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Asymptotic enumeration on self-similar graphs with two boundary vertices

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We study two graph parameters, namely the number of spanning forests and the number of connected subgraphs, for self-similar graphs with exactly two boundary vertices. In both cases, we determine the general behavior for these and related auxiliary quantities by means of polynomial recurrences and a careful asymptotic analysis. It turns out that the so-called resistance scaling factor of a graph plays an essential role in both instances, a phenomenon that was previously observed for the number of spanning trees. Several explicit examples show that our findings are likely to hold in an even more general setting.

Keywords: spanning forests, connected subgraphs, self-similar graphs, asymptotic enumeration

1 Introduction

In several recent papers [3, 4, 5, 6, 14, 15, 16] the enumeration of various graph-theoretical objects on self-similar graphs was studied, including, among others, the number of spanning trees and forests, matchings, and connected subgraphs. Some of these counting problems have a background in theoretical physics, as also explained in [3]: the number of spanning forests of a graph can be seen as a special value of the Tutte polynomial $T(G, x, y)$ of a graph (namely, the value for $x = 2$, $y = 1$). On the other hand, it also occurs as a special $q \to 0$ limit of the partition function of the q-state Potts model, see for example [13]. While this and related problems are well-studied for lattices, [3] is the first work where fractal-type structures are investigated. There, the authors were concerned with the asymptotic behavior of the number of spanning forests $|\mathsf{SF}(G_n)|$ for an increasing family of graphs G_n, and in particular with the associated asymptotic growth constant

$$\lim_{n\to\infty} \frac{\log|\mathsf{SF}(G_n)|}{|G_n|},$$

which is a quantity of physical interest. The authors of [3] were able to calculate numerical values for the growth constants of Sierpiński graphs with small dimension. In the present paper, we study the problem

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in more generality, but we have to restrict ourselves to a special type of construction that will be explained in the following section. The other counting problem that we address is the enumeration of connected subgraphs. We will see that similar techniques can be used for both problems and that the so-called *resistance scaling factor* of a graph with respect to two of its vertices plays a crucial role in both instances. This phenomenon also occurs in the enumeration of spanning trees, which was exhibited in [16].

Let us demonstrate this in the case of the aforementioned sequence of finite Sierpiński graphs which are depicted on Figure 1. The number of spanning trees in the level-n graph X_n is described by a system

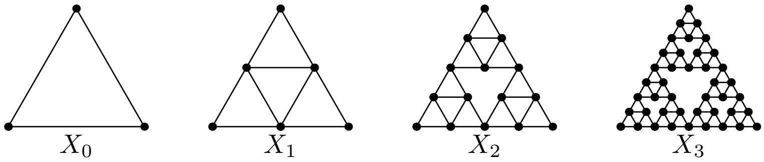

Fig. 1: Finite Sierpiński graphs X_0, X_1, X_2, X_3.

of polynomial recurrence equations:

$$\begin{aligned} a_{1,n+1} &= 6a_{1,n}^2 a_{2,n}, \\ a_{2,n+1} &= 7a_{1,n}a_{2,n}^2 + a_{1,n}^2 a_{3,n}, \\ a_{3,n+1} &= 14a_{2,n}^3 + 12a_{1,n}a_{2,n}a_{3,n}, \end{aligned}$$

where $a_{k,n}$ is the number of spanning forests in X_n with k components each of which contains exactly one "boundary vertex" of X_n, see [6, 14]. From this it is not too difficult to derive an explicit formula for the number $\tau(X_n)$ of spanning trees in X_n:

$$\tau(X_n) = \sqrt[4]{\frac{3}{20}} \left(\frac{5}{3}\right)^{-n/2} \left(\sqrt[4]{540}\right)^{3^n}.$$

In [16] it was proved that the number of spanning trees can be computed explicitly for a large class of self-similar graphs with a high degree of symmetry. Similarly, the number of connected subgraphs or connected subsets can be described by a system of seven polynomial recurrence equations:

$$\begin{aligned} \boldsymbol{b_{1,n+1}} &= \boldsymbol{12b_{1,n}b_{2,n}b_{4,n}} + \boldsymbol{14b_{2,n}^3} + 3b_{1,n}b_{5,n}^2 + \cdots, \\ \boldsymbol{b_{2,n+1}} &= \boldsymbol{b_{1,n}b_{4,n}^2} + \boldsymbol{7b_{2,n}^2 b_{4,n}} + b_{2,n}b_{4,n}^2 + 3b_{2,n}b_{5,n}^2 + \cdots, \\ b_{3,n+1} &= 2b_{1,n}b_{4,n}b_{5,n} + \cdots, \\ \boldsymbol{b_{4,n+1}} &= \boldsymbol{6b_{2,n}b_{4,n}^2} + b_{4,n}^3 + 3b_{4,n}b_{5,n}^2, \\ b_{5,n+1} &= 4b_{2,n}b_{4,n}b_{5,n} + \cdots, \\ b_{6,n+1} &= 2b_{2,n}b_{5,n}^2 + \cdots, \\ b_{7,n+1} &= 3b_{3,n}b_{5,n}^2 + \cdots. \end{aligned}$$

