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Strong chromatic index of products of graphs

Olivier Togni

LE2I, UMR CNRS 5158, Université de Bourgogne, BP 47870, 21078 Dijon Cedex, France
olivier.togni@u-bourgogne.fr

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The strong chromatic index of a graph is the minimum number of colours needed to colour the edges in such a way that each colour class is an induced matching. In this paper, we present bounds for the strong chromatic index of three different products of graphs in terms of the strong chromatic index of each factor. For the Cartesian product of paths, cycles or complete graphs, we derive sharper results. In particular, strong chromatic indices of d-dimensional grids and of some toroidal grids are given along with approximate results on the strong chromatic index of generalized hypercubes.

Keywords: Strong edge colouring; induced matching; Cartesian product; Kronecker product; strong product.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. An edge between vertex x and vertex y will be denoted by xy.

A *proper edge-colouring* is a mapping $c : E(G) \to \mathbb{N}$ satisfying $c(xy) \neq c(yz), \forall xy, yz \in E$. For any vertex $x \in V$, let $S_c(x)$ denote the set of the colours of all edges incident to x. A proper edge-colouring c is said to be *strong* if no two edges of the same colour lie on a path of length 3; that is, for any edge xy of G, $S_c(x) \cap S_c(y) = \{c(xy)\}$. Equivalently, a strong colouring also corresponds to a partition of the edges into induced matchings. The *strong chromatic index* of G, denoted by $\chi'_s(G)$, is the minimum number of colours of any strong colouring of G. A strong colouring of a graph G corresponds to a vertex colouring of $L(G)^2$, the square of the line-graph of G; where the square of a graph is obtained by adding edges between vertices at distance 2 in the graph. A colouring of the square of a graph is also referred to as a *distance-2* or $L(1, 1)$ colouring in some works.

These problems of strong edge-colouring and distance colouring have interesting applications, specifically for channel assignment in mobile multi-hop radio networks [10] and in cellular networks [4].

The problem of determining the strong chromatic index of a graph is proved to be NP-complete, even for bipartite graphs of girth at least g, for any fixed g [6]. In 1985, Erdős and Nešetřil conjectured (see [2]) that the strong chromatic index of every graph of maximum degree Δ is at most $\frac{5}{4}\Delta^2$. Later, Faudree et al. conjectured in [2] that $\chi'_s(G) \leq \Delta^2$ for bipartite graphs G of maximum degree Δ. In [7], a probabilistic argument is used to show that $\chi'_s(G) \leq 1.998\Delta(G)^2$. Other approximation results exist, for instance for cubic graphs [1], multigraphs [3] and C_4-free graphs [5]. Exact values for specific graphs are presented in [2, 8, 9].

In this paper, we study the strong chromatic index of graphs obtained by Cartesian, Kronecker and strong products (definitions are given below). In Section 2, we derive upper bounds for each of them in terms of the strong chromatic index of the two factors along with two lower bounds for the Cartesian product. In Section 3, we turn our attention on the Cartesian product to present improved bounds for products of paths, cycles or cliques. In particular, these results allow to find the exact value of the strong chromatic index for d-dimensional grids and some d-dimensional toroidal grids and approximate results for other toroidal grids and generalized hypercubes.

The following notation will be used throughout this paper. For a graph G, denote by n_G its order, by $\Delta(G)$ its maximum degree and by $\chi(G)$ its chromatic number.

The *Cartesian* product $G \square H$ of two graphs G and H has vertex set $V(G) \times V(H)$ and edge set $\{(a,x)(b,y) : ab \in E(G)$ and $x = y$ or $xy \in E(H)$ and $a = b\}$.

The *Kronecker* (sometimes called direct or categorical) product $G \times H$ has vertex set $V(G) \times V(H)$ and edge set $\{(a,x)(b,y) : ab \in E(G)$ and $xy \in E(H)\}$.

The *strong* product $G \boxtimes H$ has vertex set $V(G) \times V(H)$ and edge set $E(G \square H) \cup E(G \times H)$.

