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Progress on the Traceability Conjecture for Oriented Graphs

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A digraph is k-traceable if each of its induced subdigraphs of order k is traceable. The Traceability Conjecture is that for $k \geq 2$ every k-traceable oriented graph of order at least $2k - 1$ is traceable. The conjecture has been proved for $k \leq 5$. We prove that it also holds for $k = 6$.

Keywords: longest path, oriented graph, nontraceable, k-traceable, Traceability Conjecture, Path Partition Conjecture

1 Introduction

The set of vertices and the set of arcs of a digraph D are denoted by $V(D)$ and $A(D)$, respectively, and the order of D is denoted $n(D)$. A directed cycle (path, walk) in a digraph will simply be called a cycle (path, walk). A digraph is *hamiltonian* if it contains a cycle that visits every vertex, *traceable* if it contains a path that visits every vertex, *walkable* (or *unilaterally connected*) if it contains a walk that visits every vertex, and *strong* (or *strongly connected*) if it has a closed walk that visits every vertex.

The maximum number of vertices on a path in a digraph D is denoted by $\lambda(D)$. A digraph D of order n is *p-deficient* if $\lambda(D) = n - p$.

A maximal strong subdigraph of a digraph D is called a *strong component* of D. We say that a strong component is *trivial* if has only one vertex.

If v is a vertex of a digraph D, we denote the sets of *out-neighbours* and *in-neighbours* of v by $N^+(v)$ and $N^-(v)$ and the cardinalities of these sets by $d^+(v)$ and $d^-(v)$, respectively. The minimum degree of D, $\delta(D)$, is defined as $\min_{v \in V(D)} (d^+(v) + d^-(v))$.

If D is a digraph and $X \subset V(D)$, then $\langle X \rangle$ denotes the subdigraph induced by X in D.

A digraph of order n is *k-traceable* for some $k \leq n$ if each of its induced subdigraphs of order k is traceable. The main topic of this paper is the following conjecture, which was formulated by Morten Nielsen in 2006. It is stated in (5).

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Conjecture 1.1 (The Traceability Conjecture (TC)) *For $k \geq 2$, every k-traceable oriented graph of order at least $2k-1$ is traceable.*

The Traceability Conjecture was inspired by the Path Partition Conjecture for 1-deficient Oriented Graphs (called the OPPC(1)), which is treated in (5) and (10). The OPPC(1) is an important special case of the Path Partition Conjecture for Digraphs (DPPC), which is treated in (2) and (4). The OPPC(1) may be stated as follows in terms of traceability (see (5)).

Conjecture 1.2 (OPPC(1)) *Let a and b be integers with $1 \leq a \leq b$. If D is a 1-deficient oriented graph of order $n = a + b + 1$, then D is not $(a+1)$-traceable or D is not $(b+1)$-traceable.*

The truth of the TC would obviously imply the truth of the OPPC(1). In particular, if the TC holds for $k = t$, it would follow that the OPPC(1) holds for $a = t - 1$.

In the case of undirected graphs, it is an easy corollary of Dirac's degree condition for hamiltonicity that for $k \geq 2$ every k-traceable graph of order at least $2k - 1$ is hamiltonian. The same is not true for oriented graphs, though we do have the following result, which is proved in (5).

Theorem 1.3 *For $k = 2, 3$ or 4, every strong k-traceable oriented graph of order greater than k is hamiltonian.*

For $k \geq 5$ the situation changes dramatically. As shown in (5), for every $n \geq 5$ there exists a non-hamiltonian strong oriented graph of order n that is k-traceable for every $k \in \{5, \dots, n\}$. However, no counterexample to the TC has yet been found. In fact, we do not even know if there exists a k-traceable oriented graph of order bigger than $k + 1$ that is nontraceable.

It is shown in (5) that the TC holds for $k \leq 5$. In this paper we prove that the TC also holds for $k = 6$, i.e. every 6-traceable oriented graph of order at least 11 is traceable.

2 Auxiliary Results

First, we present some general properties of k-traceable oriented graphs. The following result concerning the minimum degree is proved in (5).

