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Leftmost Derivations of Propagating Scattered Context Grammars: A New Proof

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In 1973, V. Virkkunen proved that propagating scattered context grammars which use leftmost derivations are as powerful as context-sensitive grammars. This paper brings a significantly simplified proof of this result.

Keywords: formal languages, propagating scattered context grammars, leftmost derivations, generative power

1 Introduction

Propagating scattered context grammars, introduced in [3], represent an important type of semi-parallel rewriting systems. Since their introduction, however, the exact relationship of the family of languages they generate to the family of context-sensitive languages is unknown. The language family generated by these grammars is included in the family of context-sensitive languages; on the other hand, the question of whether this inclusion is proper represents an open problem in formal language theory. There have been several attempts to modify the definition of propagating scattered context grammars to obtain the family of context-sensitive languages (see [1, 2, 7, 9, 11]). The approach discussed in [11] allows the productions to be applied only in a leftmost way and, thereby, obtain the family of context-sensitive languages generated by these grammars. This result is of some interest as the use of context-free, context-sensitive, and unrestricted productions in a leftmost way in the corresponding grammars of the Chomsky hierarchy does not have any impact on their generative power.

The proof in [11] consists of two parts; first, two preliminary lemmas (Lemma 2 and Lemma 3) are given and then the main result, stated in Theorem 2, is presented as a straightforward corollary of these

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two lemmas. In Lemma 2 it is demonstrated how any sentence of some context-sensitive language can be derived by a propagating scattered context grammar which uses leftmost derivations. Every sentence generated in such a way contains, however, some additional symbols. Lemma 3 shows how these symbols can be removed. Together, the proof consists of six-page-long construction part and not even one-page-long basic idea of the construction which makes it extremely hard to follow. A more formal proof of the correctness of the construction is missing.

This paper aims to present the proof of this result in a much simpler and more readable way. The main difference of our proof lies (1) in the way how the symbols to be rewritten are selected and (2) the way how context-sensitive productions are simulated. Furthermore, the proof is based on a single construction instead of two. All this leads to a significantly simpler and more transparent construction.

2 Preliminaries and definitions

We assume that the reader is familiar with formal language theory (see [10]). For an alphabet V, $|V|$ denotes the cardinality of V. V^* represents the free monoid generated by V. The unit of V^* is denoted by ε. Set $V^+ = V^* - \{\varepsilon\}$. For $w \in V^*$, $|w|$ and $\mathrm{alph}(w)$ denote the length of w and the set of symbols occurring in w, respectively.

A *grammar* is a quadruple $G = (V, T, P, S)$, where V is the *total alphabet*, $T \subset V$ is the set of *terminals*, P is a finite set of *productions* of the form $x \to y$, where $x \in V^*(V-T)V^*$, $y \in V^*$, and $S \in V - T$ is the *start symbol* of G. If $u = z_1 x z_2$, $v = z_1 y z_2$, and $x \to y \in P$, where $z_1, z_2 \in V^*$, then G makes a *derivation step* from u to v according to $x \to y$, symbolically written as $u \Rightarrow_G v\ [x \to y]$ or, simply, $u \Rightarrow_G v$. Let $\Rightarrow_G^+$ and $\Rightarrow_G^*$ denote the transitive closure of $\Rightarrow_G$ and the reflexive and transitive closure of $\Rightarrow_G$, respectively. If $S \Rightarrow_G^* w$, where $w \in T^*$, $S \Rightarrow_G^* w$ is said to be a *successful derivation* of G. The *language of* G, denoted by $L(G)$, is defined as $L(G) = \{w \in T^* : S \Rightarrow_G^* w\}$. If each production of G is of the form $xAy \to xuy$, where $x, y \in V^*$, $A \in V - T$, $u \in V^+$, then G is a *context-sensitive grammar*. The family of context-sensitive languages is denoted by $\mathscr{L}(\mathit{CS})$. If each production of G is of one of the following forms: $AB \to CD$, $A \to BC$, $A \to a$, where $A, B, C, D \in V - T$, and $a \in T$, then G is a grammar in the *Kuroda normal form*.

