On Hereditary Helly Classes of Graphs

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In graph theory, the Helly property has been applied to families of sets, such as cliques, disks, bicliques, and neighbourhoods, leading to the classes of clique-Helly, disk-Helly, biclique-Helly, neighbourhood-Helly graphs, respectively. A natural question is to determine for which graphs the corresponding Helly property holds, for every induced subgraph. This leads to the corresponding classes of hereditary clique-Helly, hereditary disk-Helly, hereditary biclique-Helly and hereditary neighbourhood-Helly graphs. In this paper, we describe characterizations in terms of families of forbidden subgraphs, for the classes of hereditary biclique-Helly and hereditary neighbourhood-Helly graphs. We consider both open and closed neighbourhoods. The forbidden subgraphs are all of fixed size, implying polynomial time recognition for these classes.

Keywords: Algorithms, bicliques, Helly property, graph classes

1 Introduction

In the scope of graph theory, the Helly property has been applied to families of sets as cliques, e.g. [6], [10], [12], [14], [19], bicliques [11], neighbourhoods [5] and disks [1], [3], [4]. The corresponding graphs are the clique-Helly, biclique-Helly, neighbourhood-Helly and disk-Helly graphs, respectively. Bicliques, in general, have been considered in some different contexts, e.g. [15, 16, 17, 20].

Besidesthe interest of examining bicliques in the scope of the Helly property, these graphs should be of interest in the study of retracts [13]. In fact, retracts of bipartite graphs are related to neighbourhood-Helly graphs and the latter are related to biclique-Helly graphs in some different aspects. See [2]. We also mention that some optimization problems, as the edge modification problem, have been already studied for the class of biclique-Helly graphs [8].

None of the above classes are closed under induced subgraphs. So, a question would be to characterize the graphs for which the Helly property is preserved for every induced subgraph. It leads to the hereditary classes of clique-Helly, biclique-Helly, neighbourhood-Helly and disk-Helly graphs. Hereditary clique-Helly graphs have been characterized in [18], while [9] (see [7]) contains a characterization of hereditary disk-Helly graphs. In this work, we describe forbidden subgraph characterizations for the

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classes of hereditary biclique-Helly and hereditary neighbourhood-Helly graphs. Both open and closed neighbourhoods are considered. All graphs in these forbidden families are of fixed size. In fact they have at most 8 vertices. Consequently, the characterizations imply polynomial time recognition for all the considered classes.

Denote by G a finite simple graph, with vertex set $V(G)$ and edge set $E(G)$. For $v_i, v_j \in V(G)$, write $v_iv_j \in E(G)$ to denote an edge with ends v_i and v_j. A *complete set* is a subset $V' \subseteq V(G)$ formed by pairwise adjacent vertices while in an *independent set* $V' \subseteq V(G)$ every pair of its vertices is not adjacent. Denote by $\alpha(G)$ the size of the largest independent set of G. A *clique* is a maximal complete set. A *biclique* is a subset $B \subseteq V(G)$ inducing a maximal complete bipartite graph in G, having at least one edge. Denote by $N(v_i) = \{v_j \in V(G) | v_iv_j \in E(G)\}$, and $N[v_i] = N(v_i) \cup \{v_i\}$, the *open* and *closed neighbourhoods* of G, respectively. For each non-negative integer k and $v_i \in V(G)$, a *disk* $D_k(v_i)$ of v_i is the set of vertices lying at distance at most k from v_i. For $V' \subseteq V(G)$, denote by $G[V']$ the subgraph of G induced by V'. Let $\mathcal{B}$ be a set of bicliques of G. Represent by $G_{\mathcal{B}}$ the *biclique subgraph* of $\mathcal{B}$, that is, the subgraph of G formed exactly by the vertices and edges of $\mathcal{B}$.