Lemma 2.1 *If $k \geq 2$ and D is a k-traceable oriented graph of order $n \geq k$, then $\delta(D) \geq n - k + 1$.*

Our next result concerns k-traceable oriented graphs that are nontraceable.

Lemma 2.2 *Suppose D is a k-traceable oriented graph of order $n > k$. If D is nontraceable and v is a vertex of D with $d^+(v) = n - k + 1$ then $\langle N^+(v) \rangle$ is nontraceable. Similarly, if $d^-(v) = n - k + 1$, then $\langle N^-(v) \rangle$ is nontraceable.*

Proof: Suppose $\langle N^+(v) \rangle$ has a hamiltonian path $x_1 x_2 \dots x_{n-k+1}$. Then the graph $D - \{x_1, \dots x_{n-k}\}$ has order k and therefore has a hamiltonian path P. If P contains the arc vx_{n-k+1}, then the path obtained from P by replacing the arc vx_{n-k+1} with the path $vx_1x_2 \dots x_{n-k}x_{n-k+1}$ is a hamiltonian path of D. If P does not contain the arc vx_{n-k+1}, then v is the end-vertex of P. In this case $Px_1x_2 \dots x_{n-k}$ is a hamiltonian path of D. The proof that $\langle N^-(v) \rangle$ is nontraceable if $d^-(v) = n - k + 1$ is similar. □

The following easy observation is proved in (5).

Lemma 2.3 *If D is an oriented graph of order n that is k-traceable for some $k \in \{2, \dots, n\}$, then D is walkable.*

In view of Lemma 2.3 we shall be mainly concerned with walkable oriented graphs. The strong components of a digraph have an *acyclic ordering*, i.e. they may be labelled $D_1, \ldots, D_h$ such that if there is an arc from D_i to D_j, then $i \leq j$ (cf. (1), p. 17). If D is walkable then, for $i = 1, \ldots, h-1$ there is at least one arc from D_i to D_{i+1}, so in this case the acyclic ordering is unique. Throughout the paper we shall label the strong components of a walkable oriented graph in accordance with this unique acyclic ordering.

In the proof of our main result we shall consider oriented graphs that are strong and those that are not strong (but walkable) separately. The nonstrong case relies on the following three results concerning the strong components of k-traceable oriented graphs. The first is an obvious but useful result, proved in (5).

Lemma 2.4 *If P is a path in a digraph D, then the intersection of P with any strong component of D is either empty or a path.*

The next result follows from Theorem 1.3 and Lemma 2.4.

Lemma 2.5 *Let $k \geq 5$ and suppose D is a k-traceable oriented graph of order $n > k$. Then every nonhamiltonian nontrivial strong component of D has order at least $n - k + 5$.*

Proof: Suppose D has a nonhamiltonian nontrivial strong component X of order at most $n-(k-4)$. Then $|V(X)| \geq 4$ and $|V(D) \setminus V(X)| \geq k-4$. If $|V(X)| = 4$, then $|V(D) \setminus V(X)| = n-4 \geq k-3$. Theorem 1.3 implies that X is not 3-traceable and, if $|V(X)| > 4$, then X is also not 4-traceable. In either case, we can choose an induced subdigraph H of D of order k such that $\langle V(H) \cap V(X) \rangle$ is nontraceable. But then it follows from Lemma 2.4 that H is nontraceable, contradicting our assumption that D is k-traceable. □

The following result, which is proved in (5), is very useful in the case of k-traceable oriented graphs of large enough order.

Lemma 2.6 *Suppose D is a k-traceable oriented graph of order at least $2k-1$, $k \geq 2$. If D is nontraceable, then D has a nonhamiltonian nontrivial strong component.*

For the proof of the strong case of our main result, we shall use the following theorem, proved in (3).

Theorem 2.7 (***Chen and Manalastas***) *Every nontraceable strong digraph has independence number at least 3.*

We shall also use the following result on k-traceable strong oriented graphs, which appears as part of the proof of Theorem 3.5 in (5).

