Independent Sets in Graphs with an Excluded Clique Minor

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Let G be a graph with n vertices, with independence number α, and with no K_{t+1}-minor for some $t \geq 5$. It is proved that $(2\alpha - 1)(2t - 5) \geq 2n - 5$. This improves upon the previous best bound whenever $n \geq \frac{2}{5}t^2$.

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1 Introduction

In 1943, Hadwiger [7] made the following conjecture, which is widely considered to be one of the most important open problems in graph theory(i); see [19] for a survey.

Hadwiger's Conjecture. For every integer $t \geq 1$, every graph with no K_{t+1}-minor is t-colourable. That is, $\chi(G) \leq \eta(G)$ for every graph G.

Hadwiger's Conjecture is trivial for $t \leq 2$, and is straightforward for $t = 3$; see [4, 7, 22]. In the cases $t = 4$ and $t = 5$, Wagner [20] and Robertson et al. [16] respectively proved that Hadwiger's Conjecture is equivalent to the Four-Colour Theorem [2, 3, 6, 15]. Hadwiger's Conjecture is open for all $t \geq 6$.

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(i) All graphs considered in this note are undirected, simple and finite. Let G be a graph with vertex set $V(G)$. Let $X \subseteq V(G)$. X is *connected* if the subgraph of G induced by X is connected. X is *dominating* if every vertex of $G \setminus X$ has a neighbour in X. X is *independent* if no two vertices in X are adjacent. The *independence number* $\alpha(G)$ is the maximum cardinality of an independent set of G. X is a *clique* if every pair of vertices in X are adjacent. The *clique number* $\omega(G)$ is the maximum cardinality of a clique in G. A *k-colouring* of G is a function that assigns one of k colours to each vertex of G such that adjacent vertices receive distinct colours. The *chromatic number* $\chi(G)$ is the minimum integer k such that G is k-colourable. A *minor* of G is a graph that can be obtained from a subgraph of G by contracting edges. The *Hadwiger number* $\eta(G)$ is the maximum integer n such that the complete graph K_n is a minor of G.

Progress on the $t = 6$ case has been recently been obtained by Kawarabayashi and Toft [10] (without using the Four-Colour Theorem). The best known upper bound is $\chi(G) \leq c \cdot \eta(G)\sqrt{\log \eta(G)}$ for some constant c, independently due to Kostochka [11] and Thomason [17, 18].

Woodall [21] observed that since $\alpha(G) \cdot \chi(G) \geq |V(G)|$ for every graph G, Hadwiger's Conjecture implies that

$$\alpha(G) \cdot \eta(G) \geq |V(G)|. \tag{1}$$

Equation (1) holds for $\eta(G) \leq 5$ since Hadwiger's Conjecture holds for $t \leq 5$. For example, $\alpha(G) \geq \frac{1}{4}|V(G)|$ for every planar graph G. It is interesting that the only known proof of this result depends on the Four-Colour Theorem. The best bound not using the Four-Colour Theorem is $\alpha(G) \geq \frac{2}{9}|V(G)|$ due to Albertson [1].

Equation (1) is open for $\eta(G) \geq 6$. In general, (1) is weaker than Hadwiger's Conjecture, but for graphs with $\alpha(G) = 2$ (that is, graphs whose complements are triangle-free), Plummer et al. [13] proved that (1) is in fact equivalent to Hadwiger's Conjecture. The first significant progress towards (1) was made by Duchet and Meyniel [5] (also see [12]), who proved that

$$(2\alpha(G) - 1) \cdot \eta(G) \geq |V(G)| \ . \tag{2}$$

This result was improved by Kawarabayashi et al. [8] to

$$(2\alpha(G) - 1) \cdot \eta(G) \geq |V(G)| + \omega(G) \ . \tag{3}$$

Assuming $\alpha(G) \geq 3$, Kawarabayashi et al. [8] proved that

$$(4\alpha(G) - 3) \cdot \eta(G) \geq 2|V(G)|, \tag{4}$$

which was further improved by Kawarabayashi and Song [9] to

$$(2\alpha(G) - 2) \cdot \eta(G) \geq |V(G)|. \tag{5}$$

The following theorem is the main contribution of this note.

Theorem 1 *Every graph G with $\eta(G) \geq 5$ satisfies*

$$(2\alpha(G) - 1)(2\eta(G) - 5) \geq 2|V(G)| - 5 \ .$$

Observe that Theorem 1 represents an improvement over (2), (4) and (5) whenever $\eta(G) \geq 5$ and $|V(G)| \geq \frac{2}{5}\eta(G)^2$. For example, Theorem 1 implies that $\alpha(G) > \frac{1}{7}|V(G)|$ for every graph G with $\eta(G) \leq 6$, whereas each of (2), (4) and (5) imply that $\alpha(G) > \frac{1}{12}|V(G)|$.

