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A Determinant of Stirling Cycle Numbers Counts Unlabeled Acyclic Single-Source Automata

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We show that a determinant of Stirling cycle numbers counts unlabeled acyclic single-source automata. The proof involves a bijection from these automata to certain marked lattice paths and a sign-reversing involution to evaluate the determinant. We also give a formula for the number of acyclic automata with a given set of sources.

Keywords: Stirling cycle number, unlabeled acyclic automaton, column-marked subdiagonal path, sign-reversing involution

1 Introduction

The chief purpose of this paper is to show bijectively that a determinant of Stirling cycle numbers counts unlabeled acyclic single-source automata. Specifically, let $A_k(n)$ denote the $kn \times kn$ matrix with (i, j) entry $\left[{\lfloor \frac{i-1}{k} \rfloor + 2 \atop \lfloor \frac{i-1}{k} \rfloor + 1 + i - j}\right]$, where $\left[{i \atop j}\right]$ is the Stirling cycle number, that is, the number of permutations on $[i] := \{1, 2, \ldots, i\}$ with j cycles. For example,

$$A_2(5) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 11 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 & 11 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 10 & 35 & 50 & 24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & 35 & 50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 15 & 85 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 15 \end{pmatrix}.$$

As evident in the example, $A_k(n)$ is formed from k copies of each of rows 2 through $n + 1$ of the Stirling cycle triangle, arranged so that the first nonzero entry in each row is a 1 and, after the first row, this 1 occurs just before the main diagonal; in other words, $A_k(n)$ is a Hessenberg matrix with 1s on the infra-diagonal.

Main Theorem. *The determinant of $A_k(n)$ is the number of unlabeled acyclic single-source automata with n transient states on a $(k+1)$-letter input alphabet.*

Section 2 reviews basic terminology for automata and recurrence relations to count finite acyclic automata, and includes a new formula for the number of acyclic automata with a given set of sources. Section 3 introduces column-marked subdiagonal paths, which play an intermediate role, and a way to code them. Section 4 presents a bijection from these column-marked subdiagonal paths to unlabeled acyclic single-source automata. Finally, Section 5 evaluates $\det A_k(n)$ using a sign-reversing involution and shows that the determinant counts the codes for column-marked subdiagonal paths.

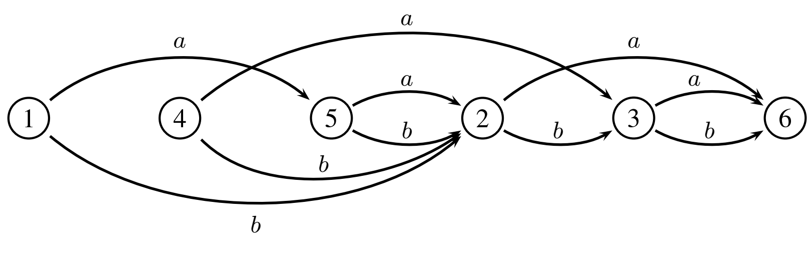
2 Automata

A (complete, deterministic) automaton consists of a set of states and an input alphabet whose letters transform the states among themselves: a letter and a state produce another state (possibly the same one). A finite automaton (finite set of states, finite input alphabet of, say, k letters) can be represented as a k-regular directed multigraph with ordered edges: the vertices represent the states and the first, second, ... edge from a vertex give the effect of the first, second, ... alphabet letter on that state. A finite automaton cannot be acyclic in the usual sense of no cycles: pick a vertex and follow any path from it. This path must ultimately hit a previously encountered vertex, thereby creating a cycle. So the term acyclic is used in the looser sense that only one vertex, called the *sink*, is involved in cycles. This means that all edges from the sink loop back to itself (and may safely be omitted) and all other paths feed into the sink.