Let $\mathcal{F}$ be a family of subsets of some set. Say that $\mathcal{F}$ is *intersecting* when the subsets of $\mathcal{F}$ pairwise intersect. On the other hand, when every intersecting subfamily of $\mathcal{F}$ has a common element then $\mathcal{F}$ is a *Helly* family. A graph G is *clique-Helly* when its family of cliques is Helly. Similarly, G is *biclique-Helly* (open neighbourhood-Helly, closed neighbourhood-Helly) when its family of bicliques (open neighbourhoods, closed neighbourhoods) is Helly. Also, say that G is *disk-Helly* when the family of disks $\{D_k(v_i) | v_i \in V(G) \text{ and } 1 \leq k < |V(G)|\}$ is Helly.

Finally, G is *hereditary clique-Helly* when every of its induced subgraphs is clique-Helly. Similarly, define *hereditary biclique-Helly*, *hereditary open neighbourhood-Helly*, *hereditary closed neighbourhood-Helly* and *hereditary disk-Helly graphs.*

Hereditary clique-Helly graphs have been characterized as follows.

Theorem 1.1 [*18*]*: A graph G is hereditary clique-Helly if and only if it does not contain as induced subgraphs the Hajós graph and any graph obtained from the latter by inserting some edges joining its vertices of degree 2.*

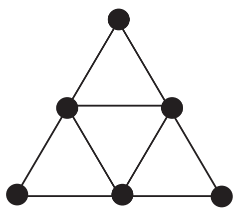

Figure 1: The Hajós graph

The following is a characterization for hereditary disk-Helly graphs.

Theorem 1.2 *[9]: A graph G is hereditary disk-Helly if and only if it is chordal and does not contain the Hajós graph as an induced subgraph.*

In Section 2, we describe a characterization for hereditary biclique-Helly graphs while the proposed characterizations for hereditary open and closed neighborhood-Helly graphs are in Section 3. Relations among these classes are formulated in the last section.

2 Hereditary biclique-Helly graphs

In this section, we describe a characterization for hereditary biclique-Helly graphs. Start with the following definition.

Let $S \subseteq V(G)$, $|S| = 3$. Denote by $\mathcal{B}_S$ the family of bicliques of G, each of them containing at least two vertices of S. Consider the graph $G_{\mathcal{B}_S}$ and denote its vertex set by $S^* \subseteq V(G)$. The induced subgraph $G[S^*]$ is called the *extension* of S. Clearly, $G_{\mathcal{B}_S}$ is a spanning subgraph of $G[S^*]$. The lemma below is useful.

Lemma 2.1 *[11]: Let G be a graph with neither triangles nor C_5's. Then each of its extensions is a bipartite graph.*

Next, the characterization is formulated.

Theorem 2.1 *: A graph G is hereditary biclique-Helly if and only if it does not contain any of the graphs of Figure 2, as induced subgraphs.*