We refer to [15] for details, including a precise definition of the quantities involved. It turns out that the asymptotically significant terms are those which are written in bold. We point out that this part of the system agrees with the system for spanning trees. Hence the number of connected subgraphs (or subsets) is asymptotically given by

$$c \cdot \left(\tfrac{5}{3}\right)^{-\frac{n}{2}} \cdot \beta^{3^n}$$

for constants $c = 6.163424\ldots$ and $\beta = 2.3032106556\ldots$ In [15] it was conjectured that a similar formula holds for other self-similar graphs as well, which we will verify for a very specific class in this paper, namely self-similar graphs with two "boundary" vertices. We obtain precise asymptotic information for the number of spanning forests and the same for the number of connected subgraphs under mild conditions on the geometry of X_n.

2 Setting and statement of results

Let $G = (VG, EG)$ be a finite (multi-)graph with vertex set VG and edge set EG. Fix two distinguished vertices v, w in G. Throughout this work we denote by $s = |EG|$ the number of edges in G, by $d = d_G(v, w)$ the distance of the distinguished vertices v and w in G, and by $\delta = |VG| - 1$ the number of edges in a spanning tree of G. Using G as model, construct a sequence $X_0, X_1, X_2, \ldots$ of self-similar (multi-)graphs as follows:

- The initial graph X_0 is given together with two distinguished vertices v_0, w_0 and is assumed to be finite.
- The graph X_{n+1} is obtained by replacing every edge of G by an isomorphic copy of X_n, where v_n, w_n are used for linkage. There are two possibilities for each replacement. Fix one of these once and forever. The distinguished vertices v_{n+1}, w_{n+1} of X_{n+1} emerge from v, w in G.

For an edge $e \in EG$ we write $\mathsf{h}_v(e) \in VG$ for the endpoint of e which is merged with v_n during the construction and call the vertex $\mathsf{h}_v(e)$ the *v-end* of the edge e. We define the *w-end* $\mathsf{h}_w(e)$ of e analogously. Thus, in every stage of the construction an edge e in G is replaced by X_n, so that the vertex $\mathsf{h}_v(e)$ is merged with v_n and $\mathsf{h}_w(e)$ is merged with w_n.

We always assume that X_0 and G are connected, so that the graphs X_n are connected, too. By considering edges of G as resistors with unit resistance we may regard G as a electrical network. With respect to the boundary v, w (source, sink) the network G is electrically equivalent to a single resistor with some resistance ρ, which is called the *resistance scaling factor* of G with respect to v and w, see for example [1, 8, 12, 15]. Notice that $\rho \leq 1$ if $d = d_G(v, w) = 1$.

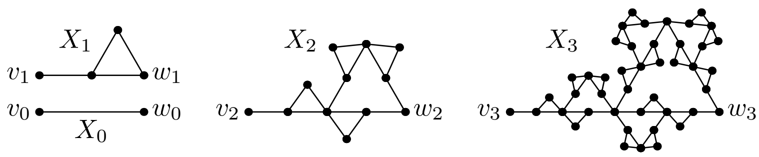

Fig. 2: The Austria graphs X_0,X_1,X_2, and X_3.

Example. The "Austria" graphs $X_0, X_1, \ldots$ were first studied in [10] with respect to volume growth and later also in [15] in the context of combinatorial enumeration (their shape resembles a map of Austria, see Figure 2). Since the initial graph is given by $X_0 = K_2$, the model graph G equals X_1. The orientation of the replacement can be seen in X_2 and X_3 in Figure 2. Obviously, $s = 4$, $d = 2$, $\delta = 3$, and $\rho = \frac{5}{3}$.

Example. In [11] spectral properties of the modified Koch curve were studied. It is a minor but interesting variation of the fractal Koch curve. The first few graphs in the associated graph sequence are depicted in Figure 3. Since $X_0 = K_2$ the model graph G is again X_1. Due to symmetry there are no choices during the replacement.