2 General bounds

2.1 Cartesian product

Theorem 1 *For any graph G and for any graph H that contains two adjacent vertices of maximum degree, we have*

$$\chi'_s(G \square H) \geq 2\Delta(G \square H).$$

Proof: Let ab be an edge of G, with a being a vertex of maximum degree and let xy be an edge of H, with both x and y being a vertex of maximum degree. Denote by S the set of all edges incident to (a,x) or (a,y) plus the edge $(b,x)(b,y)$. Then all edges of S must be coloured by distinct colours in any strong colouring. A simple count gives $|S| = 2\Delta(G) + 2\Delta(H) = 2\Delta(G \square H)$. □

Theorem 2 *Let G and H be two graphs. For the Cartesian product, we have*

$$\chi'_s(G \square H) \leq \chi'_s(G)\chi(H) + \chi'_s(H)\chi(G).$$

Proof: Let $G' = G \square H$ and let $k_G = \chi'_s(G)$ and $k_H = \chi'_s(H)$. Denote by c_G a strong colouring of G with colours from $0, 1, \dots k_G - 1$ and denote by c_H a strong colouring of H with colours from $0, 1, \dots k_H - 1$. Let v_G be a proper vertex colouring of G using the $\chi(G)$ colours $0, 1, \dots \chi(G) - 1$ and let v_H be a proper vertex colouring of H using the $\chi(H)$ colours $0, 1, \dots \chi(H) - 1$.

A colouring c' of G' is defined as follows:

For any edge ab of G, for any vertex x of H, set

$$c'((a,x)(b,x)) = c_G(ab) + k_G v_H(x),$$

for any edge xy of H, for any vertex a of G, set

$$c'((a,x)(a,y)) = c_H(xy) + k_H v_G(a) + k_G \chi(H).$$

As c_G and c_H are proper colourings and $c'((a,x)(b,x)) < k_G\chi(H)$, then c' is a proper colouring too. So it remains to show that c' is strong. For any vertex a of G and for any vertex x of H, let $I(a,x) = \{s + k_G v_H(x) : s \in S_{c_G}(a)\}$ and let $J(a,x) = \{s + k_H v_G(a) + k_G\chi(H) : s \in S_{c_H}(x)\}$.

First, consider two vertices (a,x) and (b,x) of G', with $ab \in E(G)$. By the definition of the colouring c', we have $S_{c'}((a,x)) = I(a,x) \cup J(a,x)$, and $S_{c'}((b,x)) = I(b,x) \cup J(b,x)$. Since c_G is strong, we have that $S_{c_G}(a) \cap S_{c_G}(b) = \{c_G(ab)\}$. Thus $I(a,x) \cap I(b,x) = \{c_G(ab) + k_G v_H(x)\}$. Since v_G is a proper colouring, we have $v_G(a) \neq v_G(b)$. Thus $J(a,x) \cap J(b,x) = \emptyset$. Therefore $S_{c'}((a,x)) \cap S_{c'}((b,x)) = \{c_G(ab) + k_G v_H(x)\} = \{c'((a,x)(b,x))\}$.

Next, consider two vertices (a,x) and (a,y) of G', with $xy \in E(H)$. By a similar argument, as v_H is a proper colouring and c_H is a strong colouring, we have $S_{c'}((a,x)) \cap S_{c'}((a,y)) = \{c'((a,x)(a,y))\}$.

Hence we have proved that c' is a strong colouring of $G' = G\Box H$ with $\chi'_s(G)\chi(H) + \chi'_s(H)\chi(G)$ colours. $\Box$

To see how tight the above theorem is, we present the following lower bound in relation with the fractional chromatic number. The fractional chromatic number of a graph G is $\chi_f(G) = \min \frac{k}{p}$ for which G has a p-tuple k-colouring, that is an assignment of p positive integers from a set of k integers (colours) to each vertex of G such that adjacent vertices receive disjoint sets of colours. It is known that $\omega(G) \leq \chi_f(G) \leq \chi(G)$ for any graph G, where $\omega(G)$ is the clique number of G.

Theorem 3 *Let G and H be two graphs, then*

$$\chi'_s(G\Box H) \geq \chi_f(G)\Delta(H).$$

Proof: Let $x \in V(H)$ be a vertex of degree $d(x) = \Delta(H)$ and let S^x be the subgraph of H consisting of the star of order $\Delta(H) + 1$ and center x. Remember that the product $G\Box H$ contains $n = |V(G)|$ copies $H_1, \ldots, H_n$ of H. Denote by S_i^x the copy of S^x in each copy H_i of H and let $S = \bigcup_{i=1}^m S_i^x$.