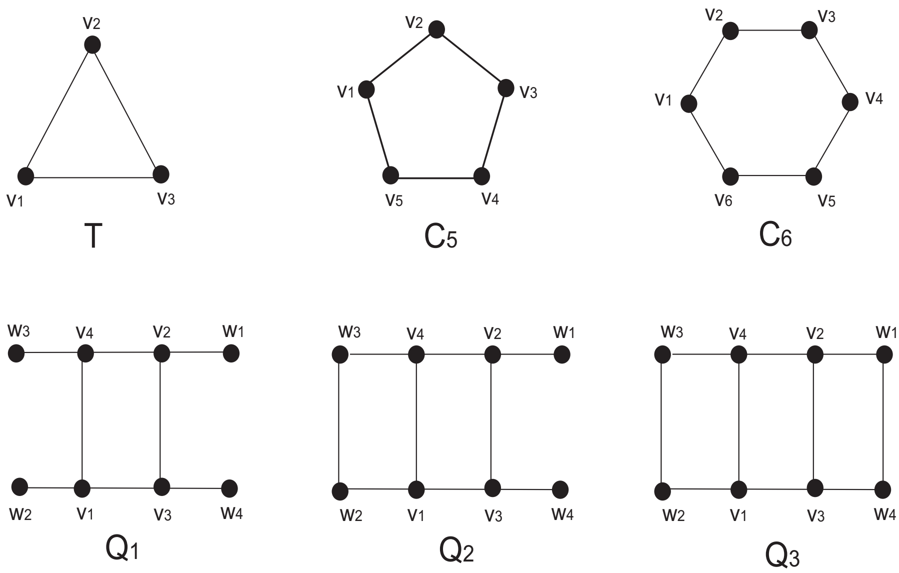

Figure 2

Proof: To prove that the graphs of Figure 2 are not biclique-Helly, we show a pairwise intersecting family $\mathcal{B}$ of bicliques with no common vertex, in each case. For the triangle, $\mathcal{B} = \{\{v_1, v_2\}$, $\{v_1, v_3\}$,$\{v_2, v_3\}\}$ and for the C_5, $\mathcal{B} = \{\{v_1, v_2, v_3\}$, $\{v_3, v_4, v_5\}$, $\{v_1, v_2, v_5\}\}$. For the C_6, the family is $\mathcal{B} = \{\{v_1, v_2, v_3\}$, $\{v_3, v_4, v_5\}$, $\{v_1, v_5, v_6\}\}$. Finally, for the graphs Q_1, Q_2 and Q_3, $\mathcal{B} = \{\{v_1, w_2, v_3, v_4\}$, $\{v_2, v_3, v_4, w_1\}$,$\{v_3, w_4, v_1, v_2\}$, $\{v_1, v_2, v_4, w_3\}\}$.

Conversely, let G be a graph that does not contain any of the graphs of Figure 2, as an induced subgraph. Suppose it is not hereditary biclique-Helly. Let H be an induced subgraph that is not biclique-Helly, and $\mathcal{B}$ a non-Helly family of bicliques of H. We can choose $\mathcal{B} = \{B_1, \ldots, B_k\}$ as a minimal such family. Clearly, $k \geq 3$. As for every i, $\mathcal{B} \setminus B_i$ is a Helly family, there exists a vertex v_i that belongs to B_j and not to B_i, for all $j \neq i$. Let $S = \{v_1, \ldots, v_k\}$ be the collection of such vertices. Write $\alpha_S = \alpha(H[S])$.

First, we show that $\alpha_S \leq 2$. Assume this condition to be false and let $\{v_1, v_2, v_3\} \subseteq S$ be an independent set of H. Since H has no triangles and $v_i \notin B_i$, there exists a vertex $w_i \in B_i$ adjacent to v_j and not to v_i, for $j \neq i$ and $1 \leq i \leq 3$. See Figure 3. Then $\{w_1, w_2, w_3\}$ is also an independent set. In the latter situation, $\{v_1, w_2, v_3, w_1, v_2, w_3\}$ induces a C_6, which is forbidden. Consequently, indeed $\alpha_S \leq 2$.

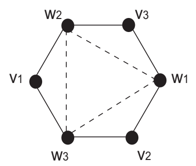

Figure 3

In the sequel, we discuss the possible values k can assume.

Let $k \geq 5$. Denote $S' = \{v_1, v_2, v_3, v_4\}$. Clearly, $S' \subseteq B_5$. For this reason and considering that $\alpha_S \leq 2$, we know that S' induces a C_4 in H. Let v_1v_3 and v_2v_4 be the non-adjacent pairs in S'. Again, because $\alpha_S \leq 2$, v_5 must be adjacent to at least one vertex of S', say adjacent to v_1. Because H has no triangles, v_5 can be adjacent neither to v_2, nor to v_4. However, in this situation, $\{v_2, v_4, v_5\}$ is an independent set of size 3, a contradiction. Consequently, $k < 5$.

