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Simultaneous generation for zeta values by the Markov-WZ method

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By application of the Markov-WZ method, we prove a more general form of a bivariate generating function identity containing, as particular cases, Koecher's and Almkvist-Granville's Apéry-like formulae for odd zeta values. As a consequence, we get a new identity producing Apéry-like series for all $\zeta(2n+4m+3)$, $n, m \geq 0$, convergent at the geometric rate with ratio 2^{-10}.

Keywords: Riemann zeta function, Apéry-like series, generating function, convergence acceleration, Markov-Wilf-Zeilberger method, Markov-WZ pair.

1 Introduction

The Riemann zeta function is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{for} \quad \mathrm{Re}(s) > 1. \tag{1}$$

Apéry's irrationality proof of $\zeta(3)$ [14] operates with the faster convergent series

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \tag{2}$$

first obtained by A. A. Markov in 1890 [10]. The general formula giving analogous series for all $\zeta(2s+3)$, $s \geq 0$, was proved by Koecher [7] (and independently in an expanded form by Leshchiner [9]). It reads

$$\sum_{s=0}^{\infty} \zeta(2s+3)x^{2s} = \sum_{k=1}^{\infty} \frac{1}{k(k^2-x^2)} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \, \frac{5k^2-x^2}{k^2-x^2} \prod_{m=1}^{k-1} \left(1 - \frac{x^2}{m^2}\right). \tag{3}$$

A similar identity generating fast convergent series for all $\zeta(4s+3)$, $s \geq 0$, which for $s > 1$ is different from Koecher's result (3), was experimentally discovered in [3] and proved by G. Almkvist and A. Granville in [1]

$$\sum_{s=0}^{\infty} \zeta(4s+3)x^{4s} = \sum_{k=1}^{\infty} \frac{k}{k^4-x^4} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\binom{2k}{k}} \frac{k}{k^4-x^4} \prod_{m=1}^{k-1} \left(\frac{m^4+4x^4}{m^4-x^4}\right). \tag{4}$$

There exists a bivariate unifying formula for identities (3) and (4),

$$\sum_{k=1}^{\infty}\frac{k}{k^4-x^2k^2-y^4}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k\binom{2k}{k}}\frac{5k^2-x^2}{k^4-x^2k^2-y^4}\prod_{m=1}^{k-1}\left(\frac{(m^2-x^2)^2+4y^4}{m^4-x^2m^2-y^4}\right), \tag{5}$$

which was first conjectured by H. Cohen and then proved by D. Bradley [5] and, independently, by T. Rivoal [15]. This identity implies (3) if $y = 0$, and gives (4) if $x = 0$. The proof of (5) given in [5, 15] relies on Borwein and Bradley's method [3] and consists of reduction of (5) to a finite non-trivial combinatorial identity which can be proved on the basis of Almkvist and Granville's work [1].

Recently, in [6] it was shown that Koecher's formula (3), and similarly Leschiner's and the identities of Bailey, Borwein and Bradley [9, 4] generating accelerated series for even zeta values $\zeta(2n+2)$, can be proved by means of the WZ method.

Formulas (3)-(5) generate accelerated series for odd zeta values and, in particular, series (2) for $\zeta(3)$ which converge at a geometric rate with ratio $1/4$. Many other more rapidly convergent expressions for $\zeta(3)$ can be proved on the basis of the WZ method. The following series, for example, convergent at the geometric rate with ratio 2^{-10},

$$\zeta(3)=\sum_{n=0}^{\infty}(-1)^n\frac{n!^{10}(205n^2+250n+77)}{64(2n+1)!^5} \tag{6}$$

was obtained by T. Amdeberhan and D. Zeilberger [2] by application of WZ-pairs. There are even faster convergent representations for $\zeta(3)$ with ratios $10^{-5}, 10^{-8}$ (see [11]). In [6] it was shown how to get such fast convergent series explicitly for other values $\zeta(n)$, $n > 3$. This can be accomplished by applying the WZ method not to the series (1) itself but to a generating function of a sequence of zeta values.

In this note, we prove a more general form of the bivariate identity (5) by application of the Markov-WZ method. We show that identity (5) and the series (6) of Amdeberhan and Zeilberger can be proved with the help of the same Markov-WZ pair, but using different summation formulas. Moreover, we get a new identity generating accelerated series for all $\zeta(2n+4m+3)$, $n, m \geq 0$, convergent at a geometric rate with ratio 2^{-10}.