As each edge of a star S_i^x must be coloured with a different colour in any strong colouring of $G\Box H$, and the colours on two adjacent stars S_i^x and $S_{i'}^x$ must be distinct too, one can see that finding a strong edge colouring of S is equivalent to finding a $\Delta(H)$-tuple colouring of G. Assume that there exists a $\Delta(H)$-tuple k-colouring of G. Then,

$$\chi'_s(G\Box H) \geq \chi'_s(S) \geq k \geq \chi_f(G)\Delta(H),$$

since by definition, $\chi_f(G) \leq \frac{k}{\Delta(H)}$. $\Box$

In view of this theorem, we can deduce that Theorem 2 gives a upper bound close to the optimal if $\chi_f(G)$ is close to $\chi(G)$ and $\chi'_s(H)$ is close to $\Delta(H)$. For instance, Theorem 2 gives the exact value of the strong chromatic index of the product of a star S_n on $n+1$ vertices by a K_2. For the product $K_n\Box S_m$, with Theorem 2 and Theorem 3, we obtain $nm \leq \chi'_s(K_n\Box S_m) \leq n(m+n-1)$. Also, for the Cartesian product of a bipartite graph by itself, if $\chi'_s(G) \leq \Delta(G)^2$ then $\chi'_s(G\Box G) \leq 4\Delta(G)^2 = \Delta(G\Box G)^2$. Thus we obtain infinite families of bipartite graphs verifying the conjecture of Faudree et al..

Nevertheless, Theorem 2 is not optimal for many product graphs. For instance, for the product of two paths P_m and P_n where $m, n \geq 3$, with Theorem 1 and Theorem 2 we obtain $8 \leq \chi'_s(P_m\Box P_n) \leq 12$. In Corollary 4 of Section 3.3, we will determine the exact value of $\chi'_s(P_m\Box P_n)$, showing that the lower bound is tight in general.

2.2 Kronecker product

Theorem 4 *Let G and H be two graphs different from K_2. For the Kronecker product $G \times H$ we have*

$$\chi'_s(G \times H) \leq \chi'_s(G)\chi'_s(H).$$

Proof: Let $G' = G \times H$ and let $k_G = \chi'_s(G)$ and $k_H = \chi'_s(H)$. Denote by c_G a strong colouring of G with colours from $0, 1, \ldots k_G - 1$ and denote by c_H a strong colouring of H with colours from $0, 1, \ldots k_H - 1$.

A colouring c' of G' is defined as follows: for any edge ab of G, for any edge xy of H, set

$$c'((a, x)(b, y)) = c_G(ab) + k_G c_H(xy).$$

This colouring is clearly proper because c_G and c_H are both proper and for any edge e of G, $c_G(e) < k_G$. We have $S_{c'}((a, x)) = \{\alpha + k_G\beta : \alpha \in S_{c_G}(a), \beta \in S_{c_H}(x)\}$ and $S_{c'}((b, y)) = \{\alpha + k_G\beta : \alpha \in S_{c_G}(b), \beta \in S_{c_H}(y)\}$. Since c_G and c_H are strong, we have $S_{c_G}(a) \cap S_{c_G}(b) = \{c_G(ab)\}$ and $S_{c_H}(x) \cap S_{c_H}(y) = \{c_H(xy)\}$. Hence $S_{c'}((a, x)) \cap S_{c'}((b, y)) = \{c_G(ab) + k_G c_H(xy)\} = \{c'((a, x)(b, y))\}$. Thus c' is a strong colouring of $G \times H$. □

This result is optimal for products of stars: If $G = S_m$ and $H = S_n$, then the product $G \times H$ contains a star S_{mn} and since $\chi'_s(S_n) = n$, Theorem 4 gives the exact value of $\chi'_s(S_m \times S_n)$. Moreover, for the Kronecker product of a bipartite graph by itself, if $\chi'_s(G) \leq \Delta(G)^2$ then this theorem gives $\chi'_s(G \times G) \leq \Delta(G)^4 = \Delta(G \times G)^2$. Thus we again obtain infinite families of bipartite graphs verifying the conjecture of Faudree et al.

2.3 Strong product

As the edge set of the strong product $G \boxtimes H$ is the union of the edge set of $G \square H$ and of $G \times H$, we shall construct a strong colouring of $G \boxtimes H$ by colouring the edges of the Cartesian product $G \square H$ using Theorem 2 and the edges of the Kronecker product $G \times H$ by a modified version of the colouring defined in proof of Theorem 4. The two next lemmas will ensure us that each of these two colourings remain strong in the final graph $G \boxtimes H$.