Next, discuss the case $k = 4$. Let $S' = \{v_1, v_2, v_3\} \subseteq S$. Since $S' \subseteq B_4$ and $\alpha_S \leq 2$, S' induces a P_3 in H. Let v_1 and v_2 be the non-adjacent vertices in S'. As $v_1, v_2, v_4 \in B_3$ and $\alpha_S \leq 2$, it follows that S must induce a C_4 in H. On the other hand, since $v_i \notin B_i$ and $v_j \in B_i$ for $j \neq i$, each B_i has an additional vertex $w_i \in B_i$, $1 \leq i \leq 4$ with the following properties: because $v_1 \notin B_1$ and H has no triangles, w_1 is adjacent to v_2 and not adjacent to v_1, v_3, v_4. Similarly, w_2 is adjacent to v_1, and not to v_2, v_3, v_4, and w_3 is adjacent to v_4 and not v_1, v_2, v_3, while w_4 is adjacent to v_3 and not v_1, v_2, v_4. See Figure 4.

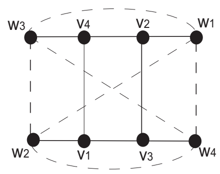

Figure 4

Let us examine the possible adjacencies among the w_i's. If w_1 and w_2 are adjacent, then H contains a C_5, which is forbidden. Similarly, if w_3 and w_4 are adjacent. So, assume these pairs are not adjacent. Let $W = \{w_1w_3, w_1w_4, w_2w_3, w_2w_4\}$ be the set of the other possible pairs of w_i's and denote $P = \{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\}$. If none of the pairs of W is adjacent, then P induces the graph Q_1. When exactly one of the pairs of W is adjacent, then P forms the graph Q_2. When precisely the pairs w_1w_3 and w_2w_4, or w_1w_4 and w_2w_3 are adjacent, then P induces Q_3. Finally, if at least two consecutive pairs of the cyclic sequence w_1, w_4, w_2, w_3, w_1 are adjacent in H, then a C_6 exists. Consequently, $k = 4$ is not possible.

Next, consider the case $k = 3$, and let $S = \{v_1, v_2, v_3\}$. Denote by $\mathcal{B}_S$ the subset of bicliques of H containing at least two vertices of S. Then $\{B_1, B_2, B_3\} \subseteq \mathcal{B}_S$. Discuss the possibilities for S. Clearly, S is not a triangle. Since $\alpha_S \leq 2$, S is not an independent set. Suppose S induces a $\overline{P_3}$, with v_1, v_2, adjacent. As $v_2, v_3 \in B_1$, there is a vertex $w_1 \in B_1$ adjacent to v_2 and v_3. Similarly, there exists $w_2 \in B_2$, with w_2 adjacent to v_1 and v_3. In this case if w_1, w_2 are adjacent, then a triangle is formed, otherwise $\{v_1, v_2, w_1, v_3, w_2\}$ induces a C_5, which is forbidden.

Let us examine the remaining alternative, that is S induces a P_3. By Lemma 1, $H[S^*]$ is bipartite. Let $X \cup Y$ be a bipartition of it. Let v_1, v_3 be the non-adjacent pair of vertices of S. Let $v_1, v_3 \in X$ and $v_2 \in Y$. Because $v_i \notin B_i$ and $v_j \in B_i$, for $j \neq i$, there is a vertex $w_i \in B_i$, $w_i \neq v_j$, where $i, j \in \{1, 3\}$, such that $w_1 \in Y$ is adjacent to v_3 and not adjacent to v_1 and v_2, while the vertex $w_3 \in Y$ is adjacent to v_1 and not to v_3 and v_2. Also, as B_2 is a biclique, there exists some vertex $w_2 \in Y$ adjacent to v_1 and v_3. Because $v_2 \notin B_2$, there is also a vertex $w_2' \in X \cap B_2$ that is adjacent to w_2 and not to v_2. Since $\{B_1, B_2, B_3\}$ is not a Helly family, w_2 is not common to these three bicliques. Without loss of generality, assume $w_2 \notin B_3$. Then there exists a vertex $w_3' \in X$ that belongs to B_3, w_3' adjacent to v_2 and w_3, and not to w_2. See Figure 5.

