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A construction of small regular bipartite graphs of girth 8

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Let q be a prime a power and k an integer such that $3 \leq k \leq q$. In this paper we present a method using Latin squares to construct adjacency matrices of k-regular bipartite graphs of girth 8 on $2(kq^2 - q)$ vertices. Some of these graphs have the smallest number of vertices among the known regular graphs with girth 8.

Keywords: Incidence matrices, Latin squares, cages, girth.

1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the books by Godsil and Royle [16] and by Lint and Wilson [21] for terminology and definitions.

Let $G = (V(G), E(G))$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The $girth$ of a graph G is the number $g = g(G)$ of edges in a smallest cycle. The *degree* of a vertex $v \in V$ is the number of vertices adjacent to v. A graph is called $regular$ if all the vertices have the same degree. A *cage* is a k-regular graph with girth g having the smallest possible number of vertices. Simply counting the numbers of vertices in the distance partition with respect to a vertex yields a lower bound $n_0(k, g)$ on the number of vertices $n(k, g)$ in a cage, with the precise form of the bound depending on whether g is even or odd.

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \ldots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \ldots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases} \tag{1}$$

As defined by Biggs [7], the *excess* of a k-regular graph G is the difference $|V(G)| - n_0(k, g)$. A (k, g)-cage with even girth g and $n_0(k, g)$ vertices is said to be a *generalized polygon graph*. Generalized polygon graphs exist if and only if $g \in \{4, 6, 8, 12\}$ [7]. The question of the construction of graphs with small excess is a difficult one. When $g = 6$, the existence of a graph with $n_0(k, 6) = 2(k^2 - k + 1)$

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vertices called *generalized triangle*, is equivalent to the existence of a projective plane of order $k-1$, that is, a symmetric $((n_0(k,6))/2, k, 1)$-design. It is known that these designs exist whenever $k-1$ is a prime power, but the existence question for other values remains unsettled. *Generalized quadrangles* when $g=8$, and *generalized hexagons* when $g=12$ are also known to exist for all prime power values of $k-1$ [5, 16, 21].

Cages have been studied intensely since they were introduced by Tutte [27] in 1947. Erdős and Sachs [10] proved the existence of a graph for any value of the regularity k and the girth g, thus most of work carried out has been focused on constructing a smallest such graph [1, 2, 4, 6, 9, 11, 13, 15, 22, 23, 24, 28, 29]. Biggs is the author of an impressive report on distinct methods for constructing cubic cages [8]. For some time, Royle [26] kept a web-site in which all the cages known so far appear. More details about constructions on cages can be found in the survey by Wong [29], the survey by Holton and Sheehan [17] or on the more recent dynamic cage survey by Exoo and Jajcay [12].

It is conjectured that cages with even girth are bipartite [25, 29]. A graph is *bipartite* if its vertex set V can be partitioned into two partite sets, V_1 and V_2, such that any edge has one end in V_1 and the other in V_2. If the vertices are ordered in such a way that the vertices of V_1 come first, then the adjacency matrix of a bipartite graph can be written in the form

$$A = \begin{pmatrix} 0 & N \\ N^\top & 0 \end{pmatrix}. \tag{2}$$

An *incidence graph* is a bipartite graph in which the elements of one part V_1 are declared as *lines* and the elements of the other part V_2 are declared as *points*. The terminology for incidence graphs is geometric. A point and a line are said to be $incident$ if they are adjacent, thus the submatrix N of (2) is called an *incidence matrix* of the bipartite graph. If the number of points and the number of lines coincide, then N is clearly a square matrix. An incidence matrix N defines a *partial plane* when

- any line has at least two points, and
- two points are incident with at most one line.

Consequently, two lines of a partial plane have at most one point in common. The corresponding bipartite graph is called the *incidence graph of the partial plane*, which clearly has even girth $g \geq 6$. Thus for simplicity we shall say that *the partial plane has girth* $g/2$ if and only if the corresponding incidence graph has girth g.

Let q be a prime power and k an integer such that $3 \leq k \leq q$. In [3], incidence matrices of $(k,6)$-bipartite graphs of order $2(kq-1)$ were given. In this paper we present a method to construct the incidence matrices of k-regular bipartite graphs of girth $g=8$ on $2q(kq-1)$ vertices.

2 Position matrices

Let S denote a set of symbols and let A be a matrix whose elements are subsets of S. Given $x \in S$ let $P_x(A)$ be a $(0,1)$-matrix of the same dimension as A that satisfies

$$(P_x(A))_{ij} = 1 \text{ if and only if } x \in A_{ij}.$$

Thus, $P_x(A)$ is called the *position matrix of the symbol* x in A. Suppose that $S = \{0, x_1, \ldots, x_n\}$. The position matrices of all the symbols in A different from 0 give rise to the following $(0,1)$-matrix $\mathcal{P}(A)$

called *position matrix of* A:

$$[\mathcal{P}(A)] = [P_{x_1}(A) \cdots P_{x_n}(A)].$$

Let $\{A^1, A^2, \ldots, A^r\}$ be a family of matrices of the same number of columns whose elements are subsets of S. Then the $(0,1)$-matrix spanned by the position matrices of all of them

$$\begin{pmatrix} \mathcal{P}(A^1) \\ \mathcal{P}(A^2) \\ \vdots \\ \mathcal{P}(A^r) \end{pmatrix} = \begin{pmatrix} P_{x_1}(A^1) & \cdots & P_{x_n}(A^1) \\ P_{x_1}(A^2) & \cdots & P_{x_n}(A^2) \\ & \vdots & \\ P_{x_1}(A^r) & \cdots & P_{x_n}(A^r) \end{pmatrix}, \tag{3}$$

is said to be *the position matrix of the family* $\mathcal{F} = \{A^1, A^2, \ldots, A^r\}$. The following example shows two matrices of order 2×2 whose elements are subsets of $S = \{0, a, b\}$ and the position matrix of them. From now on, if there is no confusion the 1–sets will be indicated as integers.