2 Statement of the main results

We start by giving several definitions, and by reviewing known facts related to the Markov-Wilf-Zeilberger theory (see [8, 10, 11, 12]).

A function $H(n,k)$, in the integer variables n and k, is called *hypergeometric* or *closed form (CF)* if the quotients

$$\frac{H(n+1,k)}{H(n,k)} \quad \text{and} \quad \frac{H(n,k+1)}{H(n,k)}$$

are both rational functions of n and k. A hypergeometric function that can be written as a ratio of products of factorials is called *pure-hypergeometric*. A pair of CF functions $F(n,k)$ and $G(n,k)$ is called a *WZ-pair* if

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k). \tag{7}$$

A *P-recursive* function is a function that satisfies a linear recurrence relation with polynomial coefficients. If for a given hypergeometric function $H(n,k)$, there exists a polynomial $P(n,k)$ in k of the form

$$P(n,k) = a_0(n) + a_1(n)k + \cdots + a_L(n)k^L,$$

for some non-negative integer L, and P-recursive functions $a_0(n), \ldots, a_L(n)$ such that $F(n,k) := H(n,k)P(n,k)$ satisfies (7) with some function G, then a pair (F,G) is called a *Markov-WZ pair* associated with the kernel $H(n,k)$ (MWZ-pair for short). We call $G(n,k)$ an *MWZ mate* of $F(n,k)$.

In 2005, M. Mohammed [11] showed that for any pure-hypergeometric kernel $H(n,k)$, there exists a non-negative integer L and a polynomial $P(n,k)$ as above such that $F(n,k) = H(n,k)P(n,k)$ has an MWZ mate $G(n,k) = F(n,k)Q(n,k)$, where $Q(n,k)$ is a ratio of two P-recursive functions.

From relation (7) we get the following summation formulas.

Proposition 1 ([11, Theorem 2(b)]) *Let (F,G) be an MWZ-pair. If $\lim_{n\to\infty} F(n,k) = 0$ for every $k \geq 0$, then*

$$\sum_{k=0}^{\infty} F(0,k) - \lim_{k\to\infty} \sum_{n=0}^{\infty} G(n,k) = \sum_{n=0}^{\infty} G(n,0), \tag{8}$$

whenever both sides converge.

Proposition 2 ([11, Cor. 2]) *Let (F,G) be an MWZ-pair. If $\lim_{k\to\infty} \sum_{n=0}^{\infty} G(n,k) = 0$, then*

$$\sum_{k=0}^{\infty} F(0,k) = \sum_{n=0}^{\infty} (F(n,n) + G(n,n+1)), \tag{9}$$

whenever both sides converge.

Formulas (8), (9) with an appropriate choice of MWZ-pairs can be used to convert a given hypergeometric series into a different rapidly converging one.

Let $(\lambda)_\nu$ be the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1, & \nu = 0; \\ \lambda(\lambda+1)\ldots(\lambda+\nu-1), & \nu \in \mathbb{N}. \end{cases}$$

Let a, b be complex numbers such that $|a| < 1$, $|b| < 1$. In Section 3, we construct a Markov-WZ pair associated with the kernel

$$H(n,k) = \frac{(1+a)_k(1-a)_k(1+b)_k(1-b)_k}{(1+a)_{n+k+1}(1-a)_{n+k+1}(1+b)_{n+k+1}(1-b)_{n+k+1}}$$

and then apply Propositions 1, 2 to get the following two theorems.

Theorem 1 *Let a, b be complex numbers, with $|a| < 1$, $|b| < 1$. Then for arbitrary complex numbers A_0, B_0, C_0 we have*

$$\sum_{k=1}^{\infty} \frac{A_0 + B_0 k + C_0 k^2}{(k^2-a^2)(k^2-b^2)} = \sum_{n=1}^{\infty} \frac{d_n}{\prod_{m=1}^{n}(m^2-a^2)(m^2-b^2)},$$