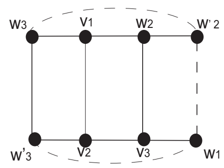

Figure 5

Let $P = \{v_1, v_2, v_3, w_1, w_2, w_2', w_3, w_3'\}$ and $W = \{w_1w_2', w_1w_3', w_3w_2'\}$. The pairs listed in W are the only possible edges, and then w_1w_3' or $w_2'w_3$ creates a C_6, and otherwise P induces a Q_3 or a Q_2 depending on the presence or absence of edge w_1w_2'. All cases lead to forbidden subgraphs.

Consequently, $k = 3$ is not possible. However, $k \geq 3$. This contradiction completes the proof. ■

3 Hereditary open and closed neighbourhood Helly graphs

We describe below characterizations for the classes of hereditary open and closed neighbourhood-Helly graphs.

Theorem 3.1 *Let G be a graph. Then G is hereditary open neighbourhood-Helly if and only if G contains neither C_6 nor a triangle as induced subgraphs.*

Proof: Let G be a hereditary open neighbourhood-Helly graph. Clearly, no subset $\{v_1, v_2, v_3\} \subseteq V(G)$ induces a triangle in G, otherwise $N(v_1)$, $N(v_2)$, $N(v_3)$ pairwise intersect, but there is no common vertex. Suppose G contains a C_6. Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the ordering of the vertices in this cycle. Since $N(v_1), N(v_3), N(v_5)$ pairwise intersect with no common vertex, we conclude that G contains no C_6.

Conversely, assume the theorem is not true. Then G contains an induced subgraph H that is not open neighbourhood-Helly. Consider a minimal family $N = \{N(v_1), N(v_2), ..., N(v_l)\}$ of pairwise intersecting neighbourhoods with no common vertex in H. Then $l \geq 3$. Because of the minimality, there is a vertex $w_i \in N(v_j)$ precisely when $i \neq j$, for all $1 \leq i \leq l$. Since $v_i \notin N(v_i)$, we conclude that $w_i \neq v_j$, for all $i \neq j$. Moreover, we show that $w_i \neq v_i$ for all i. Suppose that the latter assumption is not true and let let $w_i = v_i$. Then, v_i and v_j are adjacent. On the other hand, since $N(v_i)$ and $N(v_j)$ intersect, there is a vertex w forming a triangle with v_i and v_j, which is forbidden by hypothesis. Consequently, $w_i \neq v_j$, for $1 \leq i, j \leq l$. We claim that $w_1, ..., w_l$ form an independent set in G. The latter is true because, if w_i, w_j are adjacent, the fact that $w_i, w_j \in N(v_k)$, for $k \neq i, j$, implies that w_i, w_j, v_k form a triangle of G, contradicting the hypothesis. Similarly, $v_1, ..., v_l$ also form an independent set, otherwise $v_i \in N(v_j)$ implies that v_i, v_j, w_k, $k \neq i, j$ are vertices of a triangle. In this situation, $w_i, w_j, w_k, v_i, v_j, v_k$ induce a C_6 in G, which is impossible. The proof is complete. ■

Next we consider the closed neighbourhood-Helly class.

Theorem 3.2 *A graph G is hereditary closed neighbourhood-Helly if and only if it contains neither C_4, C_5, C_6 nor the Hajós graph as induced subgraphs.*

Proof: Suppose G contains the Hajós graph H. The family of the closed neighbourhoods of the three vertices with degree two in H is intersecting and has no common vertex. Consequently, G cannot contain the Hajós graph. Suppose G contains a C_4. The family of closed neighbourhoods of its four vertices violates the Helly Property. Therefore, no $C_4's$ can exist. Similarly, the families of the closed neighbourhoods of three vertices in a C_5, two of them non-adjacent, and the closed neighbourhoods of the three mutually non-adjacent vertices in a C_6, are both intersecting and have no common vertex. Consequently, G contains neither C_5 nor C_6.

