

# Symmetric monochromatic subsets in colorings of the Lobachevsky plane

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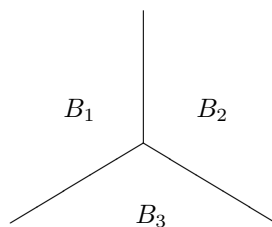
We prove that for each partition of the Lobachevsky plane into finitely many Borel pieces one of the cells of the partition contains an unbounded centrally symmetric subset.

**Keywords:** Partition, central symmetry, monochromatic set, Borel piece, Lobachevsky plane, Poincaré model, Borel  $k$ -partition, coloring

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## 1 Introduction

It follows from [B<sub>1</sub>] (see also [BP<sub>1</sub>, Theorem 1]) that for each partition of the  $n$ -dimensional space  $\mathbb{R}^n$  into  $n$  pieces one of the pieces contains an unbounded centrally symmetric subset. On the other hand,  $\mathbb{R}^n$  admits a partition  $\mathbb{R}^n = B_0 \cup \dots \cup B_n$  into  $(n + 1)$  Borel pieces containing no unbounded centrally symmetric subset. For  $n = 2$  such a partition is drawn at the picture:



Taking the same partition of the Lobachevsky plane  $H^2$ , we can see that each piece  $B_i$  does contain an unbounded centrally symmetric subset (for such a set just take any hyperbolic line lying in  $B_i$ ).

We call a subset  $S$  of the hyperbolic plane  $H^2$  *centrally symmetric* or else *symmetric with respect to a point*  $c \in H^2$  if  $S = f_c(S)$  where  $f_c : H^2 \rightarrow H^2$  is the involutive isometry of  $H^2$  assigning to each point  $x \in H^2$  the unique point  $y \in H^2$  such that  $c$  is the midpoint of the segment  $[x, y]$ . The map  $f_c$  is called the *central symmetry* of  $H^2$  with respect to the point  $c$ .

By a *partition* of a set  $X$  we understand a decomposition  $X = B_1 \cup \dots \cup B_n$  of  $X$  into pairwise disjoint subsets called the *pieces* of the partition.

The following theorem shows that the Lobachevsky plane differs dramatically from the Euclidean plane from the Ramsey point of view.

**Theorem 1.1** *For any partition  $H^2 = B_1 \cup \dots \cup B_m$  of the Lobachevsky plane into finitely many Borel pieces one of the pieces contains an unbounded centrally symmetric subset.*

## 2 Proof of Theorem 1.1

We shall prove a bit more: given a partition  $H^2 = B_1 \cup \dots \cup B_m$  of the Lobachevsky plane into  $m$  Borel pieces we shall find  $i \leq m$  and an unbounded subset  $S \subset B_i$  symmetric with respect to some point  $c$  in an arbitrarily small neighborhood of some finite set  $F \subset H^2$  depending only on  $m$ .

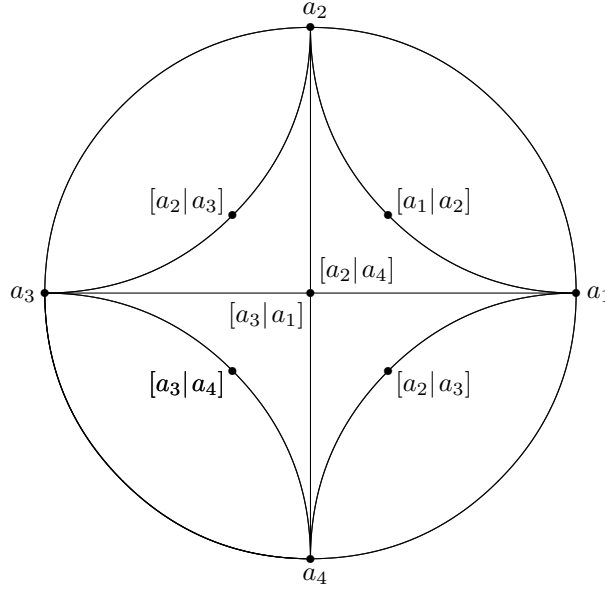
To define this set  $F$  it will be convenient to work in the Poincaré model of the Lobachevsky plane  $H^2$ . In this model the hyperbolic plane  $H^2$  is identified with the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  on the complex plane and hyperbolic lines are just segments of circles orthogonal to the boundary of  $\mathbb{D}$ . Let  $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$  be the hyperbolic plane  $\mathbb{D}$  with attached ideal line. For a real number  $R > 0$  the set  $\mathbb{D}_R = \{z \in \mathbb{C} : |z| \leq 1 - 1/R\}$  can be thought as a hyperbolic disk of increasing radius as  $R$  tends to  $\infty$ .