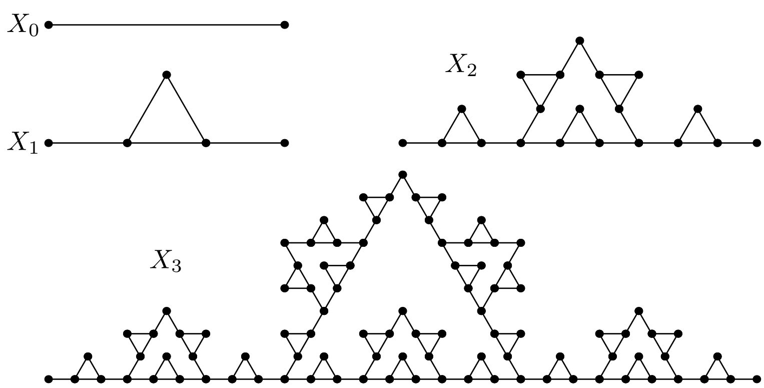

Fig. 3: The finite modified Koch graphs X_0,X_1,X_2, and X_3.

Theorem 1. *Assume that $s > 1$, then two different cases concerning the asymptotics of spanning forests in X_n may occur:*

(I) *The number of spanning forests in X_n which do not connect v_n and w_n is given by*

$$C_1\,\beta_1^{s^n}\,(1+o(1)),$$

whereas the number of spanning forests in X_n connecting v_n and w_n satisfies

$$C_2\,\beta_1^{s^n}\,\beta_2^{-d^n}\,(1+o(1))$$

for certain constants $\beta_1 > 1, \beta_2 \geq 1, C_1 > 0, C_2 > 0$.

(II) *The number of spanning forests in X_n which do not connect v_n and w_n is given by*

$$C_3\,\rho^{\frac{\delta-1}{s-1}\,n}\,\beta_3^{s^n}\,(1+o(1)),$$

whereas the number of spanning forests in X_n connecting v_n and w_n satisfies

$$C_4\,\rho^{-\frac{s-\delta}{s-1}\,n}\,\beta_3^{s^n}\,(1+o(1))$$

for certain constants $\beta_3 > 1, C_3 > 0, C_4 > 0$.

Expressions for these constants are provided in the proof. Depending on the values of ρ and d, we get the following description:

- *If $\rho \geq 1$ then Case I occurs.*

- *If $\rho < 1$ and $d = 1$ then Case II occurs.*
- *If $\rho < 1$ and $d > 1$ then both cases are possible depending on the initial graph X_0. Here it may happen that the term $\beta_1^{s^n}(1 + o(1))$ of Case I must be replaced by the weaker term $\beta_1^{s^n+O(1)}$ (this situation is described in the proof).*

The proof and an example for the case $\rho < 1$ and $d > 1$ are provided in Section 3.

Theorem 2. *Assume that $s > 1$, that v, w are leaves (i.e. vertices with degree 1) in G, that v_0, w_0 are leaves in X_0, and that G is not a path, then there are constants $\beta_4 > 1$ and $C_{5,0} > 0, C_{5,1} > 0$, so that the number of non-empty connected subgraphs of X_n is given by*

$$C_{5,k}\, \rho^{-\frac{s-\delta}{s-1}\,n}\, \beta_4^{s^n}\, \big(1 + O(2^{-n})\big),$$

where $k \in \{0, 1\}$ is the remainder of the division of n by 2.

The assumption that v, w and v_0, w_0 are leaves assures that the maximal degree X_n is uniformly bounded. Together with $s > 1$ it implies further that $\rho \geq 2$. In the case that G is a path it is easy to derive an exact formula for the number of non-empty connected subgraphs of X_n. Theorem 2 is proved in Section 4.

3 Spanning forests

This section is devoted to the proof of Theorem 1. Thus we always assume that $s > 1$. Let $\mathsf{SF}(G)$ be the set of all spanning forests in G. Introduce auxiliary sets: $\mathsf{SF}^\bullet(G)$, $\mathsf{SF}^\circ(G)$. The first set contains all spanning forests of G connecting v and w, whereas the latter denotes the set of all spanning forests of G, where v and w are not connected by the forest. Denote by $\mathsf{SF}_m(G)$ the set of all spanning forests of G with m edges and combine these notations in the obvious way, e.g. $\mathsf{SF}^\circ_m(G)$. Finally, we use $\mathsf{SF}(X_n), \mathsf{SF}^\bullet(X_n), \mathsf{SF}_m(X_n)$ analogously.