Lemma 2.8 *Let D be a k-traceable strong oriented graph and let $X = \{x_1, x_2, x_3\}$ be an independent set of vertices in D. Let*

$$A_i = V(D) \setminus \{X \cup N^-(x_i)\},\ B_i = V(D) \setminus \{X \cup N^+(x_i)\};\ i = 1, 2, 3.$$

Then $|A_i| \leq 3k - 12$ and $|B_i| \leq 3k - 12$ for $i = 1, 2, 3$.

Proof: Let i, j be any pair of distinct integers in $\{1, 2, 3\}$. If $|A_i \cap A_j| \geq k - 3$, let H be an induced subdigraph of D whose vertex set consists of x_1, x_2, x_3 and $k-3$ vertices of $A_i \cap A_j$. Then H has order k and is nontraceable, since both x_i and x_j have no in-neighbours in H. This contradiction shows that $|A_i \cap A_j| \leq k - 4$. Similarly, $|B_i \cap B_j| \leq k - 4$. Now suppose $|A_1 \cap B_2| \geq 2k - 7$. Then, since $|A_1 \cap A_3| \leq k - 4$, at most $k - 4$ vertices of $A_1 \cap B_2$ are in A_3, so at least $k - 3$ vertices of

$A_1 \cap B_2$ are in B_3, but then $|B_2 \cap B_3| \geq k-3$. This contradiction proves that $|A_1 \cap B_2| \leq 2k-8$. But $A_1 = (A_1 \cap A_2) \cup (A_1 \cap B_2)$. Hence $|A_1| \leq (k-4)+2k-8 = 3k-12$. Similarly, $|B_1| \leq 3k-12$. □

Theorem 2.7 and Lemma 2.8 were used in (5) to prove the following theorem.

Theorem 2.9 *For $k \geq 2$ every k-traceable strong oriented graph of order at least $6k-20$ is traceable.*

We now turn our attention to k-traceable oriented graphs of small order. Knowledge of their structure is actually important when considering the traceability of k-traceable oriented graphs of large order.

3 Hypotraceable oriented graphs

A digraph D of order n is called *hypotraceable* if $n \geq 3$ and D is $(n-1)$-traceable but not n-traceable. Thus a hypotraceble digraph is nontraceable but the removal of any vertex leaves a traceable digraph. Our next result shows the importance of hypotraceable oriented graphs in connection with the TC.

Lemma 3.1 *If $k > 2$ and D is a nontraceable, k-traceable oriented graph of order $n \geq k+1$, then D has a hypotraceable induced subdigraph of order h for some $k+1 \leq h \leq n$.*

Proof: Suppose $n = k+r$. Then, for any set S consisting of r vertices of D, the oriented graph $D-S$ is traceable. If D itself is not hypotraceable, then D has a vertex x_1 such that $D-x_1$ is nontraceable. We repeat this procedure until we obtain a subset $\{x_1,\ldots,x_t\}$ in D for some $t \leq r-1$ such that $D-\{x_1,\ldots,x_t\}$ is hypotraceable. □

In view of Lemma 3.1 it is important to know the possible orders of hypotraceable oriented graphs. Grötschel, Thomassen and Wakabayashi constructed an infinite family of hypotraceable oriented graphs in (6). These graphs are obtained from hypohamiltonian digraphs. The smallest hypotraceable oriented graph constructed by applying the construction in (6) to hypohamiltonian digraphs constructed by Thomassen in (9) has order 12. If there does not exist a hypotraceable oriented graph of order less than 12, then it would follow immediately from Theorem 2.9 and Lemma 3.1 that every 6-traceable oriented graph of order n, where $6 \leq n \leq 12$ is traceable. However, the best that we've managed to do so far was to show that there does not exist a hypotraceable oriented graph of order less than 8. To prove this, we need the following result, which is stated in (6) without proof.

Lemma 3.2 *If D is a hypotraceable digraph, then D does not have a vertex with indegree 1 or outdegree 1.*

Proof: Let D be a hypotraceable digraph, $x \in V(D)$ and suppose y is the only out-neighbour of x. Then every hamiltonian path of $D-\{y\}$ must end in x, hence can be extended with y, which is a contradiction. □

A strong digraph of order at least 2 cannot have a vertex of indegree 0 or outdegree 0, so the following holds.