A non-sink state is called *transient*. The *size* of an acyclic automaton is the number of transient states. An acyclic automaton of size n thus has transient states which we label $1, 2, \ldots, n$ and a sink, labeled $n+1$. Liskovets [1] uses the inclusion-exclusion principle (also used below) to obtain the following recurrence relation for the number $a_k(n)$ of acyclic automata of size n on a k-letter input alphabet ($k \geq 1$):

$$a_k(0) = 1; \qquad a_k(n) = \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n}{j} (j+1)^{k(n-j)} a_k(j), \quad n \geq 1.$$

A *source* is a vertex (state) with no incoming edges. A finite acyclic automaton has at least one source because a path traversed backward $v_1 \leftarrow v_2 \leftarrow v_3 \leftarrow \ldots$ must have distinct vertices and so cannot continue indefinitely. An *interior* vertex is one that is neither the sink nor a source. An acyclic automaton with 6 states and input alphabet a, b is represented below as a multigraph with the vertices (states) on a horizontal line.



It has two sources, 1 and 4; the interior states are 2,3, 5 and the sink is 6. Observe that the vertices are so arranged that all edges point to the right. This is always possible for a finite acyclic automaton (see Corollary 4 below).

Proposition 1 *The number* $b_k(n, j)$ *of acyclic automata of size* n *on a* k*-letter input alphabet with source set* $\{1, 2, \ldots, j\}$ $(1 \le j \le n)$ *is given by*

$$b_k(n, j) = \sum_{i=0}^{n-j} (-1)^{n-i+j} \binom{n-j}{i} (i+1)^{k(n-i)} a_k(i)$$

Remark For the case $j = 1$, this formula is a bit more succinct than the recurrence in [1, Theorem 3.2].

Proof: Consider the set $\mathcal{A}$ of acyclic automata with transient vertices $[n] = \{1, 2, \ldots, n\}$ in which the sources include $\{1, 2, \ldots, j\}$. Call $\{1, 2, \ldots, j\}$ the *principal* sources. For $X \subseteq [j + 1, n]$, let

$$\begin{aligned} f(X) &= \text{\# automata in } \mathcal{A} \text{ whose set of non-principal sources includes } X, \\ g(X) &= \text{\# automata in } \mathcal{A} \text{ whose set of non-principal sources is precisely } X. \end{aligned}$$

Then $f(X) = \sum_{Y:\, X \subseteq Y \subseteq [j+1,n]} g(Y)$ and by Möbius inversion [2] on the lattice of subsets of $[j + 1, n]$, $g(X) = \sum_{Y:\, X \subseteq Y \subseteq [j+1,n]} \mu(X, Y) f(Y)$ where $\mu(X, Y)$ is the Möbius function for this lattice. Since $\mu(X, Y) = (-1)^{|Y|-|X|}$ if $X \subseteq Y$, we have in particular that

$$g(\emptyset) = \sum_{Y \subseteq [j+1,n]} (-1)^{|\,Y\,|} f(Y). \tag{1}$$

Let us evaluate the sum in (1) by the size of Y; $|\,Y\,|$ ranges from 0 to $n - j$. For $|\,Y\,| = n - i$ with $j \le i \le n$, there are $\binom{n-j}{n-i} = \binom{n-j}{i-j}$ choices for Y. The vertices in $[n + 1]\backslash Y$ together with their incident edges form an acyclic automaton with i transient states in which $\{1, 2, \ldots, j\}$ are sources; clearly, there are $(i - j + 1)^{kj} a_k(i - j)$ such automata because the graph obtained by deleting the vertices $1, 2, \ldots, j$ and their incident edges is a subautomaton of size $i - j$. Also, all edges from the $n - i$ vertices comprising Y go directly into the set $[j + 1, n + 1]\backslash Y$: $(i - j + 1)^{k(n-i)}$ choices. Thus $f(Y) = (i - j + 1)^{kj} a_k(i - j)(i - j + 1)^{k(n-i)}$. Hence,

$$g(\emptyset) = \sum_{i=j}^{n} (-1)^{n-i} \binom{n-j}{i-j} (i - j + 1)^{k(n-i+j)} a_k(i - j). \tag{2}$$

Now, by definition, $g(\emptyset)$ is the number of automata in $\mathcal{A}$ for which $1, 2, \ldots, j$ are the only sources, that is, $g(\emptyset) = b_k(n, j)$ and thus the Proposition follows from (2) by a change in summation index. $\square$