We are interested in the number $|\mathsf{SF}(X_n)|$ of spanning forests in X_n. Obviously, the disjoint union $\mathsf{SF}^\circ(X_n) \uplus \mathsf{SF}^\bullet(X_n) = \mathsf{SF}(X_n)$ is the set of all spanning forests of X_n. For simplicity, set $a_n = |\mathsf{SF}^\bullet(X_n)|$, $b_n = |\mathsf{SF}^\circ(X_n)|$. Notice that the sequence b_n tends to infinity since $s > 1$. Since every spanning forest of X_{n+1} can be decomposed into spanning forests on each copy of X_n, we obtain the recursions

$$a_{n+1} = \sum_{F \in \mathsf{SF}^\bullet(G)} a_n^{|EF|} b_n^{|EG|-|EF|} = \sum_m |\mathsf{SF}^\bullet_m(G)|\; a_n^m\, b_n^{s-m},$$

$$b_{n+1} = \sum_{F \in \mathsf{SF}^\circ(G)} a_n^{|EF|} b_n^{|EG|-|EF|} = \sum_m |\mathsf{SF}^\circ_m(G)|\; a_n^m\, b_n^{s-m}.$$

Using the polynomials p, q, r given by

$$p(x) = x^d q(x) = \sum_{m=d}^{\delta} |\mathsf{SF}^\bullet_m(G)|\; x^m = |\mathsf{SF}^\bullet_d(G)| x^d + \cdots + |\mathsf{SF}^\bullet_\delta(G)| x^\delta,$$

$$r(x) = \sum_{m=0}^{\delta-1} |\mathsf{SF}^\circ_m(G)|\; x^m = 1 + \cdots + |\mathsf{SF}^\circ_{\delta-1}(G)| x^{\delta-1},$$

the previous recursion can be reformulated as

$$a_{n+1} = b_n^s\, p(x_n) = b_n^s\, x_n^d\, q(x_n) \qquad \text{and} \qquad b_{n+1} = b_n^s\, r(x_n), \tag{1}$$

yielding a recursion for the quotient $x_n = \frac{a_n}{b_n}$:

$$x_{n+1} = f(x_n) = x_n^d g(x_n).$$

Here the rational functions f and g are given by

$$f(x) = \frac{p(x)}{r(x)} \qquad \text{and} \qquad g(x) = \frac{q(x)}{r(x)},$$

respectively. The degrees of p and r are given by $\deg(p) = \delta$ and $\deg(r) = \delta - 1$ (recall that $\delta = |VG| - 1$). It is easy to see that $\mathsf{SF}^{\bullet}_{\delta}(G)$ is the set of spanning trees in G and $\mathsf{SF}^{\circ}_{\delta-1}(G)$ is the set of spanning forests in G with two components each of which contains exactly one distinguished vertex. In Bollobas's book [2] such spanning forests are called *thickets*. Note that asymptotic information about the number of spanning forests in X_n is closely related to the dynamical behavior of f. This idea was already used in the work [3].

Lemma 3. *Let us note two basic inequalities for* $|\mathsf{SF}^{\circ}_{m-1}(G)|$ *and* $|\mathsf{SF}^{\bullet}_{m}(G)|$*:*

- *We have* $|\mathsf{SF}^{\circ}_{m-1}(G)| \geq \rho\, |\mathsf{SF}^{\bullet}_{m}(G)|$.
- *Assume that* $d = d_G(v,w) = 1$*, then* $|\mathsf{SF}^{\circ}_{m-1}(G)| \leq |\mathsf{SF}^{\bullet}_{m}(G)|$.

Proof. In order to prove the first part of the lemma, we introduce $\mathsf{SF}^{\otimes}(G)$ to be the set of all spanning forests in $\mathsf{SF}^{\circ}(G)$, where v and w can be connected by one additional edge, and let $B \subseteq VG$ be a vertex subset with $v, w \in B$, so that the induced subgraph $G[B]$ is connected. Then there is a relation between the number $\tau(G[B])$ of spanning trees in $G[B]$ and the number $\theta(G[B])$ of thickets in $G[B]$ with respect to v, w:

$$\theta(G[B]) = \rho(G[B])\, \tau(G[B]),$$

where $\rho(G[B])$ is the resistance scaling factor of $G[B]$ with respect to v, w, see [15]; in fact this formula is a consequence of Kirchhoff's famous theorem on electrical currents and spanning trees, see [9] or [2]. Because of Rayleigh's Monotonicity Law for electrical networks (see for example [7]) we have $\rho(G[B]) \geq \rho(G) = \rho$. Now define $\mathsf{SF}^{\otimes}(G,B)$ to be the set of all forests in $\mathsf{SF}^{\otimes}(G)$, where the components of v and w have union B, and define $\mathsf{SF}^{\bullet}(G,B)$ to be the set of forests in $\mathsf{SF}^{\bullet}(G)$, where the component of v and w is B. Since $\mathsf{SF}^{\otimes}(G,B)$ and $\mathsf{SF}^{\bullet}(G,B)$ only impose restrictions with respect to v and w within the set B, it follows that