MATRICES			SYMBOLS			
			a		b	
A^1	a	a	**1**	**1**	0	0
	b	b	0	0	**1**	**1**
A^2	$\{a,b\}$	0	**1**	0	**1**	0
	0	$\{a,b\}$	0	**1**	0	**1**

(4)

As already mentioned in the Introduction, our main aim is to obtain incidence matrices of bipartite k-regular graphs of girth 8 with small excess. Such incidence matrices may be seen also as incidence matrices of partial planes which will be obtained by identifying row i of $P_z(A^\alpha)$ as *line* $i(\alpha)$, and column j of $P_z(A^\alpha)$ as *point* $j(z)$, for any matrix $A^\alpha \in \mathcal{F}$ and $z \in S - 0$. To achieve our goal we propose the following definitions.

Definition 2.1 *Let $g \geq 4$ be an even number. A family of matrices $\mathcal{F} = \{A^1, A^2, \ldots, A^r\}$ of the same number of columns whose elements are subsets of a set of symbols S is said to have girth g if the position matrix of $\mathcal{F}$ is the incidence matrix of a bipartite graph of girth g.*

Each matrix of (4) has girth $g = \infty$, and the two matrices A^1, A^2 form a family of girth 8.

Let us recall that a *Latin square* of order q is a $q \times q$ matrix with entries from a set of q symbols such that each symbol occurs exactly once in each row and exactly once in each column. A Latin square has clearly girth $g = \infty$ because the position matrices of its elements are permutation matrices yielding the incidence matrix of a partial plane consisting in a set of parallel lines (since they have no common point).

In [3] we introduced the notion of quasi row-disjoint matrices as follows.

Definition 2.2 *[3] Let A^1 and A^2 be two matrices of the same number of columns whose elements are subsets of a set of symbols S such that* $0 \in S$. *A pair (x,y) with $x, y \in S$ belongs to the cartesian product of any two rows $(A^1)_i \times (A^2)_h$ if and only if $(x,y) \in (A^1)_{ij} \times (A^2)_{hj}$ for some j. Then A^1 and A^2 are said to be* quasi row-disjoint *if and only if the cartesian product of any two rows $(A^1)_i, (A^2)_h$ contains at most one pair $(x,x) \in (A^1)_i \times (A^2)_h$ with $x \neq 0$.*

The pair of matrices of the example (4) are quasi row-disjoint matrices. Moreover, in [3] we have stated the following theorem which roughly speaking says that a family of r matrices is quasi row-disjoint if and only if its girth is at least 6.

Theorem 2.1 *[3] Let A^1 and A^2 be two matrices each one of girth at least 6 of the same number of columns and whose elements are subsets of a set of symbols S such that $0 \in S$. Then A^1 and A^2 are quasi row-disjoint if and only the family $\{A^1, A^2\}$ has girth at least 6.*

In the next theorem we give a sufficient condition for a family of matrices to have girth at least 8.

Theorem 2.2 *Let $\mathcal{F} = \{A^1, \ldots, A^r\}$ be a set of $r \geq 2$ quasi row-disjoint matrices of the same number of columns whose elements are subsets of a set of symbols S such that $0 \in S$. Let $(A^u)_i$, $(A^v)_{i'}$, $(A^w)_{i''}$ denote any three mutually distinct rows of matrices $A^u, A^v, A^w \in \mathcal{F}$. Then the girth of $\mathcal{F}$ is at least 8 if the sets $(A^u)_i \times (A^v)_{i'}, (A^v)_{i'} \times (A^w)_{i''}, (A^u)_i \times (A^w)_{i''}$ contains at most two distinct pairs (x,x), (y,y) with $x, y \neq 0$, $x, y \in S$.*

Proof: Suppose $(x,x) \in (A^u)_i \times (A^v)_{i'}$, $x \neq 0$. Therefore the position matrix of $\mathcal{F}$ has the following entries equal to 1:

$$P_x(A^u)(i,j) = P_x(A^v)(i',j) = 1, \tag{5}$$

where $P_x(A^u)$ and $P_x(A^v)$ are the position matrix of the element x in A^u and A^v respectively. Recall that for any given matrix $A^\alpha \in \mathcal{F}$ and $z \in S - 0$, row i of $P_z(A^\alpha)$ is line $i(\alpha)$, and column j of $P_z(A^\alpha)$ is point $j(z)$. Consequently, (5) means that lines $i(u)$ and $i'(v)$ have point $j(x)$ as a common point.

Analogously, $(y,y) \in (A^v)_{i'} \times (A^w)_{i''}$, $y \neq 0$, is equivalent to:

$$P_y(A^v)(i',j') = P_y(A^w)(i'',j') = 1,$$

or in other words, lines $i'(v)$ and $i''(w)$ have the point $j'(y)$ in common, with $j'(y) \neq j(x)$ because $y \neq x$. Thus if there exists $z \neq 0$, $z \neq x, y$, such that $(z,z) \in (A^u)_i \times (A^w)_{i''}$, then lines $i(u)$ and $i''(w)$ have the point $j''(z)$ in common, $j'' \neq j, j'$, yielding that the partial plane defined by the position matrix of $\mathcal{F}$ contains the triangle $j(x)j'(y)j''(z)$. In other words, the position matrix of $\mathcal{F}$ is the incidence matrix of a bipartite graph of girth less than 8. $\square$

Our immediate goal is to derive a method for constructing a family of matrices with girth 8, because the position matrix of this family will be the incidence matrix of a bipartite graph of girth 8.

3 Method

Throughout this work let $[[n]]$ denote the set of non negative integers $\{0, 1, \ldots, n\}$ and $(n]] = [[n]] \setminus \{0\}$. Let I_n be the identity matrix and denote by $(t \times F)I_n$ the matrix obtained from I_n by replacing each one with a subset $\{t\} \times F$ of $\{t\} \times [[n]]$ for some $t \in [[n]]$ and $F \subseteq [[n]]$. In the following theorem we demonstrate a method for obtaining a family of matrices with girth 8 using Latin squares.