Corollary 3.3 *If D is a strong hypotraceable digraph, then D has minimum indegree at least 2 and minimum outdegree at least 2.*

We shall also use the following corollary of Lemma 2.2.

Corollary 3.4 *Let D be a hypotraceable oriented graph. If D contains a vertex v, such that $d^+(v) = 2$ (or $d^-(v) = 2$), then the out-neighbours (or in-neighbours) of v are nonadjacent.*

The following result is proved in (7).

Lemma 3.5 (***Grötschel and Wakabayashi***) *Every nontrivial strong component of a hypotraceable oriented graph has order at least 5.*

It is stated in (6) (without proof) that there does not exist a hypotraceable digraph of order less than 7. We now use the lemmas above to extend this bound in the case of oriented graphs.

Lemma 3.6 *There does not exist a hypotraceable oriented graph of order less than 8.*

Proof: Suppose D is a hypotracable oriented graph of order n, with $3 \leq n \leq 7$.

Case 1. *D is strong*: In this case Theorem 2.7 implies that D has three independent vertices $\{x_1, x_2, x_3\}$ and it follows from Corollary 3.3 that $d^+(x_1) \geq 2$ and $d^-(x_1) \geq 2$. This is not possible if $n \leq 6$, so assume $n = 7$. Then $d^+(x_i) = d^-(x_i) = 2$ for $i = 1, 2, 3$. Let $N^+(x_1) = \{a_1, a_2\}$ and $N^-(x_1) = \{b_1, b_2\}$. By Corollary 3.4, $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are independent sets. If $D - \{b_1, b_2\}$ has a hamiltonian path Q, then Q starts at x_1 and ends at either x_2 or x_3, say x_3. But $d^+(x_3) = 2$, so x_3 is adjacent to at least one of b_1 and b_2, say b_2. But then $b_1 Q b_2$ is a hamiltonian path of D. This contradiction shows that $D - \{b_1, b_2\}$ has no hamiltonian path and hence $D - b_1$ has no hamiltonian path starting at b_2. Since D is 6-traceable, $D - b_1$ has a hamiltonian path P. The initial vertex of P cannot be x_1, otherwise $b_1 P$ is a hamiltonian path of D. Without loss of generality, we assume that the initial vertex of P is either x_2 or a_1.

Subcase 1.1 *The initial vertex of P is x_2*: In this case $b_1 \notin N^-(x_2)$, otherwise $b_1 P$ would be a hamiltonian path of D. Hence $b_1 \in N^+(x_2)$. There are now two possibilities to consider for the second vertex of P.

Subcase 1.1.1 *The second vertex of P is a_1*: Then $N^+(x_2) = \{a_1, b_1\}$, so $N^-(x_2) = \{a_2, b_2\}$. If $a_1 x_3 \in A(D)$, then $x_1 a_2 x_2 a_1 x_3$ is a hamiltonian path in $D - \{b_1, b_2\}$, but we have shown $D - \{b_1, b_2\}$ is nontraceable, so $a_1 \notin N^-(x_3)$. Also, $a_2 \notin N^-(x_3)$, otherwise $b_2 x_2 b_1 x_1 a_2 x_3 a_1$ would be a hamiltonian path of D. Hence $N^-(x_3) = \{b_1, b_2\}$. But then $b_2 x_3 a_2 x_2 b_1 x_1 a_1$ is a hamiltonian path of D.

Subcase 1.1.2 *The second vertex of P is b_2*: Then $N^+(x_2) = \{b_1, b_2\}$, so $N^-(x_2) = \{a_1, a_2\}$. Then we may assume w.l.o.g. that P is the path $x_2 b_2 x_1 a_1 x_3 a_2$. But then $b_2 x_1 a_1 x_3 a_2 x_2 b_1$ is a hamiltonian path in D.