On the boundary of the unit disk  $\mathbb{D}$  consider the  $(m + 1)$ -element set

$$A = \{z \in \mathbb{C} : z^{m+1} = 1\}.$$

For two distinct points  $x, y \in A$  let  $[x|y] \in \mathbb{D}$  denote the midpoint of the arc in  $\overline{\mathbb{D}}$  that connects the points  $x, y$  and lies on a hyperbolic line in  $H^2 = \mathbb{D}$ . Then  $F = \{[x|y] : x, y \in A, x \neq y\}$  is a finite subset of cardinality  $|F| \leq m(m + 1)/2$  in the unit disk  $\mathbb{D}$ .

For  $m = 3$  the set  $A$  consists of four points  $a_1 = 1, a_2 = i, a_3 = -1$  and  $a_4 = -i$  while  $F$  consists of five points  $[a_1|a_2], [a_2|a_3], [a_3|a_4], [a_4|a_1], [a_1|a_3] = [a_2|a_4]$  as shown at the following picture:



We claim that for any open neighborhood  $W$  of  $F$  in  $\mathbb{C}$  one of the pieces of a partition  $H^2 = B_1 \cup \dots \cup B_m$  contains an unbounded subset symmetric with respect to some point  $c \in W$ . To derive a contradiction we assume the converse: for every point  $c \in W$  and every  $i \leq m$  the set  $B_i \cap f_c(B_i)$  is bounded in  $H^2$ .

For every  $n \in \mathbb{N}$  consider the set

$$C_n = \{c \in W : \bigcup_{i=1}^m B_i \cap f_c(B_i) \subset \mathbb{D}_n\}.$$

We claim that  $C_n$  is a coanalytic subset of  $W$ , that is, the complement  $W \setminus C_n$  is analytic, which in its turn means that  $W \setminus C_n$  is the continuous image of a Polish space. Observe that

$$W \setminus C_n = \{c \in W : \text{there are } i \leq m \text{ and } x \in B_i \cap f_c(B_i) \setminus \mathbb{D}_n\} = \text{pr}_2(E),$$

where  $\text{pr}_2 : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$  is the projection on the second factor and

$$E = \bigcup_{i=1}^m \{(x, c) \in \mathbb{D} \times W : x \in B_i \cap f_c(B_i) \setminus \mathbb{D}_n\}$$

is a Borel subset of  $\mathbb{D} \times W$ . Being a Borel subset of the Polish space  $\mathbb{D} \times W$ , the space  $E$  is analytic and so is its continuous image  $\text{pr}_2(E) = W \setminus C_n$ . Then  $C_n$  is coanalytic and hence has the Baire property [Ke, 21.6], which means that  $C_n$  coincides with an open subset  $U_n$  of  $W$  modulo some meager set. The latter means that the symmetric difference  $U_n \triangle C_n$  is meager (i.e., is of the first Baire category) in  $W$ . Replacing  $U_n$  by the interior of the closure  $\bar{U}_n$  of  $U_n$  in  $W$ , if necessary, we may additionally assume that  $U_n$  is regular open, that is,  $U_n$  coincides with the interior of its closure in  $W$ .

We claim that  $C_n \subset C_{n+1}$  implies  $U_n \subset U_{n+1}$ . First we check that

$$U_n \setminus U_{n+1} \subset (U_n \triangle C_n) \cup (U_{n+1} \triangle C_{n+1})$$

is meager. Indeed, for every  $x \in U_n \setminus U_{n+1}$  we get  $x \in U_n \setminus C_n \subset U_n \triangle C_n$  if  $x \notin C_n$  and  $x \in C_{n+1} \setminus U_{n+1} \subset U_{n+1} \triangle C_{n+1}$  if  $x \in C_n \subset C_{n+1}$ . Therefore, the set  $U_n \setminus U_{n+1}$  is meager, which implies  $U_n \subset \overline{U_{n+1}}$  and hence  $U_n \subset U_{n+1}$  because the set  $U_{n+1}$  is regular open.