Lemma 1 ([4]) *For every context-sensitive grammar there exists an equivalent grammar in the Kuroda normal form.*

A *scattered context grammar* (see [1, 2, 3, 5, 6, 7, 8, 9, 11]) is a quadruple $G = (V, T, P, S)$, where V is the total alphabet, $T \subset V$ is the set of terminals, $S \in V - T$ is the start symbol of G, and P is a finite set of productions such that each production has the form $(A_1, A_2, \ldots, A_n) \to (x_1, x_2, \ldots, x_n)$, for some $n \geq 1$, where $A_i \in V - T$, and $x_i \in V^*$, for all $1 \leq i \leq n$. If each production $(A_1, A_2, \ldots, A_n) \to (x_1, x_2, \ldots, x_n) \in P$ satisfies $x_i \in V^+$ for all $1 \leq i \leq n$, then G is a *propagating scattered context grammar*. If $(A_1, A_2, \ldots, A_n) \to (x_1, x_2, \ldots, x_n) \in P$, $u = u_1 A_1 u_2 A_2 \ldots u_n A_n u_{n+1}$, and $v = u_1 x_1 u_2 x_2 \ldots u_n x_n u_{n+1}$, where $u_i \in V^*$ for all $1 \leq i \leq n+1$, then G makes a *derivation step* from u to v according to $p = (A_1, A_2, \ldots, A_n) \to (x_1, x_2, \ldots, x_n)$, symbolically written as $u \Rightarrow_G v\ [p]$ or, simply, $u \Rightarrow_G v$. In addition, if $A_i \notin \mathrm{alph}(u_i)$ for all $1 \leq i \leq n$, then the direct derivation is *leftmost*, and we write $u\ {}_{\mathrm{lm}}\!\Rightarrow_G v\ [p]$; if $A_i \notin \mathrm{alph}(u_{i+1})$ for all $1 \leq i \leq n$, then the direct derivation is *rightmost*, and we write $u\ {}_{\mathrm{rm}}\!\Rightarrow_G v\ [p]$. The language of G, denoted by $L(G)$, is defined as $L(G) = \{w \in T^* : S \Rightarrow_G^* w\}$. A propagating scattered context grammar $G = (V, T, P, S)$ uses *leftmost* or *rightmost derivations* if its language is defined as $L(G, \mathrm{lm}) = \{w \in T^* : S\ {}_{\mathrm{lm}}\!\Rightarrow_G^* w\}$ or $L(G, \mathrm{rm}) = \{w \in T^* : S\ {}_{\mathrm{rm}}\!\Rightarrow_G^*$

$w\}$, respectively. The family of languages generated by propagating scattered context grammars which use leftmost or rightmost derivations is denoted by $\mathscr{L}(\mathit{PSC}, \mathrm{lm})$ or $\mathscr{L}(\mathit{PSC}, \mathrm{rm})$, respectively.

3 Main Results

The following theorem and its proof, which represent the main result of this paper, demonstrate that propagating scattered context grammars which use leftmost derivations are equivalent to context-sensitive grammars.

Theorem 1 $\mathscr{L}(\mathit{PSC}, \mathrm{lm}) = \mathscr{L}(\mathit{CS})$.

Proof: As propagating scattered context grammars do not contain erasing productions, their derivations can be simulated by linear bounded automata. As a result, $\mathscr{L}(\mathit{PSC}, \mathrm{lm}) \subseteq \mathscr{L}(\mathit{CS})$. In what follows, we demonstrate that also $\mathscr{L}(\mathit{CS}) \subseteq \mathscr{L}(\mathit{PSC}, \mathrm{lm})$ holds true by demonstrating that for every grammar in the Kuroda normal form there exists an equivalent propagating scattered context grammar which uses leftmost derivations.

Let $G = (V, T, P, S)$ be a grammar in the Kuroda normal form. Set $N_1 = (V - T) \cup \{\bar{a} : a \in T\}$ (suppose that $(V - T) \cap \{\bar{a} : a \in T\} = \emptyset$), $\hat{N}_1 = \{\hat{A} : A \in N_1\}$. Let $n = |N_1|$; then, we denote the elements of N_1 as $\{A_1, A_2, \ldots, A_n\}$. Define the homomorphism α from V^* to N_1^* as $\alpha(A) = A$ for each $A \in V - T$, and $\alpha(a) = \bar{a}$ for each $a \in T$. Set $N_2' = \{A' : A \in V - T\}$, $N_3 = \{\langle ab \rangle : a, b \in V\}$, $N_4' = \{\langle Aa \rangle' : A \in V - T, a \in V\}$, and

$$\begin{aligned} N_5 = \{&\langle a, 0 \rangle, \langle ab, 0 \rangle : a, b \in V\} \\ &\cup \{\langle a, i, j \rangle : a \in V - T, 1 \leq i \leq 3, 1 \leq j \leq n\} \\ &\cup \{\langle ab, 4 \rangle : a, b \in T\}. \end{aligned}$$