Theorem 3.1 *Let q be a prime power, and let $\mathbb{F}_q$ be the Galois field of order q. For each $u, t \in \mathbb{F}_q$, define the $q \times q$ matrix $L^{u,t}$ by*

$$L^{u,t}(i,j) = i + uj + ut, \;\; i, j \in \mathbb{F}_q.$$

Then the following assertions hold:

(i) For all $u, v \in \mathbb{F}_q \setminus \{0\}$, $u \neq v$, and $t, t' \in \mathbb{F}_q$ the matrices $L^{u,t}$ and $L^{v,t'}$ are quasi row-disjoint Latin squares with entries from $[[q-1]]$.

(ii) For any given $u, t \in \mathbb{F}_q$, $u \neq 0$, the matrix $L^{u,t} \times L^{u+u^{-1},t}$ has q^2 distinct entries. Moreover, the position matrix of the family

$$\{L^{u,t} \times L^{u+u^{-1},t} : t \in \mathbb{F}_q\}$$

is a $(0,1)$-matrix of order $q^2 \times q^3$ considering symbol $(0,0)$ different from 0, and it is the incidence matrix of a partial plane consisting in q^2 parallel lines with q points on each.

(iii) The family $\{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\}$ has girth 8 and its position matrix has order $(q^3 - q^2) \times q^3$ and q ones in each row and $q-1$ ones in each column.

(iv) The position matrix of the family

$$\{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\} \cup \{\,(t \times [[q-1]])I_q : t \in \mathbb{F}_q\}$$

is the incidence matrix of a q-regular bipartite graph of girth 8 with q^3 vertices in each partite set.

Proof: (i) Clearly $L^{u,t}$ and $L^{v,t'}$ are $q \times q$ Latin squares on $[[q-1]]$. Let us show that they are quasi row-disjoint. Otherwise, there exists $i, i'j, j' \in \mathbb{F}_q$ with $j \neq j'$ such that

$$\begin{aligned} L^{u,t}(i,j) &= L^{v,t'}(i',j), \\ L^{u,t}(i,j') &= L^{v,t'}(i',j'). \end{aligned}$$

Equivalently,

$$\begin{aligned} i + uj + ut &= i' + vj + vt', \\ i + uj' + ut &= i' + vj' + vt'. \end{aligned}$$

Therefore $u(j - j') = v(j - j')$ implying that $u = v$ or $j = j'$, a contradiction in either case.

(ii) Note that $L^{u+u^{-1},t}$ is also a Latin square if $u + u^{-1} \neq 0$, otherwise $L^{0,t}(i,j) = i$. In either case it is very easy to check that $L^{u,t}$ and $L^{u+u^{-1},t}$ are orthogonal, which implies that $L^{u,t} \times L^{u+u^{-1},t}$ has q^2 distinct entries. Hence, the position matrix of the family $\{L^{u,t} \times L^{u+u^{-1},t} : t \in \mathbb{F}_q\}$ is a $(0,1)$-matrix of order $q^2 \times q^3$ by considering entry (0,0) different from 0.

Let us show now that for every $i, i', j, t, t' \in \mathbb{F}_q$ such that $t \neq t'$ we have

$$(L^{u,t} \times L^{u+u^{-1},t})(i,j) \neq (L^{u,t'} \times L^{u+u^{-1},t'})(i',j).$$

Otherwise we would have

$$\begin{aligned} &i + uj + ut &&= i' + uj + ut', \\ &i + (u+u^{-1})j + (u+u^{-1})t &&= i' + (u+u^{-1})j + (u+u^{-1})t', \end{aligned}$$

implying $t = t'$, which is a contradiction. As a consequence, the position matrix of the set $\{L^{u,t} \times L^{u+u^{-1},t} : t \in \mathbb{F}_q\}$ has one unique entry equal to 1 in each column. Considering the rows of this $(0,1)$-matrix as lines and the columns as points, this is equivalent to say that each point belongs to a unique line. Thus, this $(0,1)$-matrix is the incidence matrix of q^2 parallel lines with q points on each.

(iii) First, let us show that the position matrix of the family $\mathcal{M} = \{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q,\ u \neq 0\}$ has q entries equal to 1 in each row and $q-1$ entries equal to 1 in each column. By (ii) the position matrix of each set $\{L^{u,t} \times L^{u+u^{-1},t} : t \in \mathbb{F}_q\}$ contributes with a unique 1 in each column, yielding that each column of the position matrix of $\mathcal{M}$ has $q-1$ entries equal to 1. Since each matrix $L^{u,t} \times L^{u+u^{-1},t}$ has q^2 distinct entries, then the q position matrices of symbols starting with the same x for any $x \in \mathbb{F}_q$ contribute with one unique 1 in each row. So the position matrix of $\mathcal{M}$ has q entries equal to 1 in each row. Therefore, we conclude that the position matrix of $\mathcal{M}$ has order $(q^3 - q^2) \times q^3$ and q entries equal to 1 in each row and $q-1$ entries equal to 1 in each column.