A *standard* j*-source* automaton is one with precisely j sources labelled $1, 2, \ldots, j$. Let $\mathcal{B}_k(n, j)$ denote the set of standard j-source acyclic automata of size n, that is, the acyclic automata on a k-letter input alphabet with sources $1, 2, \ldots, j$, interior vertices $j + 1, j + 2, \ldots, n$ and sink $n + 1$. The *two-line representation* of an automaton in $\mathcal{B}_k(n, j)$ is the $2 \times kn$ matrix whose top row lists the vertices in order and whose columns then list the edges from each vertex in order. For example,

$$B = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 \\ 2 & 4 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 3 & 5 & 3 & 2 & 2 & 6 \end{pmatrix}$$

is in $\mathcal{B}_3(5,1)$ (here 1 is the only source) and the source-to-sink paths in B include $1 \xrightarrow{a} 2 \xrightarrow{a} 6,\ 1 \xrightarrow{b} 4 \xrightarrow{c} 3 \xrightarrow{a} 6,\ 1 \xrightarrow{b} 4 \xrightarrow{b} 5 \xrightarrow{b} 2 \xrightarrow{b} 6$, where the alphabet is $\{a,b,c\}$.

An *unlabeled* standard j-source automaton is an equivalence class of standard j-source automata under relabeling of the interior vertices. Liskovets notes [1] that, when $j = 1$, $\mathcal{B}_k(n,j)$ has no nontrivial automorphisms, that is, each of the $(n-j)!$ relabelings of the interior vertices of $B \in \mathcal{B}_k(n,j)$ produces a different automaton. We prove this for general j below. It follows that unlabeled standard j-source automata of size n on a k-letter alphabet are counted by $\frac{1}{(n-j)!}b_k(n,j)$ and, in particular, the formula for $b_k(n,j)$ in Prop. 1 always gives a number divisible by $(n-j)!$. The next result establishes a canonical representative in each relabeling class.

Proposition 2 *Each equivalence class in $\mathcal{B}_k(n,j)$ under relabeling of interior vertices has size $(n-j)!$ and contains exactly one standard j-source automaton with the "last occurrences increasing" property: the last occurrences of the interior vertices—$j+1, j+2, \ldots, n$—in the bottom row of its two-line representation occur in that order.*

Proof: The first assertion follows from the fact that the interior vertices of an automaton $B \in \mathcal{B}_k(n,j)$ can be distinguished intrinsically, that is, independent of their labeling. To see this, first mark the sources, namely $1, 2, \ldots, j$, with marks (new labels) $v_1, v_2, \ldots, v_j$ and observe that there exists at least one interior vertex whose only incoming edge(s) are from a source (the only currently marked vertices) for otherwise a cycle would be present. For each such interior vertex v, choose the last edge from a marked vertex to v where edges are in the left-to-right order of the two-line representation. This determines an order on these vertices; mark them in order $v_{j+1}, v_{j+2}, \ldots, v_\ell$. If there still remain unmarked interior vertices, at least one of them has incoming edges only from a marked vertex or again a cycle would be present. For each such vertex, use the last incoming edge from a marked vertex, where now edges are arranged in order of the subscript on the initial vertex with the built-in order breaking ties, to order and mark these vertices $v_{\ell+1}, v_{\ell+2}, \ldots$. Proceed similarly until all interior vertices are marked. For example, with $j = 1$ and

$$B = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 \\ 2 & 4 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 3 & 5 & 3 & 2 & 2 & 6 \end{pmatrix},$$

there is only one source and so $v_1 = 1$. There is just one interior vertex, namely 4, whose only incoming edge is from the source, and so $v_2 = 4$ and 4 becomes a marked vertex. Now all incoming edges to both 3 and 5 are from marked vertices and the last such edges (built-in order comes into play) are $4 \xrightarrow{b} 5$ and $4 \xrightarrow{c} 3$ putting vertices $3, 5$ in the order $5, 3$. So $v_3 = 5$ and $v_4 = 3$. Finally, $v_5 = 2$. This proves the first assertion.