with

$$d_n = \frac{(-1)^{n-1}B_0(5n^2-a^2-b^2)}{2n\binom{2n}{n}} \prod_{m=1}^{n-1} \left((m^2-a^2-b^2)^2-4a^2b^2\right) + \frac{20n+5}{2(5n^2-2a^2-2b^2)}L_n + \frac{35n^5-35n^3(a^2+b^2)+4n(3a^4+3b^4-4a^2b^2)}{4(5n^2-2a^2-2b^2)}L_{n-1},$$

where L_n *is a solution of the second order difference equation*

$$4(4n+3)(4n+5)(5n^2-2a^2-2b^2)L_{n+1}+2(n+1)p(n)L_n - n(n+1)(5(n+1)^2-2a^2-2b^2)q(n)L_{n-1}=0, \quad n=1,2,\dots$$

with initial conditions $L_0 = C_0$,

$$L_1 = \left(\frac{1}{3}-\frac{2}{15}(a^2+b^2)\right)A_0 + \left(\frac{1}{6}(a^2+b^2)-\frac{2}{15}(a^4+b^4-4a^2b^2)-\frac{1}{30}\right)C_0,$$

whose growth is described by the inequality

$$\lim_{n\to\infty}\left(\frac{|L_n|}{n!^4}\right)^{\frac{1}{n}} \le \frac{1}{4},$$

and

$$\begin{aligned} p(n) &= 30n^7+105n^6+n^5(145-52(a^2+b^2))+n^4(100-130(a^2+b^2)) \\ &+ n^3(35-124(a^2+b^2)+56(a^4+b^4)-208a^2b^2)+n^2(5-56(a^2+b^2) \\ &+ 84(a^4+b^4)-312a^2b^2)+n(80a^2b^2(a^2+b^2)-16(a^6+b^6)+48(a^4+b^4-3a^2b^2) \\ &- 14(a^2+b^2))+(10(a^2-b^2)^2-2(a^2+b^2)+40a^2b^2(a^2+b^2)-8(a^6+b^6)), \end{aligned} \tag{10}$$

$$\begin{aligned} q(n) &= n^8-6n^6(a^2+b^2)+n^4(9(a^4+b^4)+30a^2b^2) \\ &- n^2(28a^2b^2(a^2+b^2)+4(a^6+b^6))+16a^2b^2(a^2-b^2)^2. \end{aligned} \tag{11}$$

If in Theorem 1 we take $B_0=1$, $A_0=C_0=0$, then $L_n=0$ for all $n\ge 0$ and we get

$$\begin{aligned} &\sum_{k=1}^{\infty}\frac{k}{(k^2-a^2)(k^2-b^2)} \\ &= \sum_{n=1}^{\infty}\frac{(5n^2-a^2-b^2)(1+a+b)_{n-1}(1+a-b)_{n-1}(1-a+b)_{n-1}(1-a-b)_{n-1}}{2(-1)^{n-1}\,n\binom{2n}{n}(1+a)_n(1-a)_n(1+b)_n(1-b)_n}. \end{aligned} \tag{12}$$

If we now put

$$a^2=\frac{x^2+\sqrt{x^4+4y^4}}{2}, \qquad b^2=\frac{x^2-\sqrt{x^4+4y^4}}{2}, \tag{13}$$

we get the bivariate identity (5) conjectured by H. Cohen.

If $A_0 = 1$, $B_0 = C_0 = a = b = 0$, we get the following series for $\zeta(4)$ mentioned by Markov in [10, p.18]:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n!^4} \left(\frac{4n+1}{2n^2} L_n + \frac{7n^3}{4} L_{n-1} \right),$$

where $L_0 = 0$, $L_1 = 1/3$, and

$$4(4n+3)(4n+5)L_{n+1} + 2(n+1)^3(6n^3+9n^2+5n+1)L_n - n^7(n+1)^3 L_{n-1} = 0, \quad n \geq 1.$$

Theorem 2 *Let x, y be complex numbers such that $|x|^2 + |y|^4 < 1$. Then*

$$\sum_{k=1}^{\infty} \frac{k}{k^4 - x^2k^2 - y^4} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} r(n)}{n\binom{2n}{n}} \frac{\prod_{m=1}^{n-1}((m^2-x^2)^2 + 4y^4)}{\prod_{m=n}^{2n}(m^4 - x^2m^2 - y^4)}, \tag{14}$$

where

$$r(n) = 205n^6 - 160n^5 + (32 - 62x^2)n^4 + 40x^2n^3 + (x^4 - 8x^2 - 25y^4)n^2 + 10y^4n + y^4(x^2 - 2).$$

