Discrete Mathematics and Theoretical Computer Science DMTCS vol. **9**, 2007, 203–196

"Trivializing" Generalizations of some Izergin-Korepin-type Determinants

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received May 8, 2007, *revised June 13, 2007; July 31, 2007*, *accepted July 31, 2007*.

We generalize (and hence trivialize and routinize) numerous explicit evaluations of determinants and pfaffians due to Kuperberg, as well as a determinant of Tsuchiya. The level of generality of our statements render their proofs easy and routine, by using Dodgson condensation and/or Krattenthaler's factor exhaustion method.

All our matrices will be assumed to be embedded inside an infinite matrix.

The first theorem adds *parameters* to the determinant formulas found in Kuperberg [Ku, Theorem 15], as well as older determinants, mentioned there, due to Cauchy, Stembridge, Laksov–Lascoux–Thorup, and Tsuchiya [T]. This way, the formulation is suited to the method of [AZ]. Our proofs are much more **succinct** and **automatable**, since their generality enables an easy induction using Dodgson's rule [D, AZ], or by employing Krattenthaler's elegant factor exhaustion method [Kr1]. Relevant background for this paper can found in [Ku], and references thereof.

Theorem 1

$$\det\left(\frac{1}{x_i+y_j+Ax_iy_j}\right)_{i,j}^{1,n} = \frac{\prod_{i<j}(x_j-x_i)(y_j-y_i)}{\prod_{i,j}^{n}(x_i+y_j+Ax_iy_j)}.$$

$$\det\left(\frac{1}{x_i+y_j}-\frac{1}{1+x_iy_j}\right)_{i,j}^{1,n} = \frac{\prod_{i<j}(1-x_ix_j)(1-y_iy_j)(x_j-x_i)(y_j-y_i)}{\prod_{i,j}^{n}(x_i+y_j)(1+x_iy_j)}\cdot\prod_{i=1}^{n}(1-x_i)(1-y_i).$$

$$\det\left(\frac{Ay_j+Bx_i}{y_j+x_i}\right)_{i,j}^{1,n} = (A-B)^{n-1}\frac{\left(A\prod_j^n y_j+(-1)^{n-1}B\prod_i^n x_i\right)\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}^{n}(x_i+y_j)}.$$

$$\det\left(\frac{1-x_iy_j}{y_j-x_i}\right)_{i,j}^{1,n} = (-1)^{\binom{n}{2}}\frac{\left(\prod_j^n(1+x_j)(1-y_j)+\prod_i^n(1-x_i)(1+y_i)\right)\prod_{i<j}(x_i-x_j)(y_i-y_j)}{2\prod_{i,j}^{n}(y_j-x_i)}.$$

Sketch of Proof: An automatic application of Dodgson Condensation [D, AZ]. □

Corollary 1 (Cauchy, Stembridge, Laksov–Lascoux–Thorup)

$$\det\left(\frac{1}{x_i+y_j}\right)_{i,j}^{1,n} = \frac{\prod_{i<j}(x_j-x_i)(y_j-y_i)}{\prod_{i,j}^{n}(x_i+y_j)}.$$

$$\mathrm{Pf}^2 = \det\left(\frac{x_j-x_i}{x_j+x_i}\right)_{i,j}^{1,2n} = \prod_{i<j\le 2n}\frac{(x_i-x_j)^2}{(x_i+x_j)^2}.$$

$$\mathrm{Pf}^2 = \det\left(\frac{x_j-x_i}{1-x_ix_j}\right)_{i,j}^{1,2n} = \prod_{i<j\le 2n}\frac{(x_i-x_j)^2}{(1-x_ix_j)^2}.$$

Remark 1 *Notice that the latter two statements apply only to even-dimensional matrices. An error from [Ku] in the second formula has been corrected here.* Pf *stands for Pfaffian of a matrix.*

The next theorem generalizes, and presents variations of, several of the determinants that appear in Theorems 16 and 17 [Ku] (typos corrected in [Kr2, Theorems 13 and 14]). Below, Z_1, Z_2, Z_3, Z_4, Z_5 are defined by (here $\gamma(a,b)=a-b, \tau(a,b)=a+b$)

$$Z_1(p,q;x,y)_{i,j} = \frac{\gamma(q^{j-i},x^{j-i})}{\tau(p^{j-i},y^{j-i})}, \qquad Z_2(p,q;x,y)_{i,j} = \frac{\gamma(q^{1+j-i},x^{1+j-i})}{\tau(p^{1+j-i},y^{1+j-i})},$$
$$Z_3(p,q;x,y)_{i,j} = \frac{\gamma(q^{-1+j-i},x^{-1+j-i})}{\tau(p^{-1+j-i},y^{-1+j-i})}, \qquad Z_4(q,q;x,x)_{i,j} = \frac{\gamma(q^{a+j-i},x^{a+j-i})}{\tau(q^{a+j-i},x^{a+j-i})},$$
$$Z_5(q,q;x,x)_{i,j} = \frac{\tau(q^{b+j-i},x^{b+j-i})}{\gamma(q^{b+j-i},x^{b+j-i})},$$

for $a \in \mathbb{Z}, b = \pm n, \pm(n+1), \ldots$. Let $\delta_{e,n} = \frac{1+(-1)^n}{2}$ denote *Kronecker's delta* function centered at the even integers, and let $\lambda_{i,j}=1$, if $i \neq j$ and $\lambda_{i,i}=0$.