Without loss of generality, assume that the sets $N_1, \hat{N}_1, N_2', N_3, N_4', N_5, \{\bar{S}, X\}$, and T are pairwise disjoint. Define the propagating scattered context grammar

$$\bar{G} = (N_1 \cup \hat{N}_1 \cup N_2' \cup N_3 \cup N_4' \cup N_5 \cup \{\bar{S}, X\} \cup T, T, \bar{P}, \bar{S}),$$

where $\bar{P}$ is constructed as follows:

1. (a) For each $a \in L(G)$, where $a \in T$, add
   $(\bar{S}) \to (a)$ to $\bar{P}$;

   (b) For each $S \Rightarrow_G ab$, where $a, b \in V$, add
   $(\bar{S}) \to (\langle ab, 0 \rangle X)$ to $\bar{P}$;

2. For each $a, b, c \in V$, add

   (a) $(\langle a, 0 \rangle, \alpha(b)) \to (\alpha(a), \langle b, 0 \rangle)$,

   (b) $(\alpha(a), \langle b, 0 \rangle) \to (\langle a, 0 \rangle, \alpha(b))$,

   (c) $(\langle a, 0 \rangle, \langle bc \rangle) \to (\alpha(a), \langle bc, 0 \rangle)$,

   (d) $(\alpha(a), \langle bc, 0 \rangle) \to (\langle a, 0 \rangle, \langle bc \rangle)$ to $\bar{P}$;

3. For each $A \to a \in P$ and $b \in V$, add

(a) $(\langle A,0\rangle) \to (\langle a,0\rangle)$,
(b) $(\langle Ab,0\rangle) \to (\langle ab,0\rangle)$,
(c) $(\langle bA,0\rangle) \to (\langle ba,0\rangle)$ to $\bar{P}$;

4. For each $A \to BC \in P$ and $a \in V$, add
   (a) $(\langle A,0\rangle) \to (B\langle C,0\rangle)$,
   (b) $(\langle Aa,0\rangle) \to (B\langle Ca,0\rangle)$,
   (c) $(\langle aA,0\rangle) \to (\alpha(a)\langle BC,0\rangle)$ to $\bar{P}$;

5. For each $AB \to CD \in P$, $a \in V$, $E \in N_3 \cup N_4'$, $F' \in \{B', \langle Ba\rangle'\}$, $1 \le i \le n$, and $1 \le j \le n-1$, add
   (a) $(\langle AB,0\rangle) \to (\langle CD,0\rangle)$,
   (b) i. $(\langle A,0\rangle, B, X) \to (\langle A,1,1\rangle, B', A_1)$,
       ii. $(\langle A,0\rangle, \langle Ba\rangle, X) \to (\langle A,1,1\rangle, \langle Ba\rangle', A_1)$,
   (c) i. $(\langle A,1,i\rangle, A_i) \to (\langle A,2,i\rangle, \hat{A}_i)$,
       ii. $(\langle A,2,i\rangle, F', \hat{A}_i) \to (\langle A,3,i\rangle, F', A_i)$,
       iii. $(\langle A,3,j\rangle, E, A_j) \to (\langle A,1,j+1\rangle, E, A_{j+1})$,
   (d) i. $(\langle A,3,n\rangle, B', E, A_n) \to (\langle C,0\rangle, D, E, X)$,
       ii. $(\langle A,3,n\rangle, \langle Ba\rangle', A_n) \to (\langle C,0\rangle, \langle Da\rangle, X)$ to $\bar{P}$;

6. For each $a, b, c \in T$, add
   (a) $(\langle ab,0\rangle) \to (\langle ab,4\rangle)$,
   (b) $(\bar{c}, \langle ab,4\rangle) \to (c, \langle ab,4\rangle)$,
   (c) $(\langle ab,4\rangle, X) \to (a,b)$ to $\bar{P}$.

In short, productions introduced in (1) initiate the derivation, productions from (2) are used to select the nonterminal to be rewritten, productions from (3), (4), and (5) simulate G's productions of the form $A \to a$, $A \to BC$, and $AB \to CD$, respectively, and, finally, productions from (6) finish the derivation. In the following paragraphs, we describe the derivation of $\bar{G}$ in greater detail.

Every derivation starts either by a production introduced in (1a) to generate sentences $a \in L(G)$, where $a \in T$, or by a production introduced in (1b) to generate sentences $x \in L(G)$, where $|x| \ge 2$. As $\bar{S}$ does not occur on the right-hand side of any production, productions from (1) are not used during the rest of the derivation.