Let  $U = \bigcup_{n=1}^{\infty} U_n$  and  $M = \bigcup_{n=1}^{\infty} U_n \triangle C_n$ . Taking into account that  $W = \bigcup_{n=1}^{\infty} C_n$ , we conclude that

$$W \setminus U = \bigcup_{n=1}^{\infty} C_n \setminus \bigcup_{n=1}^{\infty} U_n \subset \bigcup_{n=1}^{\infty} C_n \setminus U_n \subset \bigcup_{n=1}^{\infty} C_n \triangle U_n = M$$

which implies that the open set  $U$  has meager complement and thus is dense in  $W$ .

We claim that  $F \subset h^{-1}(U)$  for some isometry  $h$  of the hyperbolic plane  $H^2 = \mathbb{D}$ .

For this consider the natural action

$$\mu : \text{Iso}(H^2) \times \mathbb{D} \rightarrow \mathbb{D}, \quad \mu : (h, x) \mapsto h(x)$$

of the isometry group  $\text{Iso}(H^2)$  of the hyperbolic plane  $H^2 = \mathbb{D}$ . It is easy to see that for every  $x \in \mathbb{D}$  the map  $\mu_x : \text{Iso}(H^2) \rightarrow \mathbb{D}, \mu_x : h \mapsto h(x)$ , is continuous and open (with respect to the compact-open topology on  $\text{Iso}(H^2)$ ). It follows that the set

$$\bigcap_{x \in F} \mu_x^{-1}(W) = \{h \in \text{Iso}(H^2) : h(F) \subset W\}$$

is an open neighborhood of the neutral element of the group  $\text{Iso}(H^2)$ .

Taking into account that  $U$  is open and dense in  $W$ , and that for every  $x \in F$  the map  $\mu_x : \text{Iso}(H^2) \rightarrow \mathbb{D}$  is open, we conclude that the preimage  $\mu_x^{-1}(U)$  is open and dense in  $\mu_x^{-1}(W) \subset \text{Iso}(H^2)$ . Then the intersection  $\bigcap_{x \in F} \mu_x^{-1}(U)$ , being an open dense subset of  $\bigcap_{x \in F} \mu_x^{-1}(W)$ , is not empty and hence contains some isometry  $h$  having the desired property:  $F \subset h^{-1}(U)$ . Since  $F$  is finite, there is  $n \in \mathbb{N}$  with  $F \subset h^{-1}(U_n)$ . For a complex number  $r \in \mathbb{D}$  consider the set  $rA = \{rz : z \in A\} \subset \mathbb{D}$  and let

$$F_r = \{[x|y] : x, y \in rA, x \neq y\} \subset \mathbb{D},$$

where  $[x|y]$  stands for the midpoint of the hyperbolic segment connecting  $x$  and  $y$  in  $H^2$ . It can be shown that for any distinct points  $x, y \in A$  the midpoint  $[rx|ry]$  tends to the midpoint  $[x|y] \in F$  as  $r$  tends to 1. Such a continuity yields a neighborhood  $O_1$  of 1 such that  $F_r \subset h^{-1}(U_n)$  for all  $r \in O_1 \cap \mathbb{D}$ .

It is clear that for any points  $x, y \in A$  the map

$$f_{x,y} : \mathbb{D} \rightarrow \mathbb{D}, \quad f_{x,y} : r \mapsto [rx|ry]$$

is open and continuous. Consequently, the preimage  $f_{x,y}^{-1}(h^{-1}(M))$  is a meager subset of  $\mathbb{D}$  and so is the union  $M' = \bigcup_{x,y \in A} f_{x,y}^{-1}(h^{-1}(M))$ . So, we can find a non-zero point  $r \in O_1 \setminus M'$  so close to 1 that the set  $rA$  is disjoint with the hyperbolic disk  $h^{-1}(\mathbb{D}_n)$  (observe that for a complex number  $r$  close to 1 the set  $rA$  is close to the set  $A$  lying in the boundary circle of  $\mathbb{D}$  and thus  $rA$  can be made disjoint with the compact subset  $h^{-1}(\mathbb{D}_n)$  of  $\mathbb{D}$ ). For this point  $r$  we shall get  $F_r \cap h^{-1}(M) = \emptyset$ .