Subcase 1.2 *The initial vertex of P is a_1*: We have shown that $D - \{b_1, b_2\}$ has no hamiltonian path, so we may assume w.l.o.g. that $D - b_1$ has the hamiltonian path $a_1 x_2 b_2 x_1 a_2 x_3$. Then $b_1 \notin N^+(x_3)$, so $N^+(x_3) = \{a_1, b_2\}$. But then $b_1 x_3 a_1 x_2 b_2 x_1 a_2$ is a hamiltonian path in D.

Case 2. *D is not strong*:

Subcase 2.1 *D has a nontrivial strong component X that is nonhamiltonian*: By Lemma 3.5, $|V(X)| \geq 5$, so $n = 6$ or 7. Since D is $(n-1)$-traceable, Lemma 2.5 now implies that $|V(X)| = 6$ and hence $n = 7$. By symmetry, we may assume that D_1 has order one and $X = D_2$. Let x be the vertex in D_1 and let $v_1 v_2 \ldots v_6$ be a path in D_2. Then $x, v_6 \notin N^-(v_1)$, so it follows from Lemma 3.2 and Corollary 3.4 that $\{v_3, v_5\} \subseteq N^-(v_1)$. Similarly, $\{v_2, v_4\} \subseteq N^+(v_6)$. Hence each of the vertices v_1, v_4 and v_6 is an initial vertex of a hamiltonian path of D_2, so x is not adjacent to v_1, v_4 or v_6. However, Lemma 3.2 implies that $d^+(x) \geq 2$ and, by Lemma 3.4, $N^+(x) \neq \{v_2, v_3\}$, so $v_5 \in N^+(x)$. Then $v_6 \notin N^+(v_1)$, otherwise

$xv_5v_1v_6v_2v_3v_4$ is a hamiltonian path in D. Hence $N^+(v_1) = \{v_2, v_4\}$, which implies that v_2 and v_4 are nonadjacent vertices. But then $N^+(v_4) = \{v_5\}$, which contradicts Lemma 3.2.

Subcase 2.2 *Every nontrivial strong component of D is hamiltonian*: In this case, if D had only two strong components, D would be traceable. Hence D has at least three strong components. By Lemma 3.5, each nontrivial strong component of D has order at least 5. Thus the only possibility is that $n = 7$ and D has two trivial strong components and one of order 5. The two trivial strong components cannot be consecutive, otherwise D would be traceable. Hence $n(D_1) = 1$, $n(D_2) = 5$ and $n(D_3) = 1$.

Let x be the vertex in D_1, let y be the vertex in D_3 and let $C = v_1v_2v_3v_4v_5v_1$ be a hamiltonian cycle of D_2. If x has only one out-neighbour v_i in D_2, then $D - v_i$ cannot be traceable, so $|N^+(x) \cap V(D_2)| \geq 2$. Similarly, $|N^-(y) \cap V(D_2)| \geq 2$. Since D is nontraceable, no predecessor of a neighbour of x on C is a neighbour of y. Thus, at least one of x and y has at most two neighbours in D_2. By symmetry, we may assume that x has only two out-neighbours in D_2, say a and b. If ab is an arc in D, then any hamiltonian path $xb \ldots v_jy$ of $D - a$ can be extended to a hamiltonian path $xab \ldots v_j$ of D. Hence a and b are nonadjacent. We may assume, w.l.o.g. that $a = v_2$ and $b = v_5$. Then $v_1, v_4 \notin N^-(y)$. But then v_1 has only four possible neighbours, namely v_2, v_3, v_4, v_5. Hence, by Lemma 3.2 and Corollary 3.4, $N^+(v_1) = \{v_2, v_4\}$ and $N^-(v_1) = \{v_3, v_5\}$. Now, if v_5 is adjacent to y, then $xv_2v_3v_1v_4v_5y$ is a hamiltonian path of D. Hence v_5 is not adjacent to y, so $N^-(y) = \{v_2, v_3\}$, which contradicts Corollary 3.4. □

Lemmas 3.1 and 3.6 imply the following.

Corollary 3.7 *If $2 \leq k \leq 7$, then every k-traceable oriented graph of order n is traceable, where $k \leq n \leq 7$.*

4 Oriented graphs that are 6-traceable

In this section we prove that the TC holds for $k = 6$, i.e. that every 6-traceable oriented graph of order at least 11 is traceable. We first prove it for oriented graphs that are not strong.