$$\frac{|\mathsf{SF}^{\otimes}_{m-1}(G,B)|}{|\mathsf{SF}^{\bullet}_{m}(G,B)|} = \frac{\theta(G[B])}{\tau(G[B])} = \rho(G[B]) \geq \rho(G) = \rho$$

holds. A simple decomposition yields

$$|\mathsf{SF}^{\circ}_{m-1}(G)| \geq |\mathsf{SF}^{\otimes}_{m-1}(G)| = \sum_{B} |\mathsf{SF}^{\otimes}_{m-1}(G,B)| \geq \sum_{B} \rho\, |\mathsf{SF}^{\bullet}_{m}(G,B)| = \rho\, |\mathsf{SF}^{\bullet}_{m}(G)|,$$

where the sums are taken over all vertex subsets $B \subseteq VG$ with $v, w \in B$, so that the induced graph $G[B]$ is connected.

For the second statement of the lemma, note that a spanning forest in $\mathsf{SF}^{\circ}_{m-1}(G)$ must not contain an edge connecting v and w. Adding such an edge to a spanning forest in $\mathsf{SF}^{\circ}_{m-1}(G)$ yields a forest in $\mathsf{SF}^{\bullet}_{m}(G)$, proving the inequality. □

Proposition 4. *The asymptotic behavior of f at 0 is given by $f(x) = O(x^d)$, whereas the expansion at ∞ is given by $f(x) = \rho^{-1}x - c + O(x^{-1})$. Here $c \geq 0$ is some constant, which is positive, unless all simple paths (i.e. paths which do not visit vertices several times) connecting v and w have length 1. In this case the function f is given by $f(x) = \rho^{-1}x$.*

Proof. Both asymptotic expansions follow from the definitions of f. The constant c is given by

$$c = \frac{|\mathsf{SF}^{\bullet}_{\delta}(G)| \cdot |\mathsf{SF}^{\circ}_{\delta-2}(G)| - |\mathsf{SF}^{\bullet}_{\delta-1}(G)| \cdot |\mathsf{SF}^{\circ}_{\delta-1}(G)|}{|\mathsf{SF}^{\circ}_{\delta-1}(G)|^2}$$

This constant is positive if and only if

$$\rho = \frac{|\mathsf{SF}^{\circ}_{\delta-1}(G)|}{|\mathsf{SF}^{\bullet}_{\delta}(G)|} < \frac{|\mathsf{SF}^{\circ}_{\delta-2}(G)|}{|\mathsf{SF}^{\bullet}_{\delta-1}(G)|}.$$

A close inspection of the proof of Lemma 3 shows that $|\mathsf{SF}^{\circ}_{\delta-2}(G)| > \rho\,|\mathsf{SF}^{\bullet}_{\delta-1}(G)|$ holds, unless all simple paths between v and w have length 1.

The last part of the statement follows from the fact that

$$|\mathsf{SF}^{\circ}_{m-1}(G)| = \rho|\mathsf{SF}^{\bullet}_{m}(G)|$$

holds, since every forest in $\mathsf{SF}^{\circ}_{m-1}(G)$ corresponds to exactly $\rho^{-1} \in \mathbb{N}$ forests in $\mathsf{SF}^{\bullet}_{m}(G)$ in this case. □

Corollary 5. *The function f satisfies $f(x) \leq \rho^{-1}x$ and thus $x_n \leq \rho^{-n}\,x_0$. If not all simple paths connecting v and w have length 1, then $f(x) < \rho^{-1}x$ for $x > 0$.*

If there is an edge connecting v and w, then $f(x) \geq x$. Furthermore, if this edge is not a bridge, then $f(x) > x$ for $x > 0$, whence $x_n \to \infty$ for $n \to \infty$.

Proof. The first statement follows easily from Lemma 3:

$$x\,r(x) = \sum_m |\mathsf{SF}^{\circ}_{m-1}(G)|x^m \geq \rho \sum_m |\mathsf{SF}^{\bullet}_{m}(G)|x^m = \rho\,p(x)$$

The improved inequality follows from the proof of Lemma 3.