The set  $rA$  consists of  $m + 1$  points. Consequently, some cell  $h^{-1}(B_i)$  of the partition  $\mathbb{D} = h^{-1}(B_1) \cup \dots \cup h^{-1}(B_m)$  contains two distinct points  $rx, ry$  of  $rA$ . Those points are symmetric with respect to the point

$$[rx|ry] \in F_r \subset h^{-1}(U_n) \setminus h^{-1}(M).$$

Then the images  $a = h(rx)$  and  $b = h(ry)$  belong to  $B_i$  and are symmetric with respect to the point  $c = h([rx|ry]) \in U_n \setminus M \subset C_n$ . It follows from the definition of  $C_n$  that  $\{a, b\} \subset B_i \cap f_c(B_i) \subset \mathbb{D}_n$ , which is not the case because  $rx, ry \notin h^{-1}(\mathbb{D}_n)$ .

### 3 Concerning partitions of $H^2$

We do not know if Theorem 1.1 is true for any finite (not necessarily Borel) partition of the Lobachevsky plane  $H^2$ . For partitions of  $H^2$  into two pieces, the Borel assumption is superfluous.

**Theorem 3.1** *There is a subset  $T \subset H^2$  of cardinality  $|T| = 3$  such that for any partition  $H^2 = A_1 \cup A_2$  of  $H^2$  into two pieces either  $A_1$  or  $A_2$  contains an unbounded subset, symmetric with respect to some point  $c \in T$ .*

**Proof:** Lemma 3.2 below allows us to find an equilateral triangle  $\Delta_{c_0 c_1 c_2}$  on the Lobachevsky plane  $H^2$  such that the composition  $f_{c_2} \circ f_{c_1} \circ f_{c_0}$  of the symmetries with respect to the points  $c_0, c_1, c_2$  coincides with the rotation by the angle  $2\pi/3$  about some point  $o \in H^2$ . Consequently  $(f_{c_2} \circ f_{c_1} \circ f_{c_0})^3$  is the identity isometry of  $H^2$ .

We claim that for any partition  $H^2 = A_1 \sqcup A_2$  of the Lobachevsky plane into two pieces one of the pieces contains an unbounded subset symmetric with respect to some point in the triangle  $T = \{c_0, c_1, c_2\}$ . Assuming the converse, we conclude that the set

$$B = \bigcup_{c \in T} \bigcup_{i=1}^2 A_i \cap f_c(A_i)$$

is bounded. It follows that two points  $x, y \in H^2 \setminus B$ , symmetric with respect to a center  $c \in T$  cannot belong to the same cell  $A_i$  of the partition.

Let  $B_0 = B$  and  $B_{i+1} = B_i \cup \bigcup_{j=0}^2 f_{c_j}^{-1}(B_i)$  for  $i \geq 0$ . By induction it can be shown that each set  $B_i$ ,  $i \geq 0$ , is bounded in  $H^2$ .

Fix any point  $x_0 \in H^2 \setminus B_9$  and consider the sequence of points  $x_1, \dots, x_9$  defined by the recursive formula:  $x_{i+1} = f_{c_{i \bmod 3}}(x_i)$ . It follows that