By construction of the vs, relabeling each interior vertex with the subscript of its corresponding v produces an automaton in $\mathcal{B}_k(n,j)$ with the "last occurrences increasing" property and is the only relabeling that does so. The example yields

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 \\ 5 & 2 & 6 & 4 & 3 & 4 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix}.$$

□

Proposition 3 *In a standard j-source finite acyclic automaton with the "last occurrences increasing" property, every edge $a \to b$ satisfies $a < b$.*

Proof: Suppose to the contrary that $a < b$ and $b \to a$ is an edge. Then the "last occurrences increasing" property implies that there is an occurrence of b to the right of a in the bottom row of the two-line representation as sketched.

$$\begin{array}{ccccc} \ldots & b & \ldots & c & \ldots \\ \ldots & a & \ldots & b & \ldots \end{array}$$

The edge $c \to b$ coming into this b necessarily has $b < c$ and, iterating, we would get an infinite sequence of vertices $a < b < c < \ldots$. □

Corollary 4 *For every finite acyclic automaton there is an ordering of the transient states with the sources coming first such that, when the vertices are arranged horizontally in this order, every edge points to the right.* □

An automaton is *single-source* (or *initially connected*) if it has only one source. Let $\mathcal{B}_k(n) := \mathcal{B}_k(n, 1)$ denote the set of single-source acyclic automata on a k-letter input alphabet with vertices $1, 2, \ldots, n+1$ where 1 is the source and $n+1$ is the sink, and set $b_k(n) = |\,\mathcal{B}_k(n)\,|$. Also, let $\mathcal{C}_k(n)$ denote the subset of $\mathcal{B}_k(n)$ representing unlabeled automata canonically as in Prop. 2; thus $|\,\mathcal{C}_k(n)\,| = \frac{1}{(n-1)!} b_k(n)$. Henceforth, we identify an unlabeled automaton with its canonical representative.

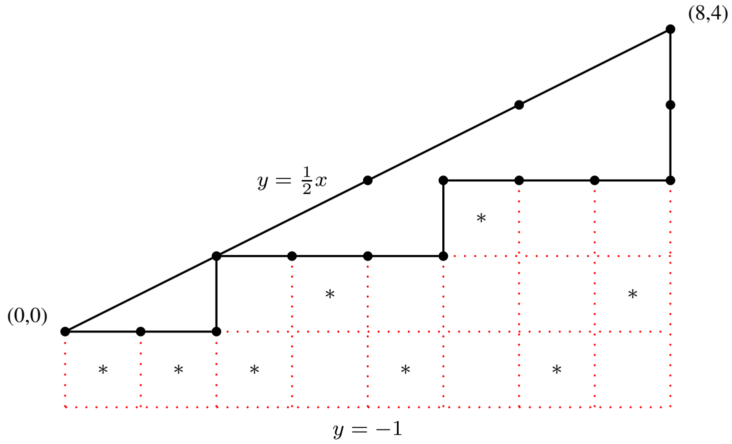
3 Column-Marked Subdiagonal Paths

A *subdiagonal* (k, n, p)-path is a lattice path of steps $E = (1, 0)$ and $N = (0, 1)$, E for east and N for north, from $(0, 0)$ to (kn, p) that never rises above the line $y = \frac{1}{k}x$. Let $\mathsf{C}_k(n, p)$ denote the set of such paths. For $k \geq 1$, it is clear that $\mathsf{C}_k(n, p)$ is nonempty only for $0 \leq p \leq n$ and the generalized ballot theorem (see, for example, [3, Theorem 3B]) implies that

$$|\,\mathsf{C}_k(n, p)\,| = \frac{kn - kp + 1}{kn + p + 1}\binom{kn + p + 1}{p}.$$

A path P in $\mathsf{C}_k(n, n)$ can be coded by the heights of its E steps above the line $y = -1$; this gives a sequence $(b_i)_{i=1}^{kn}$ subject to the restrictions $1 \leq b_1 \leq b_2 \leq \ldots \leq b_{kn}$ and $b_i \leq \lceil i/k \rceil$ for all i.

A *column-marked* subdiagonal (k, n, p)-path is one in which, for each $i \in [1, kn]$, one of the lattice squares below the ith E step and above the horizontal line $y = -1$ is marked, say with a '$*$'. Let $\mathsf{C}_k^*(n, p)$ denote the set of such marked paths.