In order to prove the second part, assume that v and w are connected by an edge. Then a similar computation as before yields $f(x) \geq x$. If the edge connecting v and w is not a bridge, then $|\mathsf{SF}^{\circ}_{m-1}(G)| < |\mathsf{SF}^{\bullet}_{m}(G)|$ for some m. Hence $f(x) > x$ for $x > 0$ in this case. □

Lemma 6. *Assume that x_n is bounded from above. Then there is a constant $\beta_1 > 1$, so that $b_n = \beta_1^{s^n+O(1)}$. If x_n converges, then the error term can be improved:*

$$b_n = C_1\beta_1^{s^n}(1 + o(1))$$

for some constant $C_1 > 0$.

Proof. Using Equation (1) we obtain

$$\log b_n = s \log b_{n-1} + \log r(x_{n-1}) = s^n \Big(\log b_0 + \sum_{k=0}^{n-1} s^{-k-1} \log r(x_k) \Big)$$

by iteration. Define the constant K_1 by

$$K_1 = \log b_0 + \sum_{k=0}^{\infty} s^{-k-1} \log r(x_k)$$

(the sum converges due to the boundedness of x_n and $r(x) \geq 1$ for $x \geq 0$). This further implies

$$\log b_n = K_1 s^n + O(1) \qquad \text{and} \qquad b_n = \beta_1^{s^n + O(1)},$$

where $\beta_1 = \exp(K_1)$. Suppose that $x_n \to x_\infty$ holds, then the second part follows from

$$\begin{aligned} \log b_n &= s^n \Big(K_1 - \sum_{k=n}^{\infty} s^{-k-1} \log r(x_k) \Big) \\ &= s^n \Big(K_1 - \sum_{k=n}^{\infty} s^{-k-1} \log r(x_\infty) + O\Big(\sum_{k=n}^{\infty} s^{-k-1} (x_k - x_\infty) \Big) \Big) \\ &= s^n \Big(K_1 - \frac{s^{-n} \log r(x_\infty)}{s-1} + o(s^{-n}) \Big) = K_1 s^n - \tfrac{1}{s-1} \log r(x_\infty) + o(1). \qquad \square \end{aligned}$$

Lemma 7. *Assume that x_n is bounded from above and that $d > 1$. Then there is a constant $\beta_2 \geq 1$, so that $x_n = \beta_2^{-d^n + O(1)}$. If x_n converges, then the error term can be improved:*

$$x_n = C_x \beta_2^{-d^n} (1 + o(1))$$

for some constant $C_x > 0$.

Proof. Since the sequence x_n is bounded from above, there is a constant c so that $x_n \in [0, c]$. For convenience set $y_n = \frac{b_n}{a_n}$. Then $y_{n+1} = y_n^d \, g(x_n)^{-1}$ and

$$\log y_n = d \log y_{n-1} - \log g(x_{n-1}) = d^n \Big(\log y_0 - \sum_{k=0}^{n-1} d^{-k-1} \log g(x_k) \Big).$$

Now define K_2 by

$$K_2 = \log y_0 - \sum_{k=0}^{\infty} d^{-k-1} \log g(x_k).$$

The sum involved in the definition of K_2 converges, since $x_n \in [0, c]$ and g is bounded on $[0, c]$ by positive constants. This yields

$$\log y_n = K_2 d^n + O(1) \qquad \text{and} \qquad x_n = \beta_2^{-d^n + O(1)},$$

where $\beta_2 = \exp(K_2)$. Since x_n is bounded, it follows that $\beta_2 \geq 1$. The improvement of the error term is obtained in the same way as in the proof of Lemma 6:

$$\log y_n = K_2 d^n + \tfrac{1}{d-1} \log g(x_\infty) + o(1),$$

where x_∞ denotes the limit of x_n. □

Lemma 8. *Assume that $\rho < 1$ and that x_n is not bounded from above, then there are constants $\beta_3 > 1, C_3 > 0, C_4 > 0$, so that*

$$\begin{aligned} a_n &= |\mathsf{SF}^\bullet(X_n)| = C_3\, \rho^{-\frac{s-\delta}{s-1}\,n}\, \beta_3^{s^n}\, (1 + O(\rho^n)), \\ b_n &= |\mathsf{SF}^\circ(X_n)| = C_4\, \rho^{\frac{\delta-1}{s-1}\,n}\, \beta_3^{s^n}\, (1 + O(\rho^n)). \end{aligned}$$