Next, let us show that the girth of $\mathcal{M}$ is at least 6. Otherwise, there exists $u, v \in \mathbb{F}_q \setminus \{0\}$, $u \neq v$ and $i, i'j, j' \in \mathbb{F}_q$ with $j \neq j'$ for which

$$\begin{array}{rcl}(L^{u,t} \times L^{u+u^{-1},t})(i,j) &=& (L^{v,t'} \times L^{v+v^{-1},t'})(i',j),\\ (L^{u,t} \times L^{u+u^{-1},t})(i,j') &=& (L^{v,t'} \times L^{v+v^{-1},t'})(i',j').\end{array}$$

This implies that $L^{u,t}$ and $L^{v,t'}$ are not quasi row-disjoint, a contradiction with item (i). Thus the family $\{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q,\ u \neq 0\}$ has girth at least 6. Next let us show that the girth is 8 applying Theorem 2.2. By way of contradiction assume that for three elements u, v, w of $\mathbb{F}_q \setminus \{0\}$ there exist three pairwise distinct columns j, j', j'' for which

$$\begin{array}{lcl}(L^{u,t} \times L^{u+u^{-1},t})(i,j) &=& (L^{v,t'} \times L^{v+v^{-1},t'})(i',j)\\ (L^{v,t'} \times L^{v+v^{-1},t'})(i',j') &=& (L^{w,t''} \times L^{w+w^{-1},t''})(i',j')\\ (L^{w,t''} \times L^{w+w^{-1},t''})(i'',j'') &=& (L^{u,t} \times L^{u+u^{-1},t})(i,j'').\end{array} \tag{6}$$

Note that u, v, w must be three distinct elements because by (ii) each matrix $L^{u,t} \times L^{u+u^{-1},t}$ has q^2 distinct entries. Then from the equality between the first coordinates we have:

$$\begin{array}{lcl}(u-v)j &=& i' - i + vt' - ut;\\ (v-w)j' &=& i'' - i' + wt'' - vt';\\ (w-u)j'' &=& i - i'' + ut - wt''.\end{array}$$

Hence

$$(v-u)j + (w-v)j' = (w-u)j''. \tag{7}$$

Moreover, from equalities between the second coordinates in (6) and taking into account (7) we obtain

$$(v^{-1} - u^{-1})j + (w^{-1} - v^{-1})j' = (w^{-1} - u^{-1})j''.$$

Multiplying this equality by uvw we get

$$w(u-v)j + u(v-w)j' = v(u-w)j''. \tag{8}$$

Multiplying (7) by w we also obtain

$$w(v-u)j + w(w-v)j' = w(w-u)j''. \tag{9}$$

Thus adding both equalities (8) and (9) we have

$$(u-w)(v-w)j' = (v-w)(u-w)j''.$$

Taking into account that u, v, w are mutually distinct we get that $j' = j''$ which is a contradiction. Therefore $\mathcal{M} = \{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\}$ has girth 8 as claimed.

(iv) By (iii) and applying Theorem 2.2, we only need to prove that for all $u, v \in \mathbb{F}_q \setminus \{0\}$ with $u \neq v$, any three matrices $L^{u,t} \times L^{u+u^{-1},t}$, $L^{v,t'} \times L^{v+v^{-1},t'}$ and $(t'' \times [[q-1]])I_q$ have girth 8. Otherwise we would have

$$\begin{array}{rcl} L^{u,t} \times L^{u+u^{-1},t}(i,j) & = & L^{v,t'} \times L^{v+v^{-1},t'}(i',j) \\ L^{v,t'} \times L^{v+v^{-1},t'}(i',j') & \in & (t'' \times [[q-1]])I_q(j',j') \\ L^{u,t} \times L^{u+u^{-1},t}(i,j') & \in & (t'' \times [[q-1]])I_q(j',j') \end{array}$$

Then $L^{u,t}(i,j) = L^{v,t'}(i',j)$ and $L^{u,t}(i,j') = L^{v,t'}(i',j') = t''$, meaning that $L^{u,t}$ and $L^{v,t'}$ are not quasi row-disjoint, contradicting item (i). Further, by (iii, we know that the position matrix of $\{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\}$ has q ones in each row and $q-1$ ones in each column. Since the rows of the position matrix of $(t \times [[q-1]])I_q$ contributes with one additional one, then the result follows. □

To illustrate the method of Theorem 3.1, both the matrices provided by this theorem and their position matrix for the first case $q = 2$ are shown in Table 1. From now on, if there is no confusion an entry (x, y) will be denoted as xy. Thus this (0,1)-matrix is the incidence matrix of a 2-regular graph of girth 8, which consists of two cycles of girth 8. In Table 2 the matrices for $q = 3$ are also depicted. The corresponding position matrix is the incidence matrix of a 3-regular graph of girth 8 on 27 vertices in each partite set. Table 3 contains the matrices for $q = 4$. Their position matrix is the incidence matrix of a 4-regular bipartite graph on 64 vertices in each partite set.

MATRICES			SYMBOLS							
			00		01		10		11	
$L^{1,0} \times L^{0,0}$	00	10	**1**	0	0	0	0	**1**	0	0
	11	01	0	0	0	**1**	0	0	**1**	0
$L^{1,1} \times L^{0,1}$	10	00	0	**1**	0	0	**1**	0	0	0
	01	11	0	0	**1**	0	0	0	0	**1**
$(0 \times [[1]])I_2$	$\{00, 01\}$	0	**1**	0	**1**	0	0	0	0	0
	0	$\{00, 01\}$	0	**1**	0	**1**	0	0	0	0
$(1 \times [[1]])I_2$	$\{10, 11\}$	0	0	0	0	0	**1**	0	**1**	0
	0	$\{10, 11\}$	0	0	0	0	0	**1**	0	**1**

Tab. 1: Case $q = 2$.

Let us call *array of* r *symbols and* n *columns* the matrix of order $r \times n$

$$O_{r,n} = \begin{pmatrix} 1 & \cdots & 1 \\ 2 & \cdots & 2 \\ \vdots & \vdots & \vdots \\ r & \cdots & r \end{pmatrix}.$$

When $r = n$ the array is denoted by O_n. It is easy to see that the position matrix of $O_{r,n}$ is the incidence matrix of a partial plane consisting in r parallel lines, each one having n points. Using the position matrices of these arrays denoted by $\mathcal{P}(O_{r,n})$ and applying Theorem 3.1, we now present the method for constructing the desired $(k, 8)$-bipartite graphs.