Since

$$\sum_{k=1}^{\infty} \frac{k}{k^4 - x^2k^2 - y^4} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n+m}{n} \zeta(2n+4m+3) x^{2n} y^{4m},$$

the formula (14) generates Apéry-like series for all $\zeta(2n+4m+3)$, $n, m \geq 0$, convergent at the geometric rate with ratio 2^{-10}. So, for example, if $x = y = 0$, we get Amdeberhan and Zeilberger's series (6) for $\zeta(3)$,

$$\zeta(3) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(205n^2 - 160n + 32)}{n^5 \binom{2n}{n}^5}.$$

If $y = 0$, we recover Theorem 4 from [6]. If $x = 0$, we find, in particular, the following expression for $\zeta(7)$:

$$\zeta(7) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (25n^2 - 10n + 2)}{n^9 \binom{2n}{n}^5} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (205n^2 - 160n + 32)}{n^5 \binom{2n}{n}^5} \left(\sum_{m=1}^{2n} \frac{1}{m^4} + \sum_{m=1}^{n-1} \frac{3}{m^4} \right).$$

3 Proof of Theorem 1.

Let a, b be complex numbers such that $|a| < 1$, $|b| < 1$. Let

$$H(n,k) = \frac{(1+a)_k(1-a)_k(1+b)_k(1-b)_k}{(1+a)_{n+k+1}(1-a)_{n+k+1}(1+b)_{n+k+1}(1-b)_{n+k+1}}$$

We are interested in finding a Markov-WZ pair associated with $H(n,k)$. For this purpose, we define the function $F(n,k) = H(n,k)P(n,k)$, where $P(n,k)$ is a polynomial in k of degree L_1 with unknown coefficients as functions of n. Then

$$F(n+1,k) - F(n,k) = \frac{(1+a)_k(1-a)_k(1+b)_k(1-b)_k}{(1+a)_{n+k+2}(1-a)_{n+k+2}(1+b)_{n+k+2}(1-b)_{n+k+2}} P_1(n,k), \tag{15}$$

where $P_1(n,k)$ is a polynomial in k of degree L_1+4. From (15) it follows that we can determine a MWZ mate of $F(n,k)$ in the form $G(n,k) = H(n,k)Q(n,k)$, where $Q(n,k)$ is a polynomial in k of degree L_2 with unknown coefficients as functions of n. Indeed, for such a choice we have

$$G(n,k+1) - G(n,k) = \frac{(1+a)_k(1-a)_k(1+b)_k(1-b)_k}{(1+a)_{n+k+2}(1-a)_{n+k+2}(1+b)_{n+k+2}(1-b)_{n+k+2}} Q_1(n,k),$$

where $Q_1(n,k)$ is a polynomial in k of degree L_2+3. Therefore, (F,G) is a Markov-WZ pair if and only if

$$P_1(n,k) = Q_1(n,k) \quad \text{identically for all} \quad n,k. \tag{16}$$

This implies that $L_2 = L_1 + 1$. On the other hand, equating coefficients of powers of k on both sides of (16), we get a system of L_1+5 linear homogeneous equations with $L_1+L_2+2 = 2L_1+3$ unknowns. In order to guarantee a solution, we should at least have that $2L_1+3 \ge L_1+5$ and hence $L_1 \ge 2$, $L_2 \ge 3$.

We now show that there is a non-zero solution of (16) with the optimal choice $L_1 = 2$, $L_2 = 3$. To see this, define two functions

$$\begin{aligned}
F(n,k) &= H(n,k)(A(n) + B(n)(k+1) + C(n)(k+1)^2),\\
G(n,k) &= H(n,k)(D(n) + E(n)k + K(n)k^2 + L(n)k^3),
\end{aligned}$$

with 7 unknown coefficients $A(n), B(n), C(n), D(n), E(n), K(n), L(n)$ as functions of n. We require that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{17}$$