Proof. Using Proposition 4, it is easy to see that x_n tends exponentially to ∞. Furthermore, it follows from Equation (1), that the quotient $y_n = \frac{b_n}{a_n}$ satisfies

$$y_{n+1} = y_n \frac{\tilde{r}(y_n)}{\tilde{q}(y_n)},$$

where $\tilde{q}$ and $\tilde{r}$ are the reversed polynomials of q and r, respectively. Since

$$\frac{\tilde{r}(0)}{\tilde{q}(0)} = \frac{|\mathsf{SF}^\circ_{\delta-1}(G)|}{|\mathsf{SF}^\bullet_\delta(G)|} = \rho,$$

the product

$$P = \prod_{k=0}^{\infty} \frac{\tilde{r}(y_k)}{\rho\, \tilde{q}(y_k)}$$

converges due to the exponential decay of y_n. This implies that $y_n = P\,\rho^n y_0\,(1+\varepsilon_n)$, where $\varepsilon_n = O(\rho^n)$. Using Equation (1) we get $a_{n+1} = a_n^s y_n^{s-\delta} \tilde{q}(y_n)$. Define ε'_n by the equation

$$\tilde{q}(y_n)(1+\varepsilon_n)^{s-\delta} = |\mathsf{SF}^\bullet_\delta(G)|(1+\varepsilon'_n).$$

Note that $\varepsilon'_n = O(\rho^n)$. Altogether we obtain $a_{n+1} = a_n^s \rho^{n(s-\delta)} C(1+\varepsilon'_n)$, where the constant C is given by $C = (Py_0)^{s-\delta} |\mathsf{SF}^\bullet_\delta(G)|$. This implies

$$\log a_n = s^n \left(\log a_0 + \sum_{k=0}^{n-1} s^{-k-1} \Big(k(s-\delta)\log\rho + \log C + \log(1+\varepsilon'_k) \Big) \right).$$

Now define the constant K_3 by the sum

$$K_3 = \log a_0 + \sum_{k=0}^{\infty} s^{-k-1} \Big(k(s-\delta)\log\rho + \log C + \log(1+\varepsilon'_k) \Big).$$

It follows that

$$\begin{aligned}\log a_n &= s^n\Big(K_3 - \sum_{k=n}^{\infty} s^{-k-1}\Big(k(s-\delta)\log\rho + \log C\Big) - \sum_{k=n}^{\infty} s^{-k-1}\log(1+\varepsilon_k')\Big)\\ &= K_3 s^n - \frac{n(s-\delta)\log\rho}{s-1} - \frac{\log C}{s-1} - \frac{(s-\delta)\log\rho}{(s-1)^2} + O(\rho^n),\end{aligned}$$

and

$$\begin{aligned}\log b_n &= \log a_n + \log y_n\\ &= K_3 s^n + \frac{n(\delta-1)\log\rho}{s-1} + \log(Py_0) - \frac{\log C}{s-1} - \frac{(s-\delta)\log\rho}{(s-1)^2} + O(\rho^n)\end{aligned}$$

proving the statement. □

Now we are prepared to prove Theorem 1:

Proof of Theorem 1. We distinguish several cases:

(A) Assume that $\rho \geq 1$. Then $x_n \leq \rho^{-n} x_0$ by Corollary 5.

- $\rho > 1$, $d > 1$: Since $\rho > 1$, the sequence x_n decays exponentially to 0. Hence Lemma 6 and Lemma 7 imply asymptotic expansions of a_n and b_n with improved error terms. Note that $r(0) = 1$ and $q(0) = |\mathsf{SF}^{\bullet}_d(G)| \geq 1$.
- $\rho = 1$, $d > 1$: Corollary 5 yields $f(x) < x$ for $x > 0$. Hence the sequence x_n tends to 0 and an application of Lemma 6 and Lemma 7 shows once again, that the strong version of the asymptotic expansions holds.
- $\rho = 1$, $d = 1$: In this case there is an edge connecting v and w, which is a bridge (otherwise $\rho < 1$). Thus $f(x) = x$ and $b_{n+1} = b_n^s r(x_0)$, whence it is easy to derive exact formulæ for a_n and b_n.

Since $d = 1$ implies $\rho \leq 1$, the case $\rho > 1$ and $d = 1$ is impossible.

(B) Assume that $\rho < 1$ and $d = 1$. Then Corollary 5 yields $f(x) > x$ for $x > 0$, so that x_n tends to ∞. In this case Lemma 8 implies the desired asymptotic expansions of a_n and b_n.