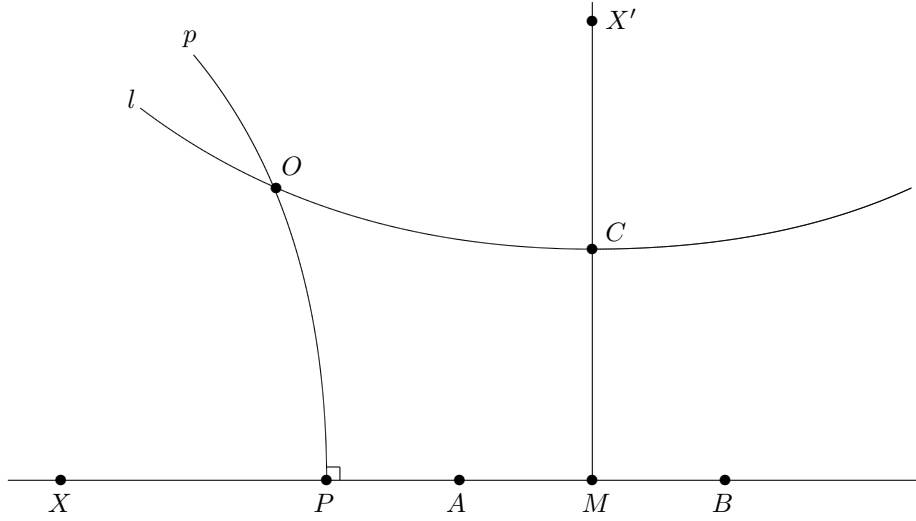
$$x_9 = (f_{c_2} \circ f_{c_1} \circ f_{c_0})^3(x_0) = x_0.$$

We claim that for every  $i \leq 9$  the point  $x_i$  does not belong to the set  $B$ . Assuming by contradiction that  $x_i \in B$ , we would conclude that  $x_{i-1} \in \bigcup_{j=0}^2 f_{c_j}^{-1}(B) \subset B_1$ . Continuing by induction, for every  $k \leq i$  we would get  $x_{i-k} \in B_k$ . In particular,  $x_0 \in B_i \subset B_9$ , which contradicts the choice of  $x_0$ .

The point  $x_0$  belongs either to  $A_1$  or to  $A_2$ . We lose no generality assuming that  $x_0 \in A_2$ . Since the points  $x_0, x_1 \notin B$  are symmetric with respect to  $c_0$  and  $x_0 \in A_2$ , we get that  $x_1 \in H^2 \setminus A_2 = A_1$ . By the same reason  $x_1, x_2$  cannot simultaneously belong to  $A_1$  and hence  $x_2 \in A_2$ . Continuing in this fashion we conclude that  $x_i$  belongs to  $A_1$  for odd  $i$  and to  $A_2$  for even  $i$ . In particular,  $x_9 \in A_1$ , which is not possible because  $x_9 = x_0 \in A_2$ .  $\square$

**Lemma 3.2** *There is an equilateral triangle  $\triangle ABC$  on the Lobachevsky plane such that the composition  $f_C \circ f_B \circ f_A$  of the symmetries with respect to the points  $A, B, C$  coincides with the rotation by the angle  $2\pi/3$  about some point  $O$ .*

**Proof:** For a positive real number  $t$  consider an equilateral triangle  $\triangle ABC$  with side  $t$  on the Lobachevsky plane. Let  $M$  be the midpoint of the side  $AB$  and  $l$  be the line through  $C$  that is orthogonal to the line  $CM$ . Consider also the line  $p$  that is orthogonal to the line  $AB$  and passes through the point  $P$  such that  $A$  is the midpoint between  $P$  and  $M$ . Observe that  $|PM| = |AB| = t$  and for sufficiently small  $t$  the lines  $p$  and  $l$  intersect at some point  $O$ .



It is easy to see that the composition  $f_B \circ f_A$  is the shift along the line  $AB$  by the distance  $2t$  and hence the image  $f_B \circ f_A(O)$  of the point  $O$  is the point symmetric to  $O$  with respect to the point  $C$ . Consequently,  $f_C \circ f_B \circ f_A(O) = O$ , which means that the isometry  $f_C \circ f_B \circ f_A$  is a rotation of the Lobachevsky plane about the point  $O$  by some angle  $\varphi_t$ .

To estimate this angle, consider the point  $X$  such that  $P$  is the midpoint between  $X$  and  $M$ . Then  $|XM| = 2t$  and consequently,  $f_B \circ f_A(X) = M$  while  $X' = f_C \circ f_B \circ f_A = f_C(M)$  is the point on the line  $CM$  such that  $C$  is the midpoint between  $X'$  and  $M$ . It follows that  $|X'X| \leq |XM| + |MX'| < 2t + 2t = 4t$ .