	$L^{1,t} \times L^{2,t}$	$L^{2,t} \times L^{1,t}$	$(t \times [[2]])I_3$		
	00 12 21	00 21 12	$\{00, 01, 02\}$	0	0
t=0	11 20 02	11 02 20	0	$\{00, 01, 02\}$	0
	22 01 10	22 10 01	0	0	$\{00, 01, 02\}$
	12 21 00	21 12 00	$\{10, 11, 12\}$	0	0
t=1	20 02 11	02 20 11	0	$\{10, 11, 12\}$	
	01 10 22	10 01 22	0	0	$\{10, 11, 12\}$
	21 00 12	12 00 21	$\{20, 21, 22\}$	0	0
t=2	02 11 20	20 11 02	0	$\{20, 21, 22\}$	0
	10 22 01	01 22 10	0	0	$\{20, 21, 22\}$

Tab. 2: Matrices for the case $q = 3$.

	$L^{1,t} \times L^{0,t}$	$L^{2,t} \times L^{1,t}$	$L^{3,t} \times L^{1,t}$	$(t \times [[3]])I_4$			
	00 10 20 30	00 21 32 13	00 31 12 23	$\{00, 01, 02, 03\}$	0	0	0
	11 01 31 21	11 30 23 02	11 20 03 32	0	$\{00, 01, 02, 03\}$	0	0
t=0	22 32 02 12	22 03 10 31	22 13 30 01	0	0	$\{00, 01, 02, 03\}$	0
	33 23 13 03	33 12 01 20	33 02 21 10	0	0	0	$\{00, 01, 02, 03\}$
	10 00 30 20	21 00 13 32	31 00 23 12	$\{10, 11, 12, 13\}$	0	0	0
t=1	01 11 21 31	30 11 02 23	20 11 32 03	0	$\{10, 11, 12, 13\}$	0	0
	32 22 12 02	03 22 31 10	13 22 01 30	0	0	$\{10, 11, 12, 13\}$	0
	23 33 03 13	12 33 20 1	02 33 10 21	0	0	0	$\{10, 11, 12, 13\}$
	20 30 00 10	32 13 00 21	12 23 00 31	$\{20, 21, 22, 23\}$	0	0	0
	31 21 11 01	23 02 11 30	03 32 11 20	0	$\{20, 21, 22, 23\}$	0	0
t=2	02 12 22 32	10 31 22 03	30 01 22 13	0	0	$\{20, 21, 22, 23\}$	0
	13 03 33 23	01 20 33 12	21 10 33 02	0	0	0	$\{20, 21, 22, 23\}$
	30 20 10 00	13 32 21 00	12 23 00 31	$\{30, 31, 32, 33\}$	0	0	0
t=3	21 31 01 11	02 23 30 11	03 32 11 20	0	$\{30, 31, 32, 33\}$	0	0
	12 02 32 22	31 10 03 22	30 01 22 13	0	0	$\{30, 31, 32, 33\}$	0
	03 13 23 33	20 01 12 33	21 10 33 02	0	0	0	$\{30, 31, 32, 33\}$

Tab. 3: Matrices for the case $q = 4$.

Theorem 3.2 *Let q be a power prime, and let $\mathbb{F}_q$ be the Galois field of order q. For each $u, t \in \mathbb{F}_q$, $u \neq 0$, let $L^{u,t}$ be the matrix $L^{u,t}(i, j) = i + uj + ut$, $i, j \in \mathbb{F}_q$, and let M be the position matrix of the family*

$$\{L^{u,t} \times L^{u+u^{-1},t} : u, t \in \mathbb{F}_q, u \neq 0\} \cup \{\ (t \times [[q-1]])I_q : t \in \mathbb{F}_q\}.$$

Then the following assertions hold:

(i) The $(0, 1)$-matrix

$$\begin{array}{|c|c|}\hline M & \mathcal{P}(O_{q^2,q})^{\top} \\ \hline\end{array} \tag{10}$$

is the incidence matrix of a bipartite graph of girth 8 with $q^3 + q^2$ vertices in one partite set having degree $q + 1$, and q^3 vertices in the other partite set having degree q.

(ii) *Let* $(L^{u,t} \times L^{u+u^{-1},t})_0$ *denote the matrix obtained from* $L^{u,t} \times L^{u+u^{-1},t}$ *by replacing each entry* $0x$ *with 0 for all* $x \in \mathbb{F}_q$*. Let* M_0 *be the position matrix of the family*

$$\{(L^{u,t} \times L^{u+u^{-1},t})_0 : u, t \in \mathbb{F}_q, u \neq 0\} \cup \{\, (t \times [[q-1]])I_q : t \in \mathbb{F}_q, t \neq 0\}.$$

Then

$$\begin{array}{|c|c|} \hline M_0 & \mathcal{P}(O_{q^2-q,q})^\top \\ \cline{2-2} & [0] \\ \hline \end{array} \tag{11}$$

is the incidence matrix of a q-regular bipartite graph of girth 8 with $q^3 - q$ *vertices in each partite set.*

(iii) *Let* k *be an integer such that* $3 \leq k \leq q-1$ *and let* $(L^{u,t} \times L^{u+u^{-1},t})_{q-k}$ *denote the matrix obtained from* $L^{u,t} \times L^{u+u^{-1},t}$ *by replacing with 0 the entries* $0y$ *and* $(x, x+s)$ *for all* $x, y, s \in \mathbb{F}_q$ *for* $s = 0, \ldots, q-1-k$*. Let* M_{q-k} *be the position matrix of*

$$\{(L^{u,t} \times L^{u+u^{-1},t})_{q-k} : u = 1, \ldots, k-1\} \cup \{\, (t \times ([[q-1]] \setminus \{t, t+1, \ldots, t+q-1-k\}))I_q : t \neq 0\}.$$