Substituting F, G into (17) and cancelling common factors we get that (17) is equivalent to the following equation of degree 6 in the variable k :

$$\begin{aligned}
&((n+k+2)^2 - a^2)((n+k+2)^2 - b^2)(A(n) + B(n)(k+1) + C(n)(k+1)^2)\\
&- A(n+1) - B(n+1)(k+1) - C(n+1)(k+1)^2 = ((n+k+2)^2 - a^2)\\
&\times ((n+k+2)^2 - b^2)(D(n) + E(n)k + K(n)k^2 + L(n)k^3) - ((k+1)^2 - a^2)\\
&\times ((k+1)^2 - b^2)(D(n) + E(n)(k+1) + K(n)(k+1)^2 + L(n)(k+1)^3).
\end{aligned} \tag{18}$$

To satisfy condition (17), all the coefficients of the powers of $(k+1)$ in the equation (18) must be identically zero. This leads to a system of first order linear recurrence equations with polynomial coefficients for $A(n), B(n), C(n), D(n), E(n), K(n), L(n)$

$$C(n) = (4n_1 - 3)L(n), \quad B(n) = (4n_1 - 2)K(n) - (10n_1^2 - 3)L(n), \quad n_1 = n+1, \tag{19}$$

$$A(n) = (4n_1 - 1)E(n) - (10n_1^2 - 1)K(n) + (20n_1^3 + 2n_1(a^2+b^2) - 1)L(n), \tag{20}$$

$$4D(n) = 10n_1E(n) - (20n_1^2 + 2a^2 + 2b^2)K(n) + (35n_1^3 + 11n_1(a^2+b^2))L(n), \tag{21}$$

$$\begin{aligned}2(4n_1+1)L(n+1) &= 2n_1(5n_1^2 - 2a^2 - 2b^2)E(n) - 2n_1^2(15n_1^2 - 6(a^2+b^2))K(n)\\ &\quad + n_1(63n_1^4 - 17n_1^2(a^2+b^2) - 4(a^4+b^4))L(n),\end{aligned} \tag{22}$$

$$\begin{aligned}&2(4n_1+2)K(n+1) - 2(10n_1^2 + 20n_1 + 7)L(n+1)\\ &= 2n_1^2(5n_1^2 - 2(a^2+b^2))E(n) - 2(16n_1^5 - 8n_1^3(a^2+b^2) + n_1(a^2-b^2)^2)K(n)\\ &+ (70n_1^6 - 31n_1^4(a^2+b^2) + n_1^2(3a^4 + 3b^4 - 14a^2b^2))L(n),\end{aligned} \tag{23}$$

$$\begin{aligned}&4(20(n_1+1)^3 + 2(n_1+1)(a^2+b^2) - 1)L(n+1) - 4(10n_1^2 + 20n_1 + 9)K(n+1)\\ &+ 4(4n_1+3)E(n+1) = (6n_1^5 - 6n_1^3(a^2+b^2) + 16a^2b^2n_1)E(n)\\ &- (20n_1^6 - 22n_1^4(a^2+b^2) + 2n_1^2(a^4+b^4+22a^2b^2))K(n)\\ &+ (45n_1^7 - 48n_1^5(a^2+b^2) + n_1^3(3a^4+3b^4+86a^2b^2) + 8a^2b^2n_1(a^2+b^2))L(n).\end{aligned} \tag{24}$$

Now multiplying equation (22) by n_1 and subtracting from (23), we get

$$2K(n+1) - 7(n_1+1)L(n+1) = -\frac{n_1((n_1^2 - a^2 - b^2)^2 - 4a^2b^2)}{2(2n_1+1)}(2K(n) - 7n_1L(n)),$$

which yields

$$K(n) - \frac{7}{2}n_1L(n) = \frac{(-1)^n(2K(0) - 7L(0))n!}{2^{n_1}}\prod_{m=1}^{n}\left(\frac{(m^2-a^2-b^2)^2 - 4a^2b^2}{2m+1}\right).$$

From (19) it follows that $2K(0) = B(0) + 7L(0)$, and therefore we have

$$K(n) = \frac{7}{2}n_1L(n) + \frac{(-1)^nB(0)n!}{2^{n_1}}\prod_{m=1}^{n}\left(\frac{(m^2-a^2-b^2)^2 - 4a^2b^2}{2m+1}\right). \tag{25}$$