Lemma 1 *For any graphs G and H, there exist a strong colouring c of $G \square H$ in $\chi'_s(G)\chi(H) + \chi'_s(H)\chi(G)$ colours that verifies the following additional property: for any edge ab of G and for any edge xy of H, $S_c((a, x)) \cap S_c((b, y)) = \emptyset$.*

Proof: The colouring c of $G \square H$ is the same as the colouring c' defined in the proof of Theorem 2. Keeping the same notation as for the proof of Theorem 2, let us see that for any edge ab of G and any edge xy of H, the equality $S_c((a, x)) \cap S_c((b, y)) = \emptyset$ holds; i-e. that any edge $(a, x)(a', x')$ incident to vertex (a, x) has a colour different from the colour of any edge $(b, y)(b', y')$ incident to vertex (b, y). We have four cases to consider:

1. $aa' \in E(G)$ and $x = x'$, $bb' \in E(H)$ and $y = y'$. Then $c'((a, x)(a', x')) = c_G(aa') + k_G v_H(x)$ and $c'((b, y)(b', y')) = c_G(bb') + k_G v_H(y)$. Since v_H is proper and $xy \in E(H)$, $v_H(x) \neq v_H(y)$ and thus $c'((a, x)(a', x')) \neq c'((b, y)(b', y'))$.

2. $xx' \in E(G)$ and $a = a'$, $yy' \in E(H)$ and $b = b'$. Then $c'((a,x)(a',x')) = c_H(xx') + k_H v_G(a) + k_G\chi(H)$ and $c'((b,y)(b',y')) = c_H(yy') + k_H v_G(b) + k_G\chi(H)$. Since v_G is proper and $ab \in E(G)$, $v_G(a) \neq v_G(b)$ and thus $c'((a,x)(a',x')) \neq c'((b,y)(b',y'))$.

3. $aa' \in E(G)$ and $x = x'$, $yy' \in E(H)$ and $b = b'$. Then $c'((a,x)(a',x')) = c_G(aa') + k_G v_H(x) \neq c_H(yy') + k_H v_G(b) + k_G\chi(H) = c'((b,y)(b',y'))$.

4. $xx' \in E(G)$ and $a = a'$, $bb' \in E(H)$ and $y = y'$. Then $c'((a,x)(a',x')) = c_H(xx') + k_H v_G(a) + k_G\chi(H) \neq c_G(bb') + k_G v_H(y) = c'((b,y)(b',y'))$.

Therefore, in all cases, $S_{c'}((a,x)) \cap S_{c'}((b,y)) = \emptyset$, which proves the lemma. □

Lemma 2 *For any graphs G and H, there exist a strong colouring c of $G \times H$ in $2\chi'_s(G)\chi'_s(H)$ colours that verifies the following additional property: for any edge ab of G and for any edge xy of H,*

$$S_c((a,x)) \cap S_c((a,y)) = \emptyset,$$

and

$$S_c((a,x)) \cap S_c((b,x)) = \emptyset.$$

Proof: The colouring c of $G \times H$ is obtained by modifying the strong colouring c' given in the proof of Theorem 4 in this way: Let $k = \chi'_s(G)\chi'_s(H)$ and let $\prec_G$ (resp. $\prec_H$) be any ordering of the vertices of G (resp. of H). For any edge ab of G with $a \prec_G b$ and any edge xy of H with $x \prec_H y$, set

$$c((a,x)(b,y)) = c'((a,x)(b,y)),$$

and

$$c((b,x)(a,y)) = c'((b,x)(a,y)) + k.$$

Keeping the same notation as for the proof of Theorem 4, let us see first that for any edge ab of G and any edge xy of H, the equality $S_c((a,x)) \cap S_c((a,y)) = \emptyset$ holds; i-e. that any edge $(a,x)(a_1,x')$ incident to vertex (a,x) has a colour different from the colour of any edge $(a,y)(a_2,y')$ incident to vertex (b,y). Let $m_1 = c_G(aa_1) + k_G c_H(xx')$ and let $m_2 = c_G(aa_2) + k_G c_H(yy')$. To have $m_1 = m_2$, one needs $c_G(aa_1) = c_G(aa_2)$ and $c_H(xx') = c_H(yy')$, which is impossible unless $a_1 = a_2$ and $x' = y$ and $y' = x$. But in that case, we obtain $c'((a,x)(a_1,x')) = m_1$ and $c'((a,y)(a_2,y')) = m_1 + k$ if $a \prec_G a_1$ and $x \prec_H y$, or else $c'((a,x)(a_1,x')) = m_1 + k$ and $c'((a,y)(a_2,y')) = m_1$. Therefore, in all cases, $c'((a,x)(a_1,x')) \neq c'((a,y)(a_2,y'))$. Hence, $S_c((a,x)) \cap S_c((a,y)) = \emptyset$.