Consider G's sentential form $a_1a_2\dots a_k$, where $a_1, a_2, \dots, a_k \in V$, for some $k \ge 2$. In $\bar{G}$, this sentential form corresponds to

$$b_1b_2\dots b_{r-1}\langle a_r,0\rangle b_{r+1}b_{r+2}\dots b_{k-2}\langle a_{k-1}a_k\rangle X,$$

where $b_i = \alpha(a_i)$ for all $i \in \{1,2,\dots,r-1,r+1,r+2,\dots,k-2\}$, for some $1 \le r \le k-2$, or to

$$b_1b_2\dots b_{k-2}\langle a_{k-1}a_k,0\rangle X,$$

where $b_i = \alpha(a_i)$ for all $1 \leq i \leq k-2$ (observe that every right-hand side of a production from (1b) represents a sentential form of this kind). To simulate a G's production, the leftmost nonterminal from its left-hand side has to be selected in the sentential form of $\bar{G}$. This is done by appending 0 to the symbol to be selected by productions from (2). Specifically, for a symbol $a \in V$, (2a) selects the leftmost symbol a immediately following the currently selected symbol and (2b) selects the leftmost symbol a preceding the currently selected symbol. Productions from (2c) and (2d) are used to select and unselect the penultimate nonterminal in $\bar{G}$'s sentential form which is composed of two symbols from V. Observe that in this way, any symbol (except for the final X) in every sentential form of $\bar{G}$ can be selected. Further, observe that during a derivation, always one symbol is selected.

After the nonterminal is selected, the use of the G's production can be simulated. Productions of the form $A \to a$ are simulated by (3a) for every selected nonterminal $a_1, a_2, \ldots, a_{k-2}$ and by (3b), (3c) if the penultimate nonterminal (which contains a_{k-1}, a_k) of the $\bar{G}$'s sentential form is selected. Analogously, productions of the form $A \to BC$ are simulated by productions from (4).

Productions from (5a) are used to simulate an application of productions of the form $AB \to CD$ within the penultimate nonterminal of $\bar{G}$'s sentential form. In what follows, we demonstrate how productions from (5b), (5c), and (5d) are used if this production is simulated within $a_1 a_2 \ldots a_{k-2}$. Suppose that the sentential form in $\bar{G}$ is of the form

$$b_1 b_2 \ldots b_{r-1} \langle a_r, 0 \rangle b_{r+1} b_{r+2} \ldots b_{k-2} \langle a_{k-1} a_k \rangle X$$

and we simulate the application of $a_r a_{r+1} \to c_r c_{r+1} \in P$. Recall that $N_1 = \{A_1, A_2, \ldots, A_n\}$ denotes the set of all symbols which may appear in $b_{r+1} b_{r+2} \ldots b_{k-2}$. First, to select $b_{r+1} = \alpha(a_{r+1})$, the production

$$(\langle a_r, 0 \rangle, b_{r+1}, X) \to (\langle a_r, 1, 1 \rangle, b'_{r+1}, A_1)$$

from (5(b)i) is applied in a successful derivation, so

$$\begin{aligned} & b_1 b_2 \ldots b_{r-1} \langle a_r, 0 \rangle b_{r+1} b_{r+2} \ldots b_{k-2} \langle a_{k-1} a_k \rangle X \\ {}_{\mathrm{lm}}\Rightarrow_{\bar{G}} \; & b_1 b_2 \ldots b_{r-1} \langle a_r, 1, 1 \rangle b'_{r+1} b_{r+2} b_{r+3} \ldots b_{k-2} \langle a_{k-1} a_k \rangle A_1. \end{aligned}$$

Observe that if b_{r+1} does not immediately follow $\langle a_r, 0 \rangle$, the leftmost $b \in \mathrm{alph}(b_{r+2} b_{r+3} \ldots b_{k-2})$ satisfying $b = b_{r+1}$ is selected by the production from (5(b)i). The purpose of productions from (5c) is to verify that the nonterminal immediately following $\langle a_r, 0 \rangle$ has been selected. First, the production

$$(\langle a_r, 1, 1 \rangle, A_1) \to (\langle a_r, 2, 1 \rangle, \hat{A}_1)$$

from (5(c)i) is applied to tag the first A_1 following $\langle a_r, 1, 1 \rangle$, so