Suppose that the q columns of $O_{kq-q,q}$ *are indexed by* $j \in \mathbb{F}_q$*. Let* $O^*_{kq-q,q}$ *be the matrix obtained from* $O_{kq-q,q}$ *by changing for 0 the entries* $(i, -u^2 s)$ *for all* $u = 1, \ldots, k-1$*,* $i = (u-1)q+1, \ldots, uq$ *and* $s = 0, \ldots, q-1-k$*. Then*

$$\begin{array}{|c|c|} \hline M_{q-k} & \mathcal{P}(O^*_{kq-q,q})^\top \\ \cline{2-2} & [0] \\ \hline \end{array} \tag{12}$$

is the incidence matrix of a k-regular bipartite graph of girth 8 with $kq^2 - q$ *vertices in each partite set.*

Proof: (i) From Theorem 3.1, it follows that M is the incidence matrix of a bipartite graph of girth 8 with q^3 lines and q^3 points. Thus we need to prove that the q^2 columns of $\mathcal{P}(O_{q^2,q})^\top$ can be added to M without decreasing the girth 8. It is readily seen that after adding these columns, the girth is at least 6, because by Theorem 3.1, for each $u \in \mathbb{F}_q \setminus \{0\}$ the set $\{L^{u,t} \times L^{u+u^{-1},t} : t \in \mathbb{F}_q, \}$ consists of q^2 parallel lines. Thus suppose that (10) contains as a sub-matrix the incidence matrix of a cycle of length 6. Then there exists $u, v \in \mathbb{F}_q \setminus \{0\}$ and $i, i', i'', j, j' \in \mathbb{F}_q$ with $u \neq v$, $i \neq i'$ and $j \neq j'$ such that

$$\begin{array}{rcl} (L^{u,t} \times L^{u+u^{-1},t})(i,j) & = & (L^{v,t'} \times L^{v+v^{-1},t'})(i'',j), \\ (L^{u,t} \times L^{u+u^{-1},t})(i',j') & = & (L^{v,t'} \times L^{v+v^{-1},t'})(i'',j'). \end{array}$$

From the equalities between their coordinates we obtain

$$\begin{array}{rcl} i - i' + u(j-j') & = & v(j-j'), \\ i - i' + (u+u^{-1})(j-j') & = & (v+v^{-1})(j-j'). \end{array}$$

Hence $u^{-1}(j-j') = v^{-1}(j-j')$, implying $u = v$ or $j = j'$, a contradiction in either case. Further if $(L^{u,t} \times L^{u+u^{-1},t})(i,j) \in (t' \times [[q-1]]I_q)(i'',j)$, and $(L^{u,t} \times L^{u+u^{-1},t})(i',j') \in (t' \times [[q-1]]I_q)(i'',j')$, then then $i'' = j = j'$ which is also a contradiction.

MATRICES			SYMBOLS						
			10	11	12	20	21	22	
0	12	21	0 0 0	0 0 0	0 **1** 0	0 0 0	0 0 **1**	0 0 0	**1** 0 0 0 0 0
11	20	0	0 0 0	**1** 0 0	0 0 0	0 **1** 0	0 0 0	0 0 0	**1** 0 0 0 0 0
22	0	10	0 0 **1**	0 0 0	0 0 0	0 0 0	0 0 0	**1** 0 0	**1** 0 0 0 0 0
12	21	0	0 0 0	0 0 0	**1** 0 0	0 0 0	0 **1** 0	0 0 0	0 **1** 0 0 0 0
20	0	11	0 0 0	0 0 **1**	0 0 0	**1** 0 0	0 0 0	0 0 0	0 **1** 0 0 0 0
0	10	22	0 **1** 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 **1**	0 **1** 0 0 0 0
21	0	12	0 0 0	0 0 0	0 0 **1**	0 0 0	**1** 0 0	0 0 0	0 0 **1** 0 0 0
0	11	20	0 0 0	0 **1** 0	0 0 0	0 0 **1**	0 0 0	0 0 0	0 0 **1** 0 0 0
10	22	0	**1** 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 **1** 0	0 0 **1** 0 0 0
0	21	12	0 0 0	0 0 0	0 0 **1**	0 0 0	0 **1** 0	0 0 0	0 0 0 **1** 0 0
11	0	20	0 0 0	**1** 0 0	0 0 0	0 0 **1**	0 0 0	0 0 0	0 0 0 **1** 0 0
22	10	0	0 **1** 0	0 0 0	0 0 0	0 0 0	0 0 0	**1** 0 0	0 0 0 **1** 0 0
21	12	0	0 0 0	0 0 0	0 **1** 0	0 0 0	**1** 0 0	0 0 0	0 0 0 0 **1** 0
0	20	11	0 0 0	0 0 **1**	0 0 0	0 **1** 0	0 0 0	0 0 0	0 0 0 0 **1** 0
10	0	22	**1** 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 **1**	0 0 0 0 **1** 0
12	0	21	0 0 0	0 0 0	**1** 0 0	0 0 0	0 0 **1**	0 0 0	0 0 0 0 0 **1**
20	11	0	0 0 0	0 **1** 0	0 0 0	**1** 0 0	0 0 0	0 0 0	0 0 0 0 0 **1**
0	22	10	0 0 **1**	0 0 0	0 0 0	0 0 0	0 0 0	0 **1** 0	0 0 0 0 0 **1**
$\{10,11,12\}$	0	0	**1** 0 0	**1** 0 0	**1** 0 0	0 0 0	0 0 0	0 0 0	0 0 0 0 0 0
0	$\{10,11,12\}$	0	0 **1** 0	0 **1** 0	0 **1** 0	0 0 0	0 0 0	0 0 0	0 0 0 0 0 0
0	0	$\{10,11,12\}$	0 0 **1**	0 0 **1**	0 0 **1**	0 0 0	0 0 0	0 0 0	0 0 0 0 0 0
$\{20,21,22\}$	0	0	0 0 0	0 0 0	0 0 0	**1** 0 0	**1** 0 0	**1** 0 0	0 0 0 0 0 0
0	$\{20,21,22\}$	0	0 0 0	0 0 0	0 0 0	0 **1** 0	0 **1** 0	0 **1** 0	0 0 0 0 0 0
0	0	$\{20,21,22\}$	0 0 0	0 0 0	0 0 0	0 0 **1**	0 0 **1**	0 0 **1**	0 0 0 0 0 0

Tab. 4: Case $q = 3$ for (ii) of Theorem 3.2. Incidence matrix of a (3,8)-graph on 48 vertices.

