum of factors, and let $n = b + 2\theta$. Let T_b be any tree on b vertices with arbitrarily assigned mutually distinct labels x_i, $i \in I$. Let T be a tree obtained from T_b by attaching t_i twigs of length two to the vertex x_i for each $i \in I$. Hence $\tau_2(x_i) = t_i$, $T \in \mathcal{T}_n^2$, and $q(T) = \prod_{i \in I} t_i$.

For $n \geq n_b$, let $\mathcal{T}_n^2(b)$ comprise $T \in \mathcal{T}_n^2$ with $b(T) = b$ and such that $q(T)$ is the largest possible.

Lemma 3 *If $T \in \mathcal{T}_n^2(b)$ and $b > 1$ then the numbers τ_2 at any two B-vertices of T differ by at most one.*

Proof: If values of τ_2 are m and k such that $m > k + 1$ then $mk < (m-1)(k+1)$ whence making the values closer one to another (by moving a twig from one B-vertex to the other) increases the value of q. □

Corollary 3 *If $T \in \mathcal{T}_n^2(b)$ and $\theta = \theta(T)$ then*

$$\begin{aligned} \theta &= (n-b)/2 \quad \textit{by} \quad (4) \textit{ and } (5),\\ q(T) &= \lfloor \theta/b \rfloor^{b - (\theta \bmod b)} \lceil \theta/b \rceil^{\theta \bmod b} && (10) \end{aligned}$$

due to (8), (9), *and Lemma* 3. □

Let $q_2(n)$ denote the *maximum of* $q(T)$ among n-vertex trees $T \in \mathcal{T}^2$ where $n = |T|$ satisfies requirement (7). Let $\mathcal{T}^{2,\max} = \{T \in \mathcal{T}^2 : q(T) = q_2(|T|)\}$, each element of $\mathcal{T}^{2,\max}$ is called a $\mathcal{T}^2$-*maximum tree*. Let $\mathcal{T}_n^{2,\max}$ stand for the subset of $\mathcal{T}^{2,\max}$ comprising n-vertex elements.

Let $b_2(n)$ be the largest $b(T)$ among $\mathcal{T}^2$-maximum n-vertex trees T, $T \in \mathcal{T}_n^{2,\max}$. Small trees $T \in \mathcal{T}_n^2$ have the unique $b(T)$, either 1 or 2 (as determined by the parity of n), and then $q_2(n)$ can be obtained from (9). Namely (see Tab. 1),

$$b_2(n) = 1 \text{ and } q_2(n) = \frac{n-1}{2} \qquad \text{for odd } n, \quad n_1 = 7 \le n < n_3 = 15,$$
$$b_2(n) = 2 \text{ and } q_2(n) = \left\lfloor \frac{n-2}{4} \right\rfloor \cdot \left\lceil \frac{n-2}{4} \right\rceil \qquad \text{for even } n, \quad n_2 = 10 \le n < n_4 = 20.$$

Then $T \in \mathcal{T}_n^{2,\max}$ is unique.

Tab. 1:

n	7	9	10	11	12	13	14	16	18
$b_2(n)$	1	1	2	1	2	1	2	2	2
$q_2(n)$	3	4	$2 \cdot 2$	5	$2 \cdot 3$	6	$3 \cdot 3$	$3 \cdot 4$	$4 \cdot 4$

For larger n, if n is fixed and T ranges over $\mathcal{T}_n^2$ then, by (3) and (5), $b = b(T)$ ranges over all natural numbers $\le n/5$ such that $b \equiv n \pmod 2$. Then $q_2(n)$ can be found by inspection using Corollary 3. Namely, for each admissible value of b, we find θ and next $q(T)$. Then $q_2(n)$ is the largest of numbers $q(T)$ thus found. Moreover, the corresponding b equals $b_2(n)$.

Let RHS stand for the product on the right-hand side of formula (10). Then $b(T)$ is the sum of two exponents of which one can be 0, θ is the sum of the single factors in RHS, cf. (8). Consequently, formula (4) gives n.