Lemma 4.1 *If D is a 6-traceable oriented graph of order $n \geq 11$ that is not strong, then D is traceable.*

Proof: It follows from Lemma 2.6 that D has a nontrivial strong component X that is nonhamiltonian, and from Lemma 2.5 that $|V(X)| = n - 1$.

By symmetry we may assume that $X = D_2$. Then D_1 has only one vertex x. Now D_2 is 5-traceable and of order $n - 1 \geq 10$. However, it is shown in (5) that the TC holds for $k = 5$, so D_2 is traceable. Let $v_1v_2 \ldots v_{n-1}$ be a hamiltonian path in D_2.

Let v_i be a vertex in D_2 that is nonadjacent to x. If $d^-(v_i) \leq n - 6$, then $D - N^-(v_i)$ has order at least 6. But if H is any subdigraph of $D - N^-(v_i)$ that has order 6 and contains both v_i and x, then H is nontraceable, since neither x nor v_i has any in-neighbour in H. This proves that every vertex in D that is not a neighbour of x has indegree at least $n - 5$. In particular, $d^-(v_1) \geq n - 5$. Hence it follows from Lemma 2.2 that $v_3, v_{n-2} \in N^-(v_1)$.

Since X is strong, v_{n-1} has an out-neighbour v_i such that $2 \leq i \leq n - 3$. If $v_{n-1} \in N^+(x)$, then $xv_{n-1}v_i \ldots v_{n-2}v_1 \ldots v_{i-1}$ is a hamiltonian path of D. This contradiction shows that v_{n-1} is not a neighbour of x and hence $d^-(v_{n-1}) \geq n - 5$.

By Lemma 2.1, $d^+(x) \geq n - 5$, so x has at least $n - 5$ neighbours in the set $v_2 \ldots, , v_{n-2}$. However, if $2 \leq i \leq n - 2$ and $v_i \in N^+(x)$, then the predecessor v_{i-1} is not in $N^-(v_{n-1})$, otherwise D has the hamiltonian path $xv_i \dots v_{n-2}v_1 \dots v_{i-1}v_{n-1}$. Thus at least $n-5$ vertices in D_2 are not in-neighbours of v_{n-1}, which implies that $d^-(v_{n-1}) \leq 3$, contradicting that $d^-(v_{n-1}) \geq n-5 \geq 6$. $\square$

For the proof of the strong case, we also need the following result concerning 6-traceable oriented graphs that are not strong.

Lemma 4.2 *If D is a 6-traceable oriented graph of order 8 that is not strong, then D is traceable.*

Proof: Suppose D is nontraceable. Then, since we have shown that there does not exist a hypotraceable oriented graph of order 7, Lemma 3.1 implies that D itself is hypotraceable. We need to consider two cases.

Case 1. *D has a nontrivial strong component X that is nonhamiltonian*: It follows from Lemma 2.5 that X has order 7. By symmetry we may assume that D_1 has order 1 and $D_2 = X$. Let x be the vertex in D_1 and let $v_1 \dots v_7$ be a path in D_2. It now follows exactly as in the proof of Lemma 4.1 that $v_3, v_6 \in N^-(v_1)$ and also that v_1 and v_7 are not neighbours of x, and $d^-(v_1) \geq 3$ and $d^-(v_7) \geq 3$.

We now consider the possible neighbourhoods of x. By Lemma 2.1, $d^+(x) \geq 3$, and by Lemma 2.2, if $d^+(x) = 3$, then $\langle N^+(x) \rangle$ is nontraceable. Moreover, if $v_i \in N^+(x)$, then $v_{i-1} \notin N^-(v_7)$.

Suppose $\{v_2, v_6\} \subseteq N^+(x)$. Then $N^+(v_1) \subseteq \{v_2, v_4, v_5\}$. But if $v_4 \in N^+(v_1)$ then $xv_2v_3v_1v_4v_5v_6v_7$ is a hamiltonian path of D. Hence, by Lemma 3.2, $v_5 \in N^+(v_1)$. But then $v_4 \in N^-(v_1)$ and $xv_2v_3v_4v_1v_5v_6v_7$ is a hamiltonian path of D. This proves that $\{v_2, v_6\} \not\subseteq N^+(x)$, so we need to consider four cases.