(ii) This item follows directly from the fact that M_0 is a sub-matrix of M obtained by deleting the first q^2 columns, which correspond to the position of the symbols starting by 0. Hence M_0 has $q^3 - q^2$ columns. Moreover, the total number of rows of M_0 is the number of matrices $L^{u,t} \times L^{u+u^{-1},t}$, (that is, $(q-1)q$) plus the number of matrices $(t \times ([[q-1]]))I_q$, $t \neq 0$, (a total of $q-1$) multiplied by q, that is, $q(q^2 - q + q - 1) = q^3 - q$. Thus M_0 is a matrix of order $(q^3 - q) \times (q^3 - q^2)$. Since $\mathcal{P}(O_{q^2-q,q})^\top$ contributes with $q^2 - q$ more columns, then (11) is a square matrix of order $q^3 - q$. Reasoning as in (iii) of Theorem 3.1, we obtain that (11) is the incidence matrix of a bipartite graph of girth 8, which has $q^3 - q$ columns and $q^3 - q$ rows both having q ones, so this item is valid. By way of example, both the matrices provided by this item (ii) and their position matrix for the case $q = 3$ are shown in Table 4. Thus this (0,1)-matrix is the incidence matrix of a 3-regular graph of girth 8 on 24 vertices in each partite set.

(iii) Note that (12) is a sub-matrix of (10), then it is the incidence matrix of a bipartite graph of girth 8. Moreover, M_{q-k} is obtained from M by deleting the first q^2 columns which corresponds to the position matrix of symbols starting by 0, and by deleting also $(q-k)(q-1)q$ columns corresponding to the symbols $(x+s, x+2s)$ for all $s = 0, 1, \ldots, q-1-k$ which have been changed for 0. Then the total number of columns of M_{q-k} is

$$q^3 - q^2 - (q-k)(q-1)q = kq(q-1).$$

The total number of rows of M_{q-k} is given by the number of matrices $(L^{u,t} \times L^{u+u^{-1},t})_{q-k}$, $u = 1, \ldots, k-1$, $t \in \mathbb{F}_q$, plus the number of matrices $(t \times ([[q-1]] \setminus \{t, t+1, \ldots, t+q-1-k\}))I_q : t \neq 0$, $t \neq 0$, (a total of $q-1$) multiplied by q, that is

$$(k-1)q^2 + (q-1)q = kq^2 - q.$$

Thus M_{q-k} is a matrix of order $(kq^2 - q) \times (kq^2 - kq)$. Since $\mathcal{P}(O^*_{kq-q,q})^\top$ contributes with $kq - q$ more columns, then (12) is a square matrix of order $kq^2 - q$.

To finish the proof of this item, we only need to show that (12) has k ones in each row and k ones in each column. To see this, let us show that given a fixed $s = 0, 1, \ldots, q-1-k$, the entries $(y, y+s)$ for all $y \in \mathbb{F}_q$, are in the same column of each matrix $L^{u,t} \times L^{u+u^{-1},t}$. Suppose $L^{u,t} \times L^{u+u^{-1},t}(i,j) = (y, y+s)$, that is

$$i + uj + ut = y, \text{ and } i + (u+u^{-1})j + (u+u^{-1})t = y + s.$$

Then $u^{-1}(j+t) = s$, which implies $j = us - t$. Thus our claim is true since the symbols $(y, y+s)$ are placed in the same column $us - t$ of the matrix $L^{u,t} \times L^{u+u^{-1},t}$. Therefore, after changing for 0 the symbols $0x$ and $(y, y+s)$ for all $x, y \in \mathbb{F}_q$ and for all $s = 0, \ldots, q-1-k$ we obtain a new matrix having k entries different from 0 in the rows i such that $i + u^2 s = 0$, and in the remaining rows $k-1$ entries different from 0. Since in the rows i of $\mathcal{P}(O^*_{qk-q,q})^\top$ such that $i + u^2 s \neq 0$ there is a 1 by hypothesis, then $\mathcal{P}(O^*_{qk-q,q})^\top$ contributes with one additional entry equal to 1 in the same rows as those having $k-1$ entries different from 0. Hence (12) is a square matrix of order $kq^2 - q$ having k ones in each row and clearly in each column.

By way of example, both the matrices provided by this item (iii) and their position matrix for the case $q = 4$ and $k = 3$ are shown in Table 5; and the corresponding matrices for $q = 5$ and $k = 3$ are shown in Table 6. Thus, these (0,1)-matrices are the incidence matrix of a 3-regular graph of girth 8 on 44 vertices in each partite set for $q = 4$ and 70 vertices in each partite set for $q = 5$. □

4 Conclusion

For q a prime power and $3 \leq k \leq q$ we have presented a method providing the incidence matrices of k-regular bipartite graphs of girth 8 with $kq^2 - q$ vertices in each partite set. Thus if $n(k, 8)$ denotes the order of a $(k, 8)$-cage, it follows from (1) that

$$2k(k^2 - 2k + 2) \quad \leq \quad n(k, 8) \quad \leq \quad 2q(kq - 1).$$

Hence the q-regular bipartite graphs constructed in this work have an excess of $4q^2 - 6q$. And the $(q-1)$-regular bipartite graphs have an excess of $8q^2 - 20q + 10$.