The results of search for two largest products, $L < R$, where both L and R are RHS corresponding to a fixed n, are presented in Tab. 2 for some values of n as stated. Consequently, for those n, $R = q_2(n)$ whence the corresponding $b = b_2(n)$.

Tab. 2: Inequalities $L < R = q_2(n)$ for some $n \ge 15$

7	$<$	2^3,	$(b = 3)$	$\underline{n = 15}$;	2^4	$<$	$4 \cdot 5$,		$n = 20$;
8	$<$	$2^2 \cdot 3$,		$n = 17$;	$2^3 \cdot 3$	$<$	5^2,		$n = 22$;
9	$<$	$2 \cdot 3^2$,		$n = 19$;	$5 \cdot 6$	$<$	$2^2 \cdot 3^2$,	$(b = 4)$	$\underline{n = 24}$;
$2^3 \cdot 3^2$	$<$	$4^2 \cdot 5$,		$n = 29$;	6^2	$<$	$2 \cdot 3^3$,		$n = 26$;
$4 \cdot 5^2$	$<$	$2^2 \cdot 3^3$,	$(b = 5)$	$\underline{n = 31}$;	$2^3 \cdot 3^3$	$<$	4^4,		$n = 36$;
5^3	$<$	$2 \cdot 3^4$,		$n = 33$;	$4^3 \cdot 5$	$<$	$2^2 \cdot 3^4$,	$(b = 6)$	$\underline{n = 38}$;
$2^2 \cdot 3^5$	$<$	4^5,		$n = 45$;	$2^2 \cdot 3^6$	$<$	$3 \cdot 4^5$,		$n = 52$;
$4^4 \cdot 5$	$<$	$2 \cdot 3^6$,	$(b = 7)$	$\underline{n = 47}$;	4^6	$<$	$2 \cdot 3^7$,	$(b = 8)$	$\underline{n = 54}$.

Tab. 2 serves two purposes. Firstly, it suggests that the function b_2 weakly increases when restricted to n of either parity. Moreover, each value of b displayed within parentheses in Tab. 2 is first attained at the corresponding underlined value of n so that $b = b_2(n)$ for that n and all larger n which are smaller than the next underlined one. Secondly, if L is displayed at $n = k$ then no $q_2(n)$ with $n \ge k$ includes L as a subproduct. Otherwise replacing L in $q_2(n)$ by the corresponding $R = q_2(k)$ gives $q(T)$ as in formula (9) where $|T| = n$, $T \in \mathcal{T}_n^2$, and $q(T) > q_2(n)$, a contradiction.

Maximum product $R = q_2(n)$ and the corresponding $b = b_2(n)$ for each admissible $n \leq 40$ are presented in Tab. 3.

Theorem 1 *For each n such that $n = 7$ or $9 \leq n \leq 40$, Tab. 3 presents both the number $q_2(n)$ of efficient dominating sets and the number $b = b_2(n)$ of B-vertices in any $T \in \mathcal{T}_n^{2,\max}$.* □

Tab. 3:

n	b	$q_2(n)$	n	b	$q_2(n)$
7	1	3			
9	1	4	10	2	$4 = 2^2$
11	1	5	12	2	$6 = 2 \cdot 3$
13	1	6	14	2	$9 = 3^2$
15	3	$8 = 2^3$	16	2	$12 = 3 \cdot 4$
17	3	$12 = 2^2 \cdot 3$	18	2	$16 = 4^2$
19	3	$18 = 2 \cdot 3^2$	20	2	$20 = 4 \cdot 5$
21	3	$27 = 3^3$	22	2	$25 = 5^2$
23	3	$36 = 3^2 \cdot 4$	24	4	$36 = 2^2 \cdot 3^2$
25	3	$48 = 3 \cdot 4^2$	26	4	$54 = 2 \cdot 3^3$
27	3	$64 = 4^3$	28	4	$81 = 3^4$
29	3	$80 = 4^2 \cdot 5$	30	4	$108 = 3^3 \cdot 4$
31	5	$108 = 2^2 \cdot 3^3$	32	4	$144 = 3^2 \cdot 4^2$
33	5	$162 = 2 \cdot 3^4$	34	4	$192 = 3 \cdot 4^3$
35	5	$243 = 3^5$	36	4	$256 = 4^4$
37	5	$324 = 3^4 \cdot 4$	38	6	$324 = 2^2 \cdot 3^4$
39	5	$432 = 3^3 \cdot 4^2$	40	6	$486 = 2 \cdot 3^5$