Similarly, one can see that $S_c((a,x)) \cap S_c((b,x)) = \emptyset$. □

Theorem 5 *Let G and H be two graphs. For the strong product $G \boxtimes H$ we have*

$$\chi'_s(G \boxtimes H) \leq \chi'_s(G)\chi(H) + \chi'_s(H)\chi(G) + 2\chi'_s(G)\chi'_s(H).$$

Proof: Remember that the edge set of $G \boxtimes H$ is the union of the edge set of $G \square H$ and of the edge set of $G \times H$. Colour the edges of $G \square H$ with a colouring c_1 satisfying Lemma 1 using a set $\mathcal{C}$ of $\chi'_s(G)\chi(H) + \chi'_s(H)\chi(G)$ colours and colour the edges of $G \times H$ with a colouring c_2 satisfying Lemma 2 and using a set $\mathcal{C}'$ disjoint with $\mathcal{C}$ of $2\chi'_s(G)\chi'_s(H)$ colours. This produces an edge-colouring of $G \boxtimes H$. Since $\mathcal{C}$ and

$\mathcal{C}'$ are disjoint, this colouring is proper and since c_1 verifies the additional property of Lemma 1 and c_2 verifies the additional property of Lemma 2, this colouring is strong. $\square$

Notice that this theorem gives the exact value of the strong chromatic index of the strong product of two complete graphs: since $K_m \boxtimes K_n = K_{mn}$ then $\chi'_s(K_{mn}) = \binom{mn}{2} = \chi'_s(K_m)\chi(K_n) + \chi'_s(K_n)\chi(K_m) + 2\chi'_s(K_m)\chi'_s(K_n)$.

3 Cartesian products of paths, cycles and cliques

In this section, we define the notion of (k,t)-colourability that leads us to improved bounds for the product of paths, cycles and complete graphs. Optimal values of the strong chromatic index are found for some graphs.

3.1 (k,t)-colourable graphs

Definition 1 *Two strong colourings c_1 and c_2 of a graph G are* compatible *if for any vertex x of G, $S_{c_1}(x) \cap S_{c_2}(x) = \emptyset$.*

Definition 2 *A graph G is (k,t)-colourable if there exist t strong colourings c_i, $1 \le i \le t$, $c_i : E(G) \to \{1, 2, \ldots, k\}$ that are pairwise compatible.*

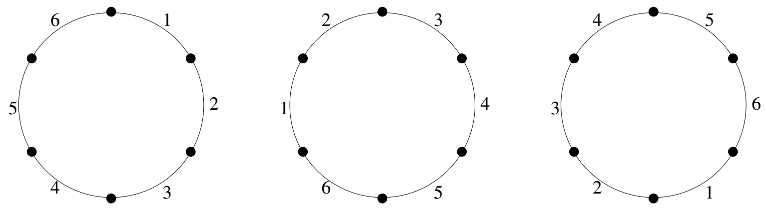

Fig. 1: Three compatible strong colourings of C_6 in six colours showing that C_6 is $(6,3)$-colourable.

Notice that a (k,t)-colourable graph is also $(\alpha k, \alpha t)$-colourable for any integer $\alpha \ge 1$. In particular, every graph G is $(t\chi'_s(G), t)$-colourable for any integer $t \ge 1$.

For the path P_n on n vertices, it is known that $\chi'_s(P_n) = 3$ for any $n \ge 4$, thus P_n is $(3t,t)$-colourable. This result can be strengthened as shown in the following proposition.