As regards to known upper bounds on $n(k, g)$, Lazebnik, Ustimenko and Woldar [19] gave the following result: Let $k \geq 2$ and $g \geq 5$ be integers, and let q denote the smallest odd prime power for which $k \leq q$. Then,

$$n(k, g) \leq 2kq^{\frac{3}{4}g - a}\ , \tag{13}$$

where $a = 4, 11/4, 7/2, 13/4$ for $g \equiv 0, 1, 2, 3 \pmod 4$, respectively. According to (13), $n(k, 8) \leq 2kq^2$, therefore the graphs provided by our method also improve this result for $g = 8$. A construction giving this upper bound (13) for $g = 8$ appeared for the first time in [18] and was used later in [14] and probably, the simplest exposition of it is in Section 2.4 of [20]. In [15], $(q, 8)$-graphs with $2q(q^2 - 2)$ vertices were constructed using geometrical techniques. But for regularities $k < q$ the graphs constructed in this paper have the smallest number of vertices among the known regular graphs with girth 8.

				SYMBOLS									
				10	12	13	20	21	23	30	31	32	
0	10	20	30	0 1 0 0	0 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
0	0	31	21	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0	1 0 0 0 0 0 0 0
0	32	0	12	0 0 0 0	0 0 0 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 1 0 0	1 0 0 0 0 0 0 0
0	23	13	0	0 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0 0 0 0 0
10	0	30	20	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
0	0	21	31	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 0	0 1 0 0 0 0 0 0
32	0	12	0	0 0 0 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 1 0 0 0 0 0 0
23	0	0	13	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 1 0 0 0 0 0 0
20	30	0	10	0 0 0 1	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
31	21	0	0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 1 0 0 0 0 0
0	12	0	32	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 1	0 0 1 0 0 0 0 0
13	0	0	23	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 1 0 0 0 0 0
30	20	10	0	0 0 1 0	0 0 0 0	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
21	31	0	0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 1 0 0 0 0
12	0	32	0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 1 0	0 0 0 1 0 0 0 0
0	13	23	0	0 0 0 0	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 1 0 0 0 0
0	21	32	13	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0 0 0 0 0
0	30	23	0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 1 0	0 1 0 0	0 0 0 0	0 0 0 0	0 0 0 0 1 0 0 0
0	0	10	31	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 0	0 0 0 0 1 0 0 0
0	12	0	20	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 1 0 0 0
21	0	13	32	0 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 0 0 0 0 0
30	0	0	23	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 1	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 1 0 0
0	0	31	10	0 0 0 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0	0 0 0 0 0 1 0 0
12	0	20	0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 1 0 0
32	13	0	21	0 0 0 0	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 1	0 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0 0 0 0 0
23	0	0	30	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 1	0 0 0 0	0 0 0 0	0 0 0 0 0 0 1 0
10	31	0	0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 0 0 0 1 0
0	20	0	12	0 0 0 0	0 0 0 1	0 0 0 0	0 1 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 1 0
13	32	21	0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 0	0 1 0 0	0 0 0 0 0 0 0 0
0	23	30	0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 1 0 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 1
31	10	0	0	0 1 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0 0 0 0 1
20	0	12	0	0 0 0 0	0 0 1 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 1
				1 0 0 0	1 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
				0 1 0 0	0 1 0 0	0 1 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
$\{10,\ 12,\ 13\}\ I_4$				0 0 1 0	0 0 1 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
				0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
				0 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
				0 0 0 0	0 0 0 0	0 0 0 0	0 1 0 0	0 1 0 0	0 1 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
$\{20,\ 21,\ 23\}\ I_4$				0 0 0 0	0 0 0 0	0 0 0 0	0 0 1 0	0 0 1 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
				0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0
				0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0	0 0 0 0 0 0 0 0
				0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 1 0 0	0 1 0 0	0 1 0 0	0 0 0 0 0 0 0 0
$\{30,\ 31,\ 32\}\ I_4$				0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 1 0	0 0 1 0	0 0 1 0	0 0 0 0 0 0 0 0
				0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 0 0 0 0 0

Tab. 5: Case $q = 4$ for $k = 3$ of Theorem 3.2 (iii). Incidence matrix of a (3,8)-graph on 88 vertices.

	t=0	**t=1**	**t=2**	**t=3**	**t=4**
	0 0 24 31 43	0 24 31 43 0	24 31 43 0 0	31 43 0 0 24	43 0 0 24 31
	0 0 30 42 0	0 30 42 0 0	30 42 0 0 0	42 0 0 0 30	0 0 0 30 42
u=1	0 0 41 0 10	0 41 0 10 0	41 0 10 0 0	0 10 0 0 41	10 0 0 41 0
	0 0 0 14 21	0 0 14 21 0	0 14 21 0 0	14 21 0 0 0	21 0 0 0 14
	0 0 13 20 32	0 13 20 32 0	13 20 32 0 0	20 32 0 0 13	32 0 0 13 20
	0 20 0 10 30	20 0 10 30 0	0 10 30 0 20	10 30 0 20 0	30 0 20 0 10
u=2	0 31 0 21 41	31 0 21 41 0	0 21 41 0 31	21 41 0 31 0	41 0 31 0 21
	0 42 0 32 0	42 0 32 0 0	0 32 0 0 42	32 0 0 42 0	0 0 42 0 32
	0 0 0 43 13	0 0 43 13 0	0 43 13 0 0	43 13 0 0 0	13 0 0 0 43
	0 14 0 0 24	14 0 0 24 0	0 0 24 0 14	0 24 0 14 0	24 0 14 0 0
		$\{10,\ 13,\ 14\}\ I_5$	$\{20,\ 21,\ 24\}\ I_5$	$\{30,\ 31,\ 32\}\ I_5$	$\{41,\ 42,\ 43\}\ I_5$

$$O^*_{10,5} = \begin{array}{|ccccc|} \hline 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 3 & 3 & 3 & 0 \\ 0 & 4 & 4 & 4 & 0 \\ 0 & 5 & 5 & 5 & 0 \\ 0 & 0 & 6 & 6 & 6 \\ 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 8 & 8 & 8 \\ 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 10 & 10 & 10 \\ \hline \end{array}$$

Tab. 6: Matrices for $q = 5$ and $k = 3$ according to Case (iii) of Theorem 3.2.

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