Substitution of (25) into (22) yields the formula

$$\begin{aligned}E(n) &= \frac{4n_1+1}{n_1(5n_1^2 - 2a^2 - 2b^2)}L(n+1) + \frac{42n_1^4 - 25n_1^2(a^2+b^2) + 4(a^4+b^4)}{2(5n_1^2 - 2a^2 - 2b^2)}L(n)\\ &\quad + \frac{3B(0)(-1)^nn_1!}{2^{n_1}}\prod_{m=1}^{n}\left(\frac{(m^2-a^2-b^2)^2 - 4a^2b^2}{2m+1}\right).\end{aligned} \tag{26}$$

Substitution of (25) and (26) into (21) gives the formula

$$\begin{aligned}D(n) &= \frac{(40n_1+10)L(n+1) + (35n_1^5 - 35n_1^3(a^2+b^2) + 4n_1(3a^4+3b^4-4a^2b^2))L(n)}{4(5n_1^2 - 2a^2 - 2b^2)}\\ &\quad + \frac{(-1)^nB(0)n!(5n_1^2 - a^2 - b^2)}{2^{n_1+1}}\prod_{m=1}^{n}\left(\frac{(m^2-a^2-b^2)^2 - 4a^2b^2}{2m+1}\right).\end{aligned} \tag{27}$$

Finally, substitution of (25), (26) into (24) gives the second-order difference equation

$$4(4n+3)(4n+5)(5n^2-2a^2-2b^2)L(n+1)+2(n+1)p(n)L(n)$$
$$-n(n+1)(5(n+1)^2-2a^2-2b^2)q(n)L(n-1)=0, \quad n=1,2,\dots,$$

with initial conditions $L(0)=C(0)$,

$$L(1)=\left(\frac{1}{3}-\frac{2}{15}(a^2+b^2)\right)A(0)+\left(\frac{1}{6}(a^2+b^2)-\frac{2}{15}(a^4+b^4-4a^2b^2)-\frac{1}{30}\right)C(0),$$

derived from (19), (20), (22), and polynomials $p(n), q(n)$ defined in (10), (11).

If we put $l(n)=L(n)/(n!)^4$, $n=0,1,2,\dots$, then it is easily seen that the sequence $l(n)$ satisfies the following recurrence equation:

$$4(4n+3)(4n+5)(5n^2-2a^2-2b^2)n^3(n+1)^3l(n+1)+2n^3p(n)l(n)$$
$$-(5(n+1)^2-2a^2-2b^2)q(n)l(n-1)=0, \quad n=1,2,\dots. \quad (28)$$

Its characteristic polynomial $64\lambda^2+12\lambda-1=0$ has two different zeros $\lambda_1=-1/4$, $\lambda_2=1/16$. Then, by Poincaré's theorem [13], for each solution $l(n)$, $n=0,1,2,\dots$, of (28), either $l(n)=0$ for all sufficiently large $n\ge n_0$, or the limit $\lim_{n\to\infty} l(n+1)/l(n)$ exists and equals one of the roots of the characteristic polynomial. Therefore, in both cases we get that the limit $\lim_{n\to\infty}|l(n)|^{1/n}$ exists and does not exceed $1/4$ or

$$\lim_{n\to\infty}\left(\frac{|L(n)|}{(n!)^4}\right)^{\frac{1}{n}}\le\frac{1}{4}. \quad (29)$$

The limit inequality (29) implies

$$\lim_{k\to\infty}\sum_{n=0}^{\infty}G(n,k)=0 \qquad \text{and} \qquad \lim_{n\to\infty}F(n,k)=0 \quad \text{for every} \quad k\ge 0,$$

and therefore, by Proposition 1, we have

$$\sum_{k=0}^{\infty}F(0,k)=\sum_{n=0}^{\infty}G(n,0),$$

yielding the theorem with $d_n=D(n-1)$, $L_n=L(n)$, $A_0=A(0)$, $B_0=B(0)$, $C_0=C(0)$. □

4 Proof of Theorem 2.

To deduce (14) from (17), take $A(0)=C(0)=0$, $B(0)=1$, and apply Proposition 2 to obtain

$$\sum_{k=0}^{\infty}F(0,k)=\sum_{n=0}^{\infty}(F(n,n)+G(n,n+1)). \quad (30)$$

Since in this case $L(n)=0$ for all $n\ge 0$, an easy computation of the right-hand side of (30) by (19), (20), (25)–(27), and substitution (13) lead to the desired conclusion. □

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