A path in $\mathsf{C}_2^*(4,4)$

A marked path P^* in $\mathsf{C}_k^*(n,n)$ can be coded by a sequence of pairs $\big((a_i,b_i)\big)_{i=1}^{kn}$ where $(b_i)_{i=1}^{kn}$ is the code for the underlying path P and $a_i \in [1,b_i]$ gives the position of the $*$ in the ith column. The example is coded by $(1,1),\ (1,1),\ (1,2),\ (2,2),\ (1,2), (3,3),\ (1,3),\ (2,3)$.

An explicit sum for $|\,\mathsf{C}_k^*(n,n)\,|$ is

$$|\,\mathsf{C}_k^*(n,n)\,| = \sum_{\substack{1 \le b_1 \le b_2 \le \ldots \le b_{kn},\\ b_i \,\le\, \lceil i/k \rceil \text{ for all } i}} b_1 b_2 \ldots b_{kn},$$

because the summand $b_1 b_2 \ldots b_{kn}$ is the number of ways to insert the '$*$'s in the underlying path coded by $(b_i)_{i=1}^{kn}$.

In the next section we exhibit a bijection from $\mathsf{C}_k^*(n,n)$ to $\mathcal{C}_{k+1}(n)$. This bijection implies, in particular, that the number of unlabeled single-source acyclic automata on a 2-letter alphabet is given by

$$|\,\mathcal{C}_2(n)\,| = |\,\mathsf{C}_1^*(n,n)\,| = \sum_{\substack{1 \le b_1 \le b_2 \le \ldots \le b_n\\ b_i \,\le\, i \text{ for all } i}} b_1 b_2 \ldots b_n = (1,3,16,127,1363,\ldots)_{n\ge 1},$$

sequence A082161 in [4].

4 Bijection from Paths to Automata

In this section we exhibit a bijection from $\mathsf{C}_k^*(n,n)$ to $\mathcal{C}_{k+1}(n)$. Using the path illustrated in the previous section as a working example with $k=2$ and $n=4$, first construct the top row of a two-line representation consisting of $k+1$ each 1s, 2s, $\ldots, n$ s and number them left to right for reference:

$$\begin{pmatrix} \overset{1}{1} & \overset{2}{1} & \overset{3}{1} & \overset{4}{2} & \overset{5}{2} & \overset{6}{2} & \overset{7}{3} & \overset{8}{3} & \overset{9}{3} & \overset{10}{4} & \overset{11}{4} & \overset{12}{4} \\ & & & & & & & & & & & \end{pmatrix}.$$

The last step in the path is necessarily an N step. For the second last, third last,... N steps in the path, count the number of steps following it. This gives a sequence $i_1, i_2, \dots, i_{n-1}$ satisfying $1 \le i_1 < i_2 < \dots < i_{n-1}$ and $i_j \le (k+1)j$ for all j. The example yields $(i_j)_{j=1}^3 = (1,5,9)$. Circle the positions $i_1, i_2, \dots, i_{n-1}$ in the two-line representation and then insert (in boldface) $2, 3, \dots, n$ in increasing order in the second row in the circled positions:

$$\begin{pmatrix} \overset{1}{1} & \overset{2}{1} & \overset{3}{1} & \overset{4}{2} & \overset{5}{2} & \overset{6}{2} & \overset{7}{3} & \overset{8}{3} & \overset{9}{3} & \overset{10}{4} & \overset{11}{4} & \overset{12}{4} \\ \mathbf{2} & & & & \mathbf{3} & & & & \mathbf{4} & & & \end{pmatrix}.$$