Case 1.1 $\{v_2, v_3, v_5\} \subseteq N^+(x)$: In this case $v_1, v_2, v_4 \notin N^-(v_7)$. Hence $N^-(v_7) = \{v_3, v_5, v_6\}$. Since $v_1 \notin N^+(v_7)$, this implies that $N^+(v_7) = \{v_2, v_4\}$. But then $xv_3v_7v_4v_5v_6v_1v_2$ is a hamiltonian path of D.

Case 1.2 $\{v_2, v_4, v_5\} \subseteq N^+(x)$: In this case $N^-(v_7) = \{v_2, v_5, v_6\}$. But then $N^+(v_7) = \{v_3, v_4\}$, which contradicts Lemma 2.2, since D is hypotraceable.

Case 1.3 $\{v_3, v_4, v_6\} \subseteq N^+(x)$: In this case $N^-(v_7) = \{v_1, v_4, v_6\}$. If either v_2 or v_3 is in $N^+(v_7)$, then D would obviously be traceable. But then $d^+(v_7) \leq 1$, contradicting Lemma 3.2, since D is hypotraceable.

Case 1.4 $\{v_3, v_5, v_6\} \subseteq N^+(x)$: In this case $N^-(v_7) = \{v_1, v_3, v_6\}$. By Lemma 3.2, $d^+(v_7) \geq 2$. But by Corollary 3.4, $N^+(v_7) \neq \{v_4, v_5\}$. Hence $v_2 \in N^+(v_7)$. Now if $v_4 \in N^-(v_1)$, then $xv_5v_6v_7v_2v_3v_4v_1$ is a hamiltonian path in D. But we have shown that $d^-(v_1) \geq 3$, hence $v_5 \in N^-(v_1)$. But then $xv_6v_7v_2v_3v_4v_5v_1$ is a hamiltonian path in D.

Case 2 *Every strong component of D is hamiltonian.* In this case D has at least three strong components, otherwise D would be traceable. By Lemma 3.5, every nontrivial strong component of D has order at least 5. Hence the only possibility is that $N(D_1) = 1$, $N(D_2) = 6$ and $N(D_3) = 1$.

Let x and y be the vertices in D_1 and D_3, respectively, and let $v_0v_1v_2v_3v_4v_5v_0$ be a hamiltonian cycle of D_2. If x has only two neighbours in D_2, then removal of those two neighbours leaves a nontraceable subdigraph of D that has order 6. Hence x has at least 3 neighbours in D_2 and, similarly, y has at least three neighbours in D_2. But if x is adjacent to v_i and $j = (i-1) \mod 6$, then v_j cannot be adjacent to y. This implies that $|N^+(x) \cap V(D_2)| = 3$ and $|N^-(y) \cap V(D_2)| = 3$ and $N^+(x) \cap N^-(y) \neq \emptyset$. A similar argument to that used in Lemma 2.2 shows that both $\langle N^+(x) \cap V(D_2) \rangle$ and $\langle N^-(y) \cap V(D_2) \rangle$ are nontraceable.

If $|N^+(x) \cup N^-(y)| \leq 3$, then D has a subdigraph of order 6 that contains at most one vertex in $N^+(x) \cap N^-(y)$. But such a subdigraph cannot be traceable. Hence $|N^+(x) \cup N^-(y)| \geq 4$. There are two possibilities to consider: $N^+(x) = \{v_0, v_1, v_3\}$, $N^-(y) = \{v_1, v_3, v_4\}$ and $N^+(x) = \{v_0, v_1, v_4\}$, $N^-(y) = \{v_1, v_2, v_4\}$. In the first case, since $D - \{v_0, v_1\}$ must be traceable, D must contain the path $xv_3v_5v_2v_4y$. But then $D - \{v_3, v_4\}$ cannot be traceable. In the second case, $D - \{v_1, v_2\}$ is nontraceable, since $D_2 - \{v_1, v_2\}$ does not have a path starting at v_0 and ending at v_4. Thus either case contradicts the 6-traceability of D. □

We are now ready to prove our main theorem.