Conversely, by hypothesis G contains neither the Hajós graph, nor any of C_4, C_5,C_6 as induced subgraphs. Assume that the theorem is false and that G is not a hereditary closed neighbourhood-Helly graph. Denote by $N = \{N[v_1], N[v_2], ..., N[v_l]\}$ a minimal such intersecting family of G with no common vertex. Clearly, $l \geq 3$. By the minimality of l, there exist vertices w_i, such that $w_i \in N[v_j]$, exactly for $i \neq j$. Compare w_i and v_j. It is clear that, $v_i \neq w_i$. Suppose $w_i = v_j$, for some i, j. Without loss of generality, let $w_1 = v_2$. Then, v_1, v_2 are not adjacent, implying $v_1 \neq w_3$. The latter means that $w_1, w_2 \neq v_3$. In this situation, if v_1, v_3 are adjacent, the vertices w_1, w_3, v_1, v_3 induce a C_4, which is forbidden. Consequently, v_1, v_3 are not adjacent. Consider vertex w_2. It follows that when w_2, w_3 are adjacent, the vertices w_1, w_3, w_2, v_3 induce a C_4, and otherwise w_1, w_3, v_1, w_2, v_3 induce a C_5 in G. Hence, this alternative cannot occur. Finally, assume $w_i \neq v_j$, for all i, j. Since G contains no C_6, $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ cannot be both independent sets. Suppose w_2, w_3 are adjacent. Since G contains neither C_4 nor C_5, we conclude that w_1 must be adjacent to both w_2, w_3. In this situation, if $\{v_1, v_2, v_3\}$ is an independent set, $w_1, w_2, w_3, v_1, v_2, v_3$ induce the Hajós graph, otherwise, they induce a C_4. In any alternative, a forbidden subgraph arises. The alternative v_2, v_3 to be adjacent instead of w_2, w_3, is similar, terminating the proof.■

4 Relations among the classes

The hereditary classes here considered can be related by employing the characterizations formulated before. The following lemma is useful.

Lemma 4.1 *[11]: Let G be an open neighbourhood-Helly graph. Then G has no triangles.*

The corollary below relates hereditary open neighbourhood-Helly graphs to the open neighbourhood-Helly graphs. It is a consequence of Theorem 3.1 and Lemma 4.1.

Corollary 4.1 *A graph is hereditary open neighbourhood-Helly if and only if it is open neighbourhood-Helly and has no* C_6.

The next lemma relates hereditary closed neighbourhood-Helly graphs to the closed neighbourhood-Helly graphs.

Lemma 4.2 *Let* G *be a closed neighbourhood-Helly graph, with no triangles. Then* G *is hereditary closed neighbourhood-Helly.*

Proof: To prove that G is hereditary closed neighbourhood-Helly, we show that G does not contain contain C_4, C_5 or C_6 as induced subgraphs. Assume the contrary, that is, G contains a C_4 as an induced subgraph. Consider the intersecting family $\{N[v_i]\}$, $1 \leq i \leq 4$. By hypothesis, it has a common intersection which implies that a triangle is formed, a contradiction. Then, C_4 is not an induced subgraph of G. Similarly, suppose G contains a C_5. The family of closed neighbourhoods of the five vertices of the C_5 intersects in a vertex, forming a triangle, again a contradiction. Finally, consider the closed neighbourhoods of the three mutually non-adjacent vertices of a C_6. As they have a common vertex, there exists an induced C_4 or a triangle, a contradiction. ∎

As a consequence of Lemma 4.2, we obtain the following relation between closed neighbourhood-Helly graphs and other hereditary Helly classes.

Corollary 4.2 *If* G *is closed neighbourhood-Helly with no triangles, then it is hereditary biclique-Helly, hereditary open neighbourhood-Helly and hereditary closed neighbourhood-Helly.*

Finally we can also conclude:

Corollary 4.3 *Let* G *be a graph with girth at least 7. Then* G *is hereditary biclique-Helly, hereditary open neighbourhood-Helly and hereditary closed neighbourhood-Helly.*

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