These will be the last occurrences of $2, 3, \dots, n$ in the second row. Working from the last column in the path back to the first, fill in the blanks in the second row *left to right* as follows. Count the number of squares from the $*$ up to the path (including the $*$ square) and add this number to the nearest boldface number to the left of the current blank entry (if there are no boldface numbers to the left, add this number to 1) and insert the result in the current blank square. In the example the numbers of squares are respectively 2,3,1,2,1,2,1,1 yielding

$$\begin{pmatrix} \overset{1}{1} & \overset{2}{1} & \overset{3}{1} & \overset{4}{2} & \overset{5}{2} & \overset{6}{2} & \overset{7}{3} & \overset{8}{3} & \overset{9}{3} & \overset{10}{4} & \overset{11}{4} & \overset{12}{4} \\ \mathbf{2} & 4 & 5 & 3 & \mathbf{3} & 5 & 4 & 5 & \mathbf{4} & 5 & 5 & \end{pmatrix}.$$

This will fill all blank entries except the last. Note that $*$ s in the bottom row correspond to sink (that is, to $n+1$) labels in the second row. Finally, insert $n+1$ into the last remaining blank space to give the image automaton:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\ 2 & 4 & 5 & 3 & 3 & 5 & 4 & 5 & 4 & 5 & 5 & 5 \end{pmatrix}.$$

This process is fully reversible and the map is a bijection. □

5 Evaluation of det $A_k(n)$

For simplicity, we treat the case $k = 1$, leaving the generalization to arbitrary k as a not-too-difficult exercise for the interested reader. Write $A(n)$ for $A_1(n)$. Thus $A(n) = \left(\left[{i+1 \atop 2i-j} \right] \right)_{1 \le i,j \le n}$. From the definition of $\det A(n)$ as a sum of signed products, we show that $\det A(n)$ is the total weight of certain lists of permutations, each list carrying weight ± 1. Then a weight-reversing involution cancels all -1 weights and reduces the problem to counting the surviving lists. These surviving lists are essentially the codes for paths in $\mathsf{C}_1^*(n,p)$, and the Main Theorem follows from §4.

To describe the permutations giving a nonzero contribution to $\det A(n) = \sum_\sigma \operatorname{sgn} \sigma \times \prod_{i=1}^n a_{i,\sigma(i)}$, define the *code* of a permutation σ on $[n]$ to be the list $\mathbf{c} = (c_i)_{i=1}^n$ with $c_i = \sigma(i) - (i-1)$. Since the (i,j) entry of $A(n)$, $\left[{i+1 \atop 2i-j} \right]$, is 0 unless $j \ge i-1$, we must have $\sigma(i) \ge i-1$ for all i. It is well known that there are 2^{n-1} such permutations, corresponding to compositions of n, with codes characterized by the following four conditions: (i) $c_i \ge 0$ for all i, (ii) $c_1 \ge 1$, (iii) each $c_i \ge 1$ is immediately followed by $c_i - 1$ zeros in the list, (iv) $\sum_{i=1}^n c_i = n$. Let us call such a list a *padded composition* of n: deleting the zeros is a bijection to ordinary compositions of n. For example, $(3,0,0,1,2,0)$ is a padded composition of 6. For a permutation σ with padded composition code $\mathbf{c}$, the nonzero entries in $\mathbf{c}$ give the cycle lengths of σ. Hence $\operatorname{sgn} \sigma$, which is the parity of "$n - \#$ cycles in σ", is given by $(-1)^{\#0\text{s in } \mathbf{c}}$.

We have $\det A(n) = \sum_\sigma \operatorname{sgn} \sigma \prod_{i=1}^n a_{i,\sigma(i)} = \sum_\sigma \operatorname{sgn} \sigma \prod_{i=1}^n \left[{i+1 \atop 2i-\sigma(i)}\right]$, and so

$$\det A(n) = \sum_{\mathbf{c}} (-1)^{\#0\text{s in }\mathbf{c}} \prod_{i=1}^{n} \left[{i+1 \atop i+1-c_i}\right], \tag{3}$$

where the sum is restricted to padded compositions **c** of n with $c_i \le i$ for all i (counted by A002083) because $\left[{i+1 \atop i+1-c_i}\right] = 0$ unless $c_i \le i$.