5 Large trees

For a $T \in \mathcal{T}_n^{2,\max}$,

$$m_k := |\{x \in B(T) \colon \tau_2(x) = k\}|. \tag{11}$$

Proposition 8 *Let $T \in \mathcal{T}_n^{2,\max}$ where $n \geq 39$ or $n = 33, 35, 37$. Then $m_k = 0$ unless $k \in \{2,3,4\}$. Hence $b(T) = m_2 + m_3 + m_4$. Additionally, $m_2\, m_4 = 0$, $m_2 \leq 1$, $m_4 \leq 5$, and m_3 is unbounded. The stated bounds on n (odd $n \geq 33$ or any $n \geq 39$) are sharp because $m_2 = 2$ if $n = 31$ or 38.*

Proof: Tab. 1 and products L together with the corresponding values of n in Tab. 2 are chosen so as to show that, for a $T \in \mathcal{T}_n^{2,\max}$, all the factors $\tau_2 = k \neq 3$ have bounded multiplicity m_k. Namely, $m_k = 0$ for $k = 7, 8, 9$ by Tab. 2 and for $k = 7{+}r \geq 10$ because then $7{+}r < 3^2 r$ with $n = 15{+}2r$ and $b \in \{1, 3\}$. Furthermore, products L at $n = 24, 26$ show that $m_6 = 0$ if n is not too small. Actually $m_6 = 0$ for $n \geq 14$ and the bound is tight because $q_2(13) = 6$ (Tab. 1). Similarly, products L at $n = 38, 47$ and R at $n = 29$ show that $m_5 = 0$ for $n \geq 30$. Analogously, products L at $n = 45, 52$ show that $m_2 \leq 1$ for large

n. On the other hand, products R at $n = 47, 54$ show that $m_2 = 1$ is possible for large n. Products R at $n = 45, 52$ show that $m_4 \geq 5$ is allowed for large n but, due to L at $n = 54$, $m_4 = 5$ at most. Lemma 3 implies that $m_2 m_4 = 0$. The value $m_2 = 2$ in R for $n = 31$ and 38 shows sharpness of the stated bounds for n. □

A tree T is called a *large tree* if its order $|T| \geq 39$ or $|T| = 33, 35, 37$.

Theorem 2 *Let T be a large tree from $\mathcal{T}_n^{2,\max}$, $b = b(T)$, and $s = m_4 - m_2$ whence $s = -1, 0, \ldots, 5$, $s = -1$ if $m_2 = 1$, otherwise $s = m_4$. Then s uniquely depends on n (Tab. 4),*

$$s = \begin{cases} -1 & \textit{if } n \bmod 7 = 5, \\ \frac{1}{2} \cdot \textit{even}\,(n \bmod 7,\ 7 + (n \bmod 7)) & \textit{otherwise,} \end{cases} \tag{12}$$

$$b = b_2(n) = \frac{n - 2s}{7} \tag{13}$$

$$= \begin{cases} \frac{n+2}{7} & \textit{if } n \bmod 7 = 5, \\ \frac{n-(n \bmod 7)}{7} & \textit{if } n \bmod 7 = 0, 2, 4, 6, \\ \frac{n-7-(n \bmod 7)}{7} & \textit{if } n \bmod 7 = 1, 3, \end{cases} \tag{14}$$