Observe that for small  $t$  the point  $X'$  is near to the point, symmetric to  $X$  with respect to  $O$ , which means that the angle  $\varphi_t = \angle XOX'$  is close to  $\pi$  for  $t$  close to zero. On the other hand, for very large  $t$  the lines  $p$  and  $l$  on the Lobachevsky plane do not intersect. So we can consider the smallest upper bound  $t_0$  of numbers  $t$  for which the lines  $l$  and  $p$  meet. For values  $t < t_0$  near to  $t_0$  the point  $O$  tends to infinity as  $t$  tends to  $t_0$ . Since the length of the side  $XX'$  of the triangle  $\triangle XOX'$  is bounded by  $4t_0$  the angle  $\varphi_t = \angle XOX'$  tends to zero as  $O$  tends to infinity. Since the angle  $\varphi_t$  depends continuously on  $t$  and decreases from  $\pi$  to zero as  $t$  increases from zero to  $t_0$ , there is a value  $t$  such that  $\varphi_t = 2\pi/3$ . For such  $t$  the composition  $f_C \circ f_B \circ f_A$  is the rotation around  $O$  on the angle  $2\pi/3$ .  $\square$

## 4 Some comments and open problems

In contrast with Theorem 1.1, Theorem 3.1 is true for the Euclidean plane  $E^2$  even in a stronger form: for any subset  $C \subset E^2$  not lying on a line and any partition  $E^2 = A_1 \cup A_2$  one of the cells of the partition contains an unbounded subset symmetric with respect to some center  $c \in C$ , see [B<sub>2</sub>].

Having in mind this result let us call a subset  $C$  of a Lobachevsky or Euclidean space  $X$  *central for (Borel)  $k$ -partitions* if for any partition  $X = A_1 \cup \dots \cup A_k$  of  $X$  into  $k$  (Borel) pieces one of the pieces contains an unbounded monochromatic subset  $S \subset X$ , symmetric with respect to some point  $c \in C$ . By  $c_k(X)$  (resp.  $c_k^B(X)$ ) we shall denote the smallest size of a subset  $C \subset X$ , central for (Borel)  $k$ -partitions of  $X$ . If no such set  $C$  exists, then we put  $c_k(X) = \infty$  (resp.  $c_k^B(X) = \infty$ ) where  $\infty$  is assumed to be greater than any cardinal number. It follows from the definition that  $c_k^B(X) \leq c_k(X)$ .

We have a lot of information about the numbers  $c_k^B(E^n)$  and  $c_k(E^n)$  for Euclidean spaces  $E^n$ , see [B<sub>2</sub>]. In particular, we known that

1.  $c_2(E^n) = c_2^B(E^n) = 3$  for all  $n \geq 2$ ;
2.  $c_3(E^3) = c_3^B(E^3) = 6$ ;
3.  $12 \leq c_4^B(E^4) \leq c_4(E^4) \leq 14$ ;
4.  $n(n+1)/2 \leq c_n^B(E^n) \leq c_n(E^n) \leq 2^n - 2$  for every  $n \geq 3$ .

Much less is known about the numbers  $c_k^B(H^n)$  and  $c_k(H^n)$  in the hyperbolic case. Theorem 3.1 yields the upper bound  $c_2(H^2) \leq 3$ . In fact, 3 is the exact value of  $c_2(H^n)$  for all  $n \geq 2$ .

**Proposition 4.1**  $c_2^B(H^n) = c_2(H^n) = 3$  for all  $n \geq 2$ .

**Proof:** The upper bound  $c_2(H^n) \leq c_2(H^2) \leq 3$  follows from Theorem 3.1. The lower bound  $3 \leq c_2^B(H^n)$  will follow as soon as for any two points  $c_1, c_2 \in H^n$  we construct a partition  $H^n = A_1 \cup A_2$  in two Borel pieces containing no unbounded set, symmetric with respect to a point  $c_i$ . To construct such a partition, consider the line  $l$  containing the points  $c_1, c_2$  and decompose  $l$  into two half-lines  $l = l_1 \sqcup l_2$ . Next, let  $H$  be an  $(n-1)$ -hyperplane in  $H^n$ , orthogonal to the line  $l$ . Let  $S$  be the unit sphere in  $H$  centered at the intersection point of  $l$  and  $H$ . Let  $S = B_1 \cup B_2$  be a partition of  $S$  into two Borel pieces such that no antipodal points of  $S$  lie in the same cell of the partition. For each point  $x \in H^n \setminus l$  consider the hyperbolic plane  $P_x$  containing the points  $x, c_1, c_2$ . The complement  $P_x \setminus l$  decomposes into two half-planes  $P_x^+ \cup P_x^-$  where  $P_x^+$  is the half-plane containing the point  $x$ . The plane  $P_x$  intersects the hyperplane  $H$  by a hyperbolic line containing two points of the sphere  $S$ . Finally put