2 Proof of Theorem 1

Theorem 1 employs the following lemma by Duchet and Meyniel [5]. The proof is included for completeness.

Lemma 1 ([5]) *Every connected graph G has a connected dominating set D and an independent set $S \subseteq D$ such that $|D| = 2|S| - 1$.*

Proof: Let D be a maximal connected set of vertices of G such that D contains an independent set S of G and $|D| = 2|S| - 1$. There is such a set since $D := S := \{v\}$ satisfies these conditions for each vertex v. We claim that D is dominating. Otherwise, since G is connected, there is a vertex v at distance 2 from D, and there is a neighbour w of v at distance 1 from D. Let $D' := D \cup \{v, w\}$ and $S' := S \cup \{v\}$. Thus D' is connected and contains an independent set S' such that $|D'| = 2|S'| - 1$. Hence D is not maximal. This contradiction proves that D is dominating. □

The next lemma is the key to the proof of Theorem 1.

Lemma 2 *Suppose that for some integer $t \geq 1$ and for some real number $p \geq t$, every graph G with $\eta(G) \leq t$ satisfies $p \cdot \alpha(G) \geq |V(G)|$. Then every graph G with $\eta(G) \geq t$ satisfies*

$$\alpha(G) \geq \frac{2|V(G)| - p}{4\eta(G) + 2p - 4t} + \frac{1}{2} \ .$$

Proof: We proceed by induction on $\eta(G) - t$. If $\eta(G) = t$ the result holds by assumption. Let G be a graph with $\eta(G) > t$. We can assume that G is connected. By Lemma 1, G has a connected dominating set D and an independent set $S \subseteq D$ such that $|D| = 2|S| - 1$. Now $\alpha(G) \geq |S| = \frac{|D|+1}{2}$. Thus we are done if

$$\frac{|D| + 1}{2} \geq \frac{2|V(G)| - p}{4\eta(G) + 2p - 4t} + \frac{1}{2} \ . \tag{6}$$

Now assume that (6) does not hold. That is,

$$|D| \leq \frac{2|V(G)| - p}{2\eta(G) + p - 2t} \ .$$

Thus

$$|V(G \setminus D)| = |V(G)| - |D| \geq \frac{(2\eta(G) + p - 2t - 2)|V(G)| + p}{2\eta(G) + p - 2t} \ .$$

Since D is dominating and connected, $\eta(G \setminus D) \leq \eta(G) - 1$. Thus by induction,

$$\begin{aligned}
\alpha(G) \geq \alpha(G \setminus D) &\geq \frac{2|V(G \setminus D)| - p}{4\eta(G \setminus D) + 2p - 4t} + \frac{1}{2} \\
&\geq \frac{2(2\eta(G) + p - 2t - 2)|V(G)| + 2p}{(2\eta(G) + p - 2t)(4\eta(G) - 4 + 2p - 4t)} - \frac{p}{4\eta(G) - 4 + 2p - 4t} + \frac{1}{2} \\
&= \frac{2|V(G)| - p}{4\eta(G) + 2p - 4t} + \frac{1}{2} \ .
\end{aligned}$$

This completes the proof. □

Lemma 3 *Suppose that Hadwiger's Conjecture is true for some integer t. Then every graph G with $\eta(G) \geq t$ satisfies*

$$(2\eta(G) - t)(2\alpha(G) - 1) \geq 2|V(G)| - t \ .$$

Proof: If Hadwiger's Conjecture is true for t then $t \cdot \alpha(G) \geq |V(G)|$ for every graph G with $\eta(G) \leq t$. Thus Lemma 2 with $p = t$ implies that every graph G with $\eta(G) \geq t$ satisfies

$$\alpha(G) \geq \frac{2|V(G)| - t}{4\eta(G) - 2t} + \frac{1}{2} \ ,$$

which implies the result. □

Theorem 1 follows from Lemma 3 with $t = 5$ since Hadwiger's Conjecture holds for $t = 5$ [16].

3 Concluding Remarks

The proof of Theorem 1 is substantially simpler than the proofs of (3)–(5), ignoring its dependence on the proof of Hadwiger's Conjecture with $t = 5$, which in turn is based on the Four-Colour Theorem. A bound that still improves upon (2), (4) and (5) but with a completely straightforward proof is obtained from Lemma 3 with $t = 3$: Every graph G with $\eta(G) \geq 3$ satisfies $(2\eta(G) - 3)(2\alpha(G) - 1) \geq 2|V(G)| - 3$.

We finish with an open problem. The method of Duchet and Meyniel [5] was generalised by Reed and Seymour [14] to prove that the fractional chromatic number $\chi_f(G) \leq 2\eta(G)$. For sufficiently large $\eta(G)$, is $\chi_f(G) \leq 2\eta(G) - c$ for some constant $c \geq 1$?

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