$$q_2(n) = \begin{cases} 2 \cdot 3^{b-1} & \textit{if } n \bmod 7 = 5, \\ 3^{b-s} \cdot 4^s & \textit{otherwise} \end{cases}$$

$$= \begin{cases} 2 \cdot 3^{\frac{n-5}{7}} & \textit{if } n \bmod 7 = 5, \\ 3^{(n-9(7+(n \bmod 7))/2)/7} \cdot 4^{(7+(n \bmod 7))/2} & \textit{if } n \bmod 7 = 1, 3, \\ 3^{(n-9(n \bmod 7)/2)/7} \cdot 4^{(n \bmod 7)/2} & \textit{otherwise.} \end{cases} \tag{15}$$

Tab. 4:

s	-1	0	1	2	3	4	5
$n \bmod 7$	5	0	2	4	6	1	3

Proof: By Proposition 8, $m_3 = b - m_2 - m_4$. Hence, by (4) and (8), $n = b + 2\theta = b + 4m_2 + 6m_3 + 8m_4 = 7b + 2(m_4 - m_2) = 7b + 2s$ whence (13). Consequently $7|n - 2s$. Hence, by Proposition 8, we get Tab. 4 whence (12) and (14) follow. On the other hand,

$$q_2(n) = 2^{m_2}\, 3^{m_3}\, 4^{m_4},$$

where $m_2 m_4 = 0$, which implies (15). □

Theorem 3 *For $n \geq 38$, $q_2(n+1) > q_2(n)$.*

Proof: The inequality holds for $n = 38$, see Tab. 3. Otherwise, using formula (15) we get the following quotients all larger than 1.

$$\frac{q_2(n+1)}{q_2(n)} = \begin{cases} 256/243 & \text{if} \quad n \bmod 7 = 0, 2, \\ 9/8 & \text{if} \quad n \bmod 7 = 4, \\ 32/27 & \text{if} \quad n \bmod 7 = 5, \\ 81/64 & \text{if} \quad n \bmod 7 = 1, 3, 6. \end{cases}$$

In order to see this result, consider the case $n \bmod 7 = r, r = 0, \ldots, 6$. Next on finding $(n+1) \bmod 7$, use formula (15). $\square$

The function q_2 is not monotone because of its small values precisely at $n = 15, 22, 29$. Moreover, $q_2(n+1) = q_2(n)$ precisely for $n = 23, 30, 37$. Nevertheless, the following important result holds.

Lemma 4 *For $n \geq 7$ and $n \neq 8$, $q_2(n+3) > q_2(n)$.*

Proof: The result follows from Tab. 3 for $n \leq 37$; otherwise from Theorem 3. $\square$

6 Main results

For maximum trees T with $|T| \leq 10$, see Proposition 4.

Theorem 4 *Maximum trees T with $|T| \geq 7$ and $|T| \neq 8$ are $\mathcal{T}^2$-maximum.*

Proof: Theorem is true for $n = 7, 9, 10$ by Proposition 4. Let T be a maximum tree of order $n \geq 11$. Then $q(T) \geq q_2(n) \geq 5$ by Tab. 3 and Lemma 4. Due to Corollary 2, for even $n \geq 10$ each n-vertex tree T with $b(T) = 1$ has $q(T) \leq \frac{n-4}{2} < q_2(n)$ by Tab. 3 and Theorem 2. Therefore $b(T) = 1$ is possible only if n is odd and then by Corollary 2 maximum tree is $\mathcal{T}^2$-maximum.

We proceed by induction on n. Consider the case $b(T) \geq 2$. For any fixed $x \in B(T)$, let $\tau_i = \tau_i(x)$, $i = 0, 1, 2$.

A. All twigs in T have length 1 or 2 (whence $\tau_0 = 0$). Due to Lemma 1, there is no $x \in PB(T)$ with $\tau_1 \geq 2$ and $\tau_2 \geq 1$. Hence precisely two following subcases A1, A2 are possible.