Theorem 2 *Write* $\gamma(a,b)=a-b, \tau(a,b)=a+b$, *we have the matrix determinants*

$$\det\left(\frac{\gamma(q^{n+j-i},x^{n+j-i})}{\gamma(p^{n+j-i},y^{n+j-i})}\right)_{i,j}^{1,n} = (py)^{\binom{n}{3}}\gamma(q,x)^n\frac{\prod_{j>i}\gamma(p^{j-i},y^{j-i})^2\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{\prod_{i,j}\gamma(p^{n+j-i},y^{n+j-i})}.$$

$$\det\left(\frac{\tau(q^{j-i},x^{j-i})}{\tau(p^{j-i},y^{j-i})}\right)_{i,j}^{1,n} = \frac{\prod_{2|j-i>0}\gamma(p^{j-i},y^{j-i})^2\prod_{2\nmid j-i>0}\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{(qx)^{\lfloor n^2/4\rfloor}\prod_{j>i}\tau(p^{j-i},y^{j-i})^2}.$$

$$\det(Z_1)_{i,j}^{1,n} = \delta_{e,n}\frac{\gamma(q,x)^n(py)^{n/2}}{(qx)^{\lfloor n^2/4\rfloor}}\frac{\prod_{2|j-i>0}\gamma(p^{j-i},y^{j-i})^2\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{\prod_{j>i}\tau(p^{j-i},y^{j-i})^2}.$$

$$\det(Z_2)_{i,j}^{1,n} = \frac{\gamma(q,x)^n}{(qx)^{\lfloor (n-1)^2/4 \rfloor}} \frac{\prod_{2|j-i>0} \gamma(p^{j-i}, y^{j-i})^2 \gamma(qp^{j-i}, xy^{j-i}) \gamma(xp^{j-i}, qy^{j-i})}{\tau(p^n, y^n)^{1-\delta_{e,n}} \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2}.$$

$$\det(Z_3)_{i,j}^{1,n} = \frac{(-py)^n \gamma(q,x)^n}{(qx)^{\lfloor (n+1)^2/4 \rfloor}} \frac{\prod_{2|j-i>0} \gamma(p^{j-i}, y^{j-i})^2 \gamma(qp^{j-i}, xy^{j-i}) \gamma(xp^{j-i}, qy^{j-i})}{\tau(p^n, y^n)^{1-\delta_{e,n}} \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2}.$$

$$\det(Z_4)_{i,j}^{1,n} = 2^{n-1} \frac{q^{na} + (-1)^n x^{na}}{(qx)^{n(n-1)(n+1-3a)/6}} \frac{\prod_{j \neq i} \gamma(q^{|j-i|}, x^{|j-i|})}{\prod_{i,j} \tau(q^{a+j-i}, x^{a+j-i})}.$$

$$\det(Z_5)_{i,j}^{1,n} = (-1)^{\binom{n}{2}} 2^{n-1} \frac{q^{nb} + x^{nb}}{(qx)^{n(n-1)(n+1-3b)/6}} \frac{\prod_{j \neq i} \gamma(q^{|j-i|}, x^{|j-i|})}{\prod_{i,j} \gamma(q^{b+j-i}, x^{b+j-i})}.$$

$$\mathrm{Pf}^2 = \det\left(\lambda_{i,j} \frac{\gamma(q^{j-i}, x^{j-i}) \gamma(r^{j-i}, z^{j-i})}{\gamma(p^{j-i}, y^{j-i})}\right)_{i,j}^{1,2n} = \frac{(yp)^n}{(qxrz)^{n^2}} \frac{\prod_{j \neq i}^{1,n} \gamma(p^{|j-i|}, y^{|j-i|})^2}{\prod_{i,j}^{1,n} \gamma(p^{n+j-i}, y^{n+j-i})^2} \times$$
$$\prod_{i,j}^{1,n} \gamma(qp^{|j-i|}, xy^{|j-i|}) \gamma(xp^{|j-i|}, qy^{|j-i|}) \gamma(rp^{|j-i|}, zy^{|j-i|}) \gamma(zp^{|j-i|}, ry^{|j-i|}).$$

Sketch of Proof: Identities Z_4 and Z_5 are directly amenable to Dodgson's Condensation technique [AZ]. For the remaining assertions, use the factor exhaustion method [Kr1] (see also [Ku]): the essential idea is to compare zeros and poles on both sides of the equation at hand. We leave the straightforward details to the reader. □

Acknowledgments

The authors wish to thank Christian Krattenthaler for helpful suggestions.

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