Proposition 1 *For any integers $n \ge 2$ and $t \ge 1$, the path P_n is $(4t, 2t)$-colourable.*

Proof: We only have to show that P_n is $(4,2)$-colourable, i-e. that there exist two compatible strong colourings of P_n with 4 colours. The first strong colouring c_1 is defined by giving the colours $1, 2, 3, 4, 1, 2, 3, 4, \ldots$ to the edges of P_n, starting from an end-vertex and going to the other end-vertex. The second colouring c_2 is defined by giving the colours $3, 4, 1, 2, 3, 4, 1, 2, \ldots$ to the edges of P_n starting from the same end-vertex than for the first colouring. $\square$

Proposition 2 *For the cycle C_n, the following holds:*

- *for any $k \ge 3$ and for any $n \ge 1$, C_{kn} is $(kt, \lfloor \frac{k}{2} \rfloor t)$-colourable,*

- *for any $n \geq 5$, $n \neq 6$, C_n is $(5t, 2t)$-colourable.*

Proof: First, observe that, given a strong colouring c of C_n, if for any colour i, any two edges of colour i are separated by at least $d-1$ edges along the shortest path (we call such a colouring d-distant), then there exist $\lfloor \frac{d}{2} \rfloor$ compatible strong colourings. These colourings can be obtained from the cycle C_n with its colouring c by a rotation of $\frac{4k\pi}{n}$, for $k = 0, 1, \ldots, \lfloor \frac{d}{2} \rfloor - 1$.

For the first assertion, a strong k-distant colouring is given by assigning cyclically the colours $1, 2, \ldots, k$ to the edge of the cycle C_{kn} (see Figure 1 for an example with $k = 6$).

For the second assertion, observe first that C_7 and C_{11} are $(5, 2)$-colourable, as can be seen by the colourings $1, 2, 3, 4, 1, 3, 5$ and $3, 4, 1, 2, 5, 4, 2$ for C_7 and $1, 2, 3, 4, 1, 2, 3, 4, 1, 3, 5$ and $3, 4, 1, 2, 3, 4, 1, 2, 5, 4, 2$ for C_{11}. Now, for the remaining cases, we have to find a 4-distant colouring using at most 5 colours. Let $n = 4p + i$, $0 \leq i \leq 3$. Observe that, as $n > 7$ and $n \neq 11$, we have $p \geq i$. The colouring of C_n is obtained by repeating i times the pattern $1, 2, 3, 4, 5$ then $p - i$ times the pattern $1, 2, 3, 4$ along the cycle. □

Proposition 3 *For any integers $n \geq 3$ and $t \geq 1$, the complete graph K_n is $(\frac{n(n-1)}{2}t, \lfloor \frac{n}{2} \rfloor t)$-colourable.*

Proof: Consider first the case of even n and let $n = 2p$. K_{2p} is decomposable into $2p - 1$ perfect matchings M_i, $0 \leq i \leq 2p - 2$. Order the edges of each matching M_i and denote by e_i^j the j^{th} edge of matching M_i, $0 \leq j \leq p - 1$. Then define the p strong colourings c_k of K_n as follows: for any edge e of K_n, for any k, $0 \leq k \leq p - 1$, set

$$c_k(e_i^j) = (i + (j + k)(2p - 1)) \bmod p(2p - 1).$$

We treat the case of odd n in a similar way. Let $n = 2p + 1$. K_{2p+1} is decomposable into $2p + 1$ matchings M_i, $0 \leq i \leq 2p$, each containing p edges $e_i^0, e_i^1, \ldots, e_i^{p-1}$. Then set

$$c_k(e_i^j) = (i + (j + k)(2p + 1)) \bmod p(2p + 1).$$

Because of the choice of the modulo in the definition of the colouring ($p(2p-1)$ for even n and $p(2p+1)$ for odd n), any colour is always in the same matching in all the p strong colourings. This is the reason why the p colourings are pairwise compatible (details of this part of the proof are left to the reader). □

3.2 Cartesian products of (k, t)-colourable graphs

Theorem 6 *Let G be a (k_G, t_G)-colourable graph and let H be a (k_H, t_H)-colourable graph, with $t_G \geq \chi(H)$ and $t_H \geq \chi(G)$. Then $G \square H$ is $(k_G + k_H, min(t_G, t_H))$-colourable.*

Proof: Let $t' = \min(t_G, t_H)$. As G is (k_G, t_G)-colourable, there exist t_G compatible strong colourings c_i, $1 \leq i \leq t_G$ of G; $c_i : E(G) \to \{0, 1, \ldots, k_G - 1\}$. Similarly, as H is (k_H, t_H)-colourable, there exist t_H compatible strong colourings d_i, $1 \leq i \leq t_H$ of H. Assume moreover that these colourings d_i use different colours than the colourings of G: $d_i : E(H) \to \{k_G, k_G + 1, \ldots, k_G + k_H - 1\}$.