A1. There exists an $x \in PB(T)$ such that (i) $\tau_1 \geq 2$ and $\tau_2 = 0$ or (ii) $\tau_1 = 1$ and $\tau_2 \geq 1$. Let

$$T' = \begin{cases} T - \{\text{two leaves at } x\} & \text{in case } (i), \\ T - \{\text{twig of length 2 at } x\} & \text{in case } (ii). \end{cases}$$

Then $|T'| = |T| - 2 \geq 9$. We consider two subcases $\tau_1 = 2$, $\tau_1 > 2$ in case (i), two ones $\tau_2 = 1$, $\tau_2 > 1$ in case (ii), we use Lemma 1, and we see that $q(T) \leq q(T')$. Let T'' be a maximum tree of order $|T'|$. Then $T'' \in \mathcal{T}^2$. Attaching a twig of length 2 to a PB-vertex of T'' gives a tree $T^\star$ of order $|T|$ such that $T^\star \in \mathcal{T}^2$ and, by (9), $q(T^\star) > q(T'') \geq q(T') \geq q(T)$, a contradiction.

A2. $\tau_1 = 0$ and $\tau_2 \geq 2$ for each $x \in PB(T)$. Let $n = 11$. Consider an auxiliary 6-vertex graph H of the capital letter H. Then $T = S(H)$, the subdivision of H, is the only possibility. However, $q(T) = 1 < q_2(11) = 5$, a contradiction, cf. Tab. 3.
Let $n \geq 12$. Assume that $x \in PB(T)$ and degree of x is as small as possible. Let $T' = T - *x$. Then $|T| - |T'| = 2\tau_2(x) + 1$ which is odd and ≥ 5. Hence, by the choice of x, $|T'| \geq |T|/2$ and $|T'| \geq 7$. Moreover, by Lemma 2, $q(T) \leq \tau_2 \cdot q(T')$ where the equality holds whenever the attachment vertex $a(x)$ is EDS-avoided in T' and T' is a maximum tree since so is T. Therefore if $|T'| \neq 8$ then, by induction hypothesis, $T' \in \mathcal{T}^2$ whence, by Proposition 7, $T \in \mathcal{T}^2$ (as required). Otherwise $|T'| = 8$, each maximum T' has an EDS-avoided vertex and $q(T') = 2$, cf. Proposition 4. Consequently, $q(T) = 2\tau_2(x)$. Hence, by Observation 2, $q(T) = q(T'')$ for some $T'' \in \mathcal{T}^2$ with $b(T'') = 2$ and $|T''| = |T| - 3$, a contradiction in view of Lemma 4.

B. T has a twig of length 3 or more. Assume that a tree T' is a 3-reduction of T whence $|T'| = |T| - 3 \geq 8$. By Proposition 2, $q(T') = q(T)$. Moreover, T' is clearly a maximum tree on $|T| - 3$ vertices. If $|T'| = 8$ then $q(T') = 2$ by Proposition 4. However, $|T| = 11$ and $q(T) \geq 5$ by Tab. 3 whence $q(T) \neq q(T')$, a contradiction. Otherwise $|T'| \geq 9$ and $T' \in \mathcal{T}^2$ by the induction hypothesis. Due to Lemma 4, $q_2(|T'| + 3) > q(T')$, a contradiction to $q(T) = q(T')$.

□

Theorem 5 *The number of maximum n-vertex trees is bounded below by the number $|\mathcal{T}_b|$ of b-vertex trees where $b = b(n) \sim n/7$ as $n \to \infty$. The lower bound is exponential in n and is attained if $7|n$ and $n \geq 35$.*