Theorem 4.3 *Every 6-traceable oriented graph of order at least 11 is traceable.*

Proof: Suppose, to the contrary that D is a 6-traceable oriented graph of order n that is nontraceable, for some $n \geq 11$. By Lemma 4.1 we may assume that D is strong. Thus Theorem 2.9 implies that $n \leq 15$. By Theorem 2.7, D has three independent vertices x_1, x_2, x_3. Let $A = N^+(x_1)$ and $B = V(D) \setminus (A \cup \{x_1, x_2, x_3\})$. Lemma 2.8 implies that $|A| \leq 6$ and $|B| \leq 6$. Hence, if $n = 12$, then $|A| \geq 3$ and $|B| \geq 3$. If $n = 11$, then it follows from Lemma 2.2 and the 6-traceability of D that $|A| \neq 6$ and $|B| \neq 6$, so in this case $3 \leq |A| \leq 5$ and $3 \leq |B| \leq 5$.

If $|B| = 3$, then put $S_1 = B \cup \{x_1, x_2, x_3\}$ and $S_2 = \{x_1\} \cup A$. If $B \geq 4$, then put $S_1 = B \cup \{x_1, x_2\}$ and $S_2 = \{x_1, x_3\} \cup A$. In either case, $6 \leq |S_1| \leq 8$ and $6 \leq |S_2| \leq 8$, $S_1 \cup S_2 = V(D)$ and $S_1 \cap S_2 = \{x_1\}$. Moreover, x_1 has no in-neighbours in S_2 and no out-neighbours in S_1, so neither $\langle S_1 \rangle$ nor $\langle S_2 \rangle$ is strong. Hence, it follows from the 6-traceability of D and Theorem 3.6 and Lemma 4.2 that S_1 has a hamiltonian path ending in x_1 and S_2 has a hamiltonian path starting in x_1. But then D has a hamiltonian path, contradicting our assumption. □

Corollary 4.4 *The OPPC(1) holds for $a \leq 5$.*

Corollary 4.5 *The OPPC(1) holds for oriented graphs of order at most 12.*

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References

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications.* Springer-Verlag, London, 2002.

[2] J. Bang-Jensen, M. Hegner Nielsen, A. Yeo, Longest path partitions in generalizations of tournaments. *Discrete Math.* **306** (2006) 1830-1839.

[3] C.C. Chen and P. Manalastas Jr., Every finite strongly connected digraph of stability 2 has a Hamiltonian path. *Discrete Math.* **44** (1983) 243–250.

[4] M. Frick, S. van Aardt, G. Dlamini, J. Dunbar and O. Oellermann, The directed path partition conjecture. *Discuss. Math. Graph Theory.* **25** (2005) 331–343.

[5] M. Frick, S.A. van Aardt, M.H. Nielsen, O. Oellermann and J.E. Dunbar, A Traceability conjecture for oriented graphs. Submitted.

[6] M. Grötschel, C. Thomassen and Y. Wakabayashi, Hypotraceable digraphs, *J. Graph Theory* **4** (1980) 377–381.

[7] M. Grötschel and Y. Wakabayashi, Constructions of hypotraceable digraphs, *Mathematical Programming*, Eds. R.W. Cottle, M.L. Kelmanson and B. Korte, Elsevier Science Publishers B.V. (North Holland), 1984.

[8] J.M. Laborde, C. Payan and N.H. Xuong, Independent sets and longest directed paths in digraphs, in: *Graphs and other combinatorial topics* (Prague,1982), 173–177 (Teubner-Texte Math. 59, 1983.)

[9] C. Thomassen, Hypohamiltonian graphs and digraphs. In *Theory and Applications of graphs, Michigan* 1976. Springer Lecture Notes, **642** (1978) 557 - 571.

[10] C.A. Whitehead, M. Frick and S.A. van Aardt, Significant differences between path partitions in directed and undirected graphs. To appear in *Utilitas Math.*