Henceforth, let us write all permutations in standard cycle form whereby the smallest entry occurs first in each cycle and these smallest entries increase left to right. Thus, with dashes separating cycles, 154-2-36 is the standard cycle form of the permutation $\left(\begin{smallmatrix} 1&2&3&4&5&6 \\ 5&2&6&1&4&3 \end{smallmatrix}\right)$. We define a *nonfirst* entry to be one that does not start a cycle. Thus the preceding permutation has 3 nonfirst entries: 5,4,6. Note that the number of nonfirst entries is 0 only for the identity permutation. We denote an identity permutation (of any size) by ϵ.

By definition of Stirling cycle number, the product in (3) counts lists $(\pi_i)_{i=1}^n$ of permutations where π_i is a permutation on $[i+1]$ with $i+1-c_i$ cycles, equivalently, with $c_i \le i$ nonfirst entries. This observation motivates us to define $\mathcal{L}_n$ to be the set all lists of permutations $\pi = (\pi_i)_{i=1}^n$ where π_i is a permutation on $[i+1]$, # nonfirst entries in π_i is $\le i$, π_1 is the transposition (1,2), each nonidentity permutation π_i is immediately followed by $c_i - 1$ ϵ's where $c_i \ge 1$ is the number of nonfirst entries in π_i (so the total number of nonfirst entries is n). Assign a weight to $\pi \in \mathcal{L}_n$ by $\text{wt}(\pi) = (-1)^{\#\,\epsilon\text{'s in }\pi}$. Then

$$\det A(n) = \sum_{\pi \in \mathcal{L}_n} \text{wt}(\pi).$$

We now define a weight-reversing involution on (most of) $\mathcal{L}_n$. Given $\pi \in \mathcal{L}_n$, scan the list of its component permutations $\pi_1 = (1,2), \pi_2, \pi_3, \ldots$ left to right. Stop at the first one that either (i) has more than one nonfirst entry, or (ii) has only one nonfirst entry, b say, and $b >$ maximum nonfirst entry m of the next permutation in the list. Say π_k is the permutation where we stop (assuming that we do stop).

In case (i) decrement (i.e. decrease by 1) the number of ϵ's in the list by splitting π_k into two nonidentity permutations as follows. Let m be the largest nonfirst entry of π_k and let ℓ be its predecessor. Replace π_k and its successor in the list (necessarily an ϵ) by the following two permutations: first the transposition (ℓ, m) and second the permutation obtained from π_k by erasing m from its cycle and turning it into a singleton. Here are two examples of this case (recall permutations are in standard cycle form and, for clarity, singleton cycles are not shown).

i	1	2	3	4	5	6
π_i	12	13	23	14-253	ϵ	ϵ

$\longrightarrow$

i	1	2	3	4	5	6
π_i	12	13	23	25	14-23	ϵ

and

i	1	2	3	4	5	6
π_i	12	23	14	13-24	ϵ	23

$\longrightarrow$

i	1	2	3	4	5	6
π_i	12	23	14	24	13	23

The reader may readily check that this sends case (i) to case (ii).

In case (ii), π_k is a transposition (a, b) with $b >$ maximum nonfirst entry m of π_{k+1}. In this case, increment the number of ϵ's in the list by combining π_k and π_{k+1} into a single permutation followed by an ϵ as follows: in π_{k+1}, b is a singleton; delete this singleton and insert b immediately after a in π_{k+1} (in the same cycle). The reader may check that this reverses the result in the two examples above and, in general, sends case (ii) to case (i). Since the map alters the number of ϵ's in the list by 1, it is clearly weight-reversing. The map fails only for lists that both consist entirely of transpositions and have the form

$$(a_1, b_1),\ (a_2, b_2),\ \ldots,\ (a_n, b_n) \quad \text{with } b_1 \le b_2 \le \ldots \le b_n.$$

Such lists have weight 1. Hence $\det A(n)$ is the number of lists $\big((a_i, b_i)\big)_{i=1}^{n}$ satisfying $1 \le a_i < b_i \le i+1$ for $1 \le i \le n$, and $b_1 \le b_2 \le \ldots \le b_n$. After subtracting 1 from each b_i, these lists code the paths in $\mathsf{C}_1^*(n,n)$ and, using §4, $\det A(n) = |\, \mathsf{C}_1^*(n,n)\,| = |\,\mathcal{C}_2(n)\,|$.

References

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