$$A_i = l_i \cup \{x \in H^2 \setminus l : P_x^+ \cap B_i \neq \emptyset\}$$

for  $i \in \{1, 2\}$ . It is easy to check that  $A_1 \sqcup A_2 = H^n$  is the desired partition of the hyperbolic space into two Borel pieces none of which contains an unbounded subset symmetric with respect to one of the points  $c_1, c_2$ .  $\square$

The preceding proposition implies that the cardinal numbers  $c_2(H^n)$  are finite.

**Problem 4.2** For which numbers  $k, n$  are the cardinal numbers  $c_k(H^n)$  and  $c_k^B(H^n)$  finite? Is it true for all  $k \leq n$ ?

Except for the equality  $c_2(E^n) = 3$ , we have no information on the numbers  $c_k(E^n)$  with  $k < n$ .

**Problem 4.3** Calculate (or at least evaluate) the numbers  $c_k(E^n)$  and  $c_k(H^n)$  for  $2 < k < n$ .

In all the cases where we know the exact values of the numbers  $c_k(E^n)$  and  $c_k^B(E^n)$  we see that those numbers are equal.

**Problem 4.4** Are the numbers  $c_k(E^n)$  and  $c_k^B(E^n)$  (resp.  $c_k(H^n)$  and  $c_k^B(H^n)$ ) equal for all  $k, n$ ?

Having in mind that each subset not lying on a line is central for 2-partitions of the Euclidean plane, we may ask about the same property of the Lobachevsky plane.

**Problem 4.5** Is any subset  $C \subset H^2$  not lying on a line central for (Borel) 2-partitions of the Lobachevsky plane  $H^2$ ?

Finally, let us ask about the numbers  $c_k^B(H^2)$  and  $c_k(H^2)$ . Observe that Theorem 1.1 guarantees that  $c_k^B(H^2) \leq \mathfrak{c}$  for all  $k \in \mathbb{N}$ . Inspecting the proof we can see that this upper bound can be improved to  $c_k^B(H^2) \leq \text{non}(\mathcal{M})$  where  $\text{non}(\mathcal{M})$  is the smallest cardinality of a non-meager subset of the real line. It is clear that  $\aleph_1 \leq \text{non}(\mathcal{M}) \leq \mathfrak{c}$ . The exact location of the cardinal  $\text{non}(\mathcal{M})$  on the interval  $[\aleph_1, \mathfrak{c}]$  depends on axioms of Set Theory, see [Bl]. In particular, the inequality  $\aleph_1 = \text{non}(\mathcal{M}) < \mathfrak{c}$  is consistent with ZFC.

**Problem 4.6** Is the inequality  $c_k^B(H^2) \leq \aleph_1$  provable in ZFC? Are the cardinals  $c_k^B(H^2)$  countable? finite?

The last problem asks if  $H^2$  contains a countable (or finite) central set for Borel  $k$ -partitions of the Lobachevsky plane. Inspecting the proof of Theorem 1.1 we can see that it gives an “approximate” answer to this problem:

**Proposition 4.7** For any  $k \in \mathbb{N}$  there is a finite subset  $C \subset H^2$  of cardinality  $|C| \leq k(k+1)/2$  such that for any partition  $H^2 = B_1 \cup \dots \cup B_k$  of  $H^2$  into  $k$  Borel pieces and for any open neighborhood  $O(C) \subset H^2$  of  $C$  one of the pieces  $B_i$  contains an unbounded subset  $S \subset B_i$  symmetric with respect to some point  $c \in O(C)$ .

**Remark 4.8** For further results and open problems related to symmetry and colorings see the surveys [BP<sub>2</sub>], [BVV] and the list of problems [BBGRZ, §4].

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