$$\begin{aligned} & b_1 b_2 \ldots b_{r-1} \langle a_r, 1, 1 \rangle b'_{r+1} b_{r+2} b_{r+3} \ldots b_{k-2} \langle a_{k-1} a_k \rangle A_1 \\ {}_{\mathrm{lm}}\Rightarrow_{\bar{G}} \; & b_1 b_2 \ldots b_{r-1} \langle a_r, 2, 1 \rangle b'_{r+1} y_1 \langle a_{k-1} a_k \rangle d_1, \end{aligned}$$

where either

$$y_1 = b_{r+2} b_{r+3} \ldots b_{m-1} \hat{A}_1 b_{m+1} b_{m+2} \ldots b_{k-2}, \; d_1 = A_1,$$

satisfying $A_1 \notin \mathrm{alph}(b_{r+2} b_{r+3} \ldots b_{m-1})$, for some $1 \leq m \leq k-2$, or

$$y_1 = b_{r+2} b_{r+3} \ldots b_{k-2}, \; d_1 = \hat{A}_1,$$

satisfying $A_1 \notin \mathrm{alph}(y_1)$. Then, the production

$$(\langle a_r, 2, 1\rangle, b'_{r+1}, \hat{A}_1) \rightarrow (\langle a_r, 3, 1\rangle, b'_{r+1}, A_1)$$

from (5(c)ii) is applied to untag the first symbol $\hat{A}_1$ following b'_{r+1}, so

$$\begin{aligned} & b_1 b_2 \dots b_{r-1}\langle a_r, 2, 1\rangle b'_{r+1} y_1 \langle a_{k-1} a_k\rangle d_1, \\ {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ & b_1 b_2 \dots b_{r-1}\langle a_r, 3, 1\rangle b'_{r+1} b_{r+2} b_{r+3} \dots b_{k-2}\langle a_{k-1} a_k\rangle A_1. \end{aligned}$$

This means that if A_1 occurs between $\langle a_r, 2, 1\rangle$ and b'_{r+1}, it is tagged by the production from (5(c)i) but it cannot be untagged by any production from (5(c)ii), so the derivation is blocked. Finally, the production

$$(\langle a_r, 3, 1\rangle, \langle a_{k-1} a_k\rangle, A_1) \rightarrow (\langle a_r, 1, 2\rangle, \langle a_{k-1} a_k\rangle, A_2)$$

from (5(c)iii) is applied, so

$$\begin{aligned} & b_1 b_2 \dots b_{r-1}\langle a_r, 3, 1\rangle b'_{r+1} b_{r+2} b_{r+3} \dots b_{k-2}\langle a_{k-1} a_k\rangle A_1 \\ {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ & b_1 b_2 \dots b_{r-1}\langle a_r, 1, 2\rangle b'_{r+1} b_{r+2} b_{r+3} \dots b_{k-2}\langle a_{k-1} a_k\rangle A_2, \end{aligned}$$

and the same verification continues for A_2. This verification proceeds for all symbols from N_1 so this part of the derivation can be expressed as

$$\begin{array}{lll} \quad u_1 & {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ v_1\ [p_{11}] & {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ w_1\ [p_{12}] \\ {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ u_2\ [p_{13}] & {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ v_2\ [p_{21}] & {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ w_2\ [p_{22}] \\ \quad \vdots & & \\ {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ u_n\ [p_{(n-1)3}] & {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ v_n\ [p_{n1}] & {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ w_n\ [p_{n2}] \end{array}$$

with

$$\begin{aligned} u_i &= b_1 b_2 \dots b_{r-1}\langle a_r, 1, i\rangle b'_{r+1} b_{r+2} b_{r+3} \dots b_{k-2}\langle a_{k-1} a_k\rangle A_i, \\ v_i &= b_1 b_2 \dots b_{r-1}\langle a_r, 2, i\rangle b'_{r+1} y_i \langle a_{k-1} a_k\rangle d_i, \\ w_i &= b_1 b_2 \dots b_{r-1}\langle a_r, 3, i\rangle b'_{r+1} b_{r+2} b_{r+3} \dots b_{k-2}\langle a_{k-1} a_k\rangle A_i, \end{aligned}$$

where p_{i1}, p_{i2}, and p_{j3} are productions from (5(c)i), (5(c)ii), and (5(c)iii), respectively, for all $1 \leq i \leq n$, $1 \leq j \leq n-1$, and either