Proof: Let n be a natural number, $n \geq 39$ or $n = 33, 35, 37$. Then $n = 7b + 2s$ with $s = -1, 0, \ldots, 5$ and $b \geq 5$ being uniquely defined in (12) and (13), respectively. Due to Theorem 4, any maximum n-vertex tree, T, is an element of $\mathcal{T}_n^{2,\max}$. The structural parameters m_k of T, defined in formula (11), are determined in Proposition 8 and Theorem 2 in terms of s. Hence, due to very definition of the class $\mathcal{T}^2$ in Sect. 4 and Observation 2, the tree T is obtained in the following way. Choose any tree T' on b vertices, select a subset S of $|s|$ vertices in T'; for $s \neq 0$, attach two or four twigs of length two to each vertex in S according as $s = -1$ or $s > 0$, and, finally, attach three such twigs to each of remaining vertices in T'. Then the resulting graph includes T' as an induced subgraph and has really $7b + 2s$ vertices as required. Define $r_t(T')$ to be the number of distinct selections of t vertices in T', $t \leq b$, two such selections being distinct if no automorphism of T' transforms one selection into another. Therefore

$$|\mathcal{T}_n^{\max}| = \sum_{T' \in \mathcal{T}_b} r_{|s|}(T'). \tag{16}$$

If $n \bmod 7 = 0$ and $n \geq 35$ then, by Theorem 2, $b = \frac{n}{7}$ and T has exactly three twigs of length two at every B-vertex. Hence $|\mathcal{T}_n^{\max}| = |\mathcal{T}_b|$. In remaining cases, by formula (16), the number of maximum n-vertex trees is greater than $|\mathcal{T}_b|$. Therefore $|\mathcal{T}_n^{\max}| \geq |\mathcal{T}_b|$. The lower bound is exponential in n since so is the cardinality of the class of b-vertex unlabeled trees [6, Sect. 9.5]. □

Corollary 4 *Formula* (16) *gives the number of maximum n-vertex trees for large enough n.* □

7 Concluding remarks

Let T_n be an n-vertex tree with largest possible number of EDS'. Let $n \geq 7$ and $n \neq 8$. Then $T_n \in \mathcal{T}_n^{2,\max}$ and T_n has the following properties. Each leaf is on a twig of length 2. At least two and at most six twigs are attached to each branch vertex of T_n. Numbers of twigs at different branch vertices differ by 0 or 1 only. Branch vertices induce a subtree, say T_n", $T_n" = \langle B(T_n) \rangle$, and the number, b, of branch vertices depends on n only, see Tab. 3 and formula (14) for values of b. Moreover, each tree T_b' of order $b \geq 1$ is T_n" for a certain T_n. Let $b \geq 5$ and let $n = 7b-2, 7b, \ldots, 7b+10$. Then there is a T_n with $T_n" \cong T_b'$ for each T_b' and each of listed n only. Furthermore, if $s := (n-7b)/2$ then $s \in \{-1, 0, \ldots, 5\}$ and if three twigs are attached to each of any $n - |s|$ vertices of any T_b', and $3 + \mathrm{sign}(s)$ twigs (2 or 4) to each of remaining $|s|$ vertices, then the resulting tree is a T_n. Conversely, for each of listed n's and any T_n, T_n" is of order b.

Notice in this context the importance of Observation 2. It shows how to pass from an integer q to a variety of trees T such that $q(T) = q$ (but to a single T only, if q is a prime).

We conclude this paper with a few open problems that we find interesting. Let $d_G(x, y)$ stand for the distance between vertices x and y in G. Let p be a positive integer, $p \geq 1$. A vertex subset S^*, $S^* \subseteq V(G)$, is called an *efficient p-dominating set* of G if the following two properties are satisfied.

- for each $x \notin S^*$, there exists $y \in S^*$ at distance at most p from x, $d_G(x, y) \leq p$,
- the distance between any two vertices of S^* is at least $2p + 1$ (i.e. $d_G(x, y) \geq 2p + 1$), if $x, y \in S^*$ and $x \neq y$.

Thus an efficient dominating set is efficient 1-dominating.

Open problems. For any integer $p \geq 2$, characterize trees

(1) which have an efficient p-dominating set,

(2) which have a fixed order and the largest number of efficient p-dominating sets.

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