Let v_G be a proper vertex colouring of G using the $\chi(G)$ colours $0, 1, \ldots \chi(G) - 1$ and let v_H be a proper vertex colouring of H using the $\chi(H)$ colours $0, 1, \ldots \chi(H) - 1$.

We define the t' colourings c'_i, $0 \leq i \leq t' - 1$ of G' as follows:

For any edge ab of G, for any vertex x of H, let $m(x) = (v_H(x) + i) \bmod t_G$ and set

$$c'_i((a,x)(b,x)) = c_{m(x)}(ab),$$

for any edge xy of H, for any vertex a of G, let $p(a) = (v_G(a) + i) \bmod t_H$ and set

$$c'_i((a,x)(a,y)) = d_{p(a)}(xy).$$

Notice that each colouring c'_i uses at most $k_G + k_H$ colours.

Let us show that each c'_i is a strong colouring of G':

First, consider two adjacent vertices (a,x) and (b,x) of G'. Then $S_{c'_i}((a,x)) \cap S_{c'_i}((b,x)) = (S_{c_{m(x)}}(a) \cup S_{d_{p(a)}}(x)) \cap (S_{c_{m(x)}}(b) \cup S_{d_{p(b)}}(x))$. Since v_G is proper and $t_G \geq \chi(H)$, we have that $p(a) \neq p(b)$ and since $d_{p(a)}$ and $d_{p(b)}$ are compatible then we obtain $S_{d_{p(a)}}(x) \cap S_{d_{p(b)}}(x) = \emptyset$. Moreover, as $c_{m(x)}$ is strong, then we have $S_{c_{m(x)}}(a) \cap S_{c_{m(x)}}(b) = \{c_{m(x)}(ab)\}$. Consequently, $S_{c'_i}((a,x)) \cap S_{c'_i}((b,x)) = \{c_{m(x)}(ab)\} = \{c'_i((a,x)(b,x))\}$.

Next, for two adjacent vertices (a,x) and (a,y) of G', a similar argument gives $S_{c'_i}((a,x)) \cap S_{c'_i}((a,y)) = \{d_{p(a)}(xy)\} = \{c'_i((a,x)(a,y))\}$.

Now let us show that for any $i_1, i_2, 0 \leq i_1, i_2 \leq t' - 1$, $i_1 \neq i_2$, c'_{i_1} and c'_{i_2} are compatible: Let (a,x) be a vertex of G'. We have

$S_{c'_{i_1}}((a,x)) = S_{c_{m_1}}(a) \cup S_{d_{p_1}}(x)$, with $m_1 = (v_H(x) + i_1) \bmod t_G$ and $p_1 = (v_G(a) + i_1) \bmod t_H$, and

$S_{c'_{i_2}}((a,x)) = S_{c_{m_2}}(a) \cup S_{d_{p_2}}(x)$, with $m_2 = (v_H(x) + i_2) \bmod t_G$ and $p_2 = (v_G(a) + i_2) \bmod t_H$.

We have $m_1 \neq m_2$ and $p_1 \neq p_2$, therefore $S_{c_{m_1}}(a) \cap S_{c_{m_2}}(a) = \emptyset$ and $S_{d_{p_1}}(x) \cap S_{d_{p_2}}(x) = \emptyset$. Moreover, since for any i and k, the colourings c_i and d_k do not use the same sets of colours, we conclude that $S_{c'_{i_1}}((a,x)) \cap S_{c'_{i_2}}((a,x)) = \emptyset$. □

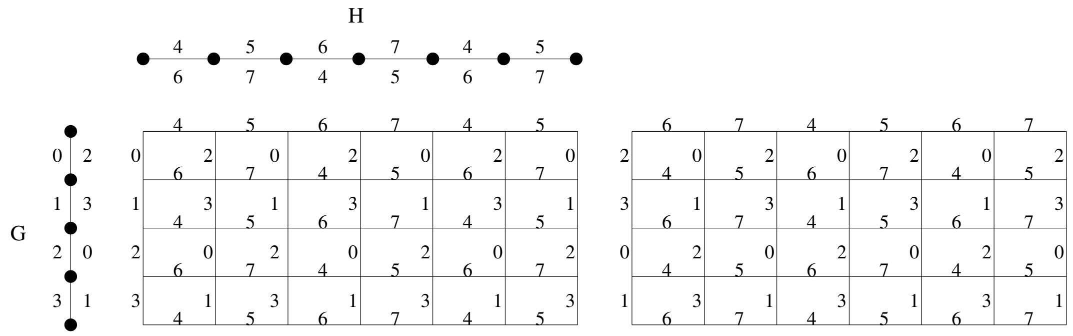

Fig. 2: Two compatible optimal strong colourings of $P_5 \square P_7$ in 8 colours.