$$y_i = b_{r+2} b_{r+3} \dots b_{i_{m-1}} \hat{A}_i b_{i_{m+1}} b_{i_{m+2}} \dots b_{k-2},\ d_i = A_i,$$

satisfying $A_i \notin \mathrm{alph}(b_{r+2} b_{r+3} \dots b_{i_{m-1}})$, for some $1 \leq i_m \leq k-2$, or

$$y_i = b_{r+2} b_{r+3} \dots b_{k-2},\ d_i = \hat{A}_i,$$

satisfying $A_i \notin \mathrm{alph}(y_i)$. After the verification is finished, the application of $a_r a_{r+1} \rightarrow c_r c_{r+1} \in P$ is simulated by

$$(\langle a_r, 3, n\rangle, b'_{r+1}, \langle a_{k-1} a_k\rangle, A_n) \rightarrow (\langle c_r, 0\rangle, c_{r+1}, \langle a_{k-1} a_k\rangle, X)$$

from (5(d)i), so

$$\begin{aligned} & b_1 b_2 \dots b_{r-1}\langle a_r, 3, n\rangle b'_{r+1} b_{r+2} b_{r+3} \dots b_{k-2}\langle a_{k-1} a_k\rangle A_n \\ {}_{\mathrm{lm}}\Rightarrow_{\bar{G}}\ & b_1 b_2 \dots b_{r-1}\langle c_r, 0\rangle c_{r+1} b_{r+2} b_{r+3} \dots b_{k-2}\langle a_{k-1} a_k\rangle X. \end{aligned}$$

Observe that in order to simulate a production of the form $AB \to CD$ within $a_{k-2}a_{k-1}$, productions from (5(b)ii) and (5(d)ii) have to be used instead of productions from (5(b)i) and (5(d)i) in the simulation described above. The details are left to the reader.

Finally, consider a G's sentence $a_1 a_2 \dots a_k \in T^+$. This corresponds to

$$\bar{a}_1\bar{a}_2 \dots \bar{a}_{r-1}\langle a_r, 0\rangle \bar{a}_{r+1}\bar{a}_{r+2} \dots \bar{a}_{k-2}\langle a_{k-1}a_k\rangle X,$$

or

$$\bar{a}_1\bar{a}_2 \dots \bar{a}_{k-2}\langle a_{k-1}a_k, 0\rangle X$$

in $\bar{G}$ after finishing the simulation. To enter the final phase in $\bar{G}$, we need the sentential form to be in the second above described form. This can be achieved by applying a production from (2c) to the first sentential form. The rest of the derivation can be expressed as

$$\begin{array}{lll} & \bar{a}_1\bar{a}_2 \dots \bar{a}_{k-2}\langle a_{k-1}a_k, 0\rangle X & \\ {}_{\mathrm{lm}}\!\Rightarrow_{\bar{G}} & \bar{a}_1\bar{a}_2 \dots \bar{a}_{k-2}\langle a_{k-1}a_k, 4\rangle X & [p_{6a}] \\ {}_{\mathrm{lm}}\!\Rightarrow_{\bar{G}}^{k-2} & a_1 a_2 \dots a_{k-2}\langle a_{k-1}a_k, 4\rangle X & [\Xi_{6b}] \\ {}_{\mathrm{lm}}\!\Rightarrow_{\bar{G}} & a_1 a_2 \dots a_{k-2}a_{k-1}a_k & [p_{6c}], \end{array}$$

where p_{6a} and p_{6c} are productions introduced in steps (6a) and (6c), respectively, and Ξ_{6b} is a sequence of $k-2$ productions from (6b). As a result, $x \in L(\bar{G}, \mathrm{lm})$ if and only if $x \in L(G)$. Therefore, $\mathscr{L}(\mathit{CS}) \subseteq \mathscr{L}(\mathit{PSC}, \mathrm{lm})$.

As $\mathscr{L}(\mathit{PSC}, \mathrm{lm}) \subseteq \mathscr{L}(\mathit{CS})$ and $\mathscr{L}(\mathit{CS}) \subseteq \mathscr{L}(\mathit{PSC}, \mathrm{lm})$, we obtain $\mathscr{L}(\mathit{PSC}, \mathrm{lm}) = \mathscr{L}(\mathit{CS})$, so the theorem holds. □

Next, we state the following corollary.

Corollary 1 $\mathscr{L}(\mathit{PSC}, \mathrm{rm}) = \mathscr{L}(\mathit{CS})$.

Proof: This corollary can be proved by a straightforward modification of the proof of Theorem 1 and its proof is, therefore, left to the reader. □

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