An example of two compatible strong colourings of $P_5 \square P_7$ as defined in the above proof is given in Figure 2 (the two compatible strong colourings of P_5 and of P_7 appear on each graph but the vertex colourings are not given).

Corollary 1 *Let G be a (k_G, t_G)-colourable graph and let H be a (k_H, t_H)-colourable graph. Then $G \square H$ is $(k_G \lceil \frac{\chi(H)}{t_G} \rceil + k_H \lceil \frac{\chi(G)}{t_H} \rceil, min(\chi(G), \chi(H)))$-colourable.*

Proof: By definition, G is $(k_G\lceil\frac{\chi(H)}{t_G}\rceil, \chi(H))$-colourable and H is $(k_H\lceil\frac{\chi(G)}{t_H}\rceil, \chi(G))$-colourable. Theorem 6 then gives the result. $\square$

Corollary 2 *Let G be a $(k, 2)$-colourable graph. Then $G\square K_2$ is $(k + \chi(G), 2)$-colourable.*

3.3 Products of paths, cycles and cliques

The *d-dimensional grid* $M_{l_1,l_2,...,l_d}$ is the Cartesian product of d paths: $M_{l_1,l_2,...,l_d} = P_{l_1}\square P_{l_2}\square \ldots \square P_{l_d}$.
When all lengths are equal: $l_1 = l_2 = \ldots = l_d = n$, we denote the grid by M_n^d.
The *d-dimensional toroidal grid* $TM_{l_1,...,l_d}$ is the Cartesian product of d cycles: $TM_{l_1,...,l_d} = C_{l_1}\square \ldots \square C_{l_d}$.
When all cycle lengths are equal: $l_1 = l_2 = \ldots = l_d = n$, we denote the toroidal grid by TM_n^d.
The *d-dimensional generalized hypercube* (also known as Hamming graph) K_n^d is the Cartesian product of K_n by itself d times: $K_n^d = K_n\square K_n\square \ldots \square K_n$. The hypercube H_d is the graph K_2^d.
By Proposition 1 and Proposition 2, P_n and C_{4n} are $(4, 2)$-colourable and C_{2n} is $(5, 2)$-colourable for $n \neq 3$. By Proposition 3, K_{2p} is $(2p(2p-1), 2p)$-colourable and K_{2p+1} is $(3p(2p+1), 2p+1)$-colourable. Therefore, Theorem 6 and Corollary 2 give the following results:

Corollary 3 *For any $d \geq 2$,*

- *the d-dimensional hypercube H_d is $(2d, 2)$-colourable,*
- *the d-dimensional grid $M_{l_1,l_2,...,l_d}$ and toroidal grid TM_{4n}^d are $(4d, 2)$-colourable,*
- *the d-dimensional toroidal grid $TM_{2l_1,2l_2,...,2l_d}$ is $(5d, 2)$-colourable for $l_i \neq 3$,*
- *the d-dimensional generalized hypercube K_{2p}^d is $(2dp(2p-1), 2p)$-colourable,*
- *the d-dimensional generalized hypercube K_{2p+1}^d is $(3dp(2p+1), 2p+1)$-colourable.*

With Theorem 1, we obtain exact or approximate values for the strong chromatic index of some Cartesian products:

Corollary 4 *For any $d \geq 2$,*

- $\chi'_s(H_d) = 2d$,
- $\chi'_s(M_{l_1,l_2,...,l_d}) = \chi'_s(TM_{4n}^d) = 4d$, *for* $l_i \geq 3$,
- $4d \leq \chi'_s(TM_{2l_1,2l_2,...,2l_d}) \leq 5d$, *for* $l_i \neq 3$,
- $2p(d-1)(2p-1) \leq \chi'_s(K_{2p}^d) \leq 2dp(2p-1)$,
- $2p(d-1)(2p+1) \leq \chi'_s(K_{2p+1}^d) \leq 3dp(2p+1)$.

Note that the exact value of the strong chromatic index of hypercubes was known before [2].

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