(C) Finally, assume that $\rho < 1$ and $d > 1$. The behavior of x_n depends on the initial value x_0. Due to Proposition 4 both 0 and ∞ are attracting fixed points of f. Thus we have to distinguish further:

- If $x_n \to 0$, then once again asymptotic expansions of a_n and b_n in strong forms are obtained from Lemma 6 and Lemma 7.
- If $x_n \to \infty$, then asymptotic expansions of a_n and b_n are implied by Lemma 8.
- If x_n is not caught by the fixpoints 0 or ∞, then x_n is bounded by positive constants. This is the only case where we only obtain weak forms of asymptotic expansions for a_n and b_n by Lemma 6 and Lemma 7 in general. If x_n converges in this case, then the error terms improve to the strong forms. □

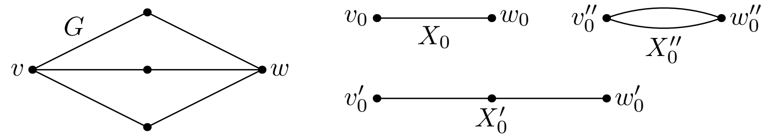

Fig. 4: A model graph G and three initial graphs X_0, X_0', X_0''.

Example. Consider the model graph G and the initial graphs X_0, X_0' and X_0'' given in Figure 4. The evolution of the sequence X_n is shown in Figure 5. In this case $s = 6$, $d = 2$, $\delta = 4$, and $\rho = \frac{2}{3}$. Furthermore, it is easy to compute the polynomials p and r:

$$p(x) = 3x^2 + 12x^3 + 12x^4 \qquad \text{and} \qquad r(x) = 1 + 6x + 12x^2 + 8x^3.$$

The rational function $f : x \mapsto \frac{p(x)}{r(x)}$ is strictly monotone and convex on $[0, \infty)$. Hence $x = 1$ is the only fixpoint in $(0, \infty)$ and this fixpoint is repelling. Finally, it is easy to compute that the ratio x_0 is given by

$$x_0 = \begin{cases} 1 & \text{if we take the initial graph } X_0, \\ \frac{1}{3} & \text{if we take the initial graph } X_0', \\ 2 & \text{if we take the initial graph } X_0''. \end{cases}$$

Thus, depending on the initial graph, we obtain different types of asymptotic expansion for the number of spanning forests. Especially, these expansions are given as follows: In the first case

$$a_n = b_n = 3^{3/5(6^n-1)} \approx 0.51728185 \cdot 1.93318204^{6^n},$$

in the second case (note that $x_\infty = 0$)

$$a_n \approx \tfrac{1}{3} \cdot 3.99222335^{6^n} \cdot 1.43574175^{-2^n} \quad \text{and} \quad b_n \approx 3.99222335^{6^n},$$

whereas in the third case we obtain

$$a_n \approx 0.55221996 \cdot \left(\tfrac{2}{3}\right)^{-2n/5} \cdot 2.67591200^{6^n},$$
$$b_n \approx 0.76288569 \cdot \left(\tfrac{2}{3}\right)^{3n/5} \cdot 2.67591200^{6^n}.$$

We conjecture that the function f is always strictly monotone and convex in $[0, \infty)$. As a consequence f has at most one fixpoint in $(0, \infty)$, which is then repelling, so that the sequence x_n always converges improving the error term always to $o(1)$.

4 Connected subgraphs

Denote by $\mathsf{SG}(G)$ the set of all non-empty (not necessarily induced) subgraphs of G and define the following subsets of $\mathsf{SG}(G)$: For $\nu \in \{0, v, w, 2\}$ let $\mathsf{SG}^\nu(G)$ be the set of connected subgraphs in $\mathsf{SG}(G)$ which contain

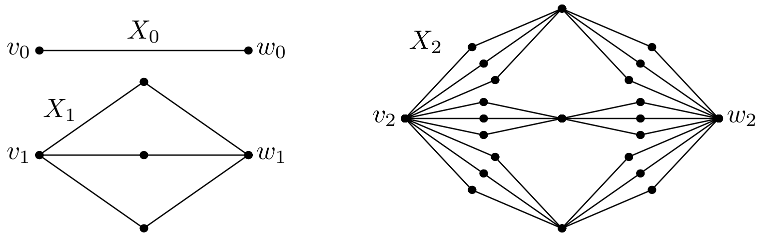

Fig. 5: The graphs X_0, X_1, X_2.

- no distinguished vertices, if $\nu = 0$,
- the vertex v but not w, if $\nu = v$,
- the vertex w but not v, if $\nu = w$,
- both distinguished vertices, if $\nu = 2$.

Finally let $\mathsf{SG}^x(G)$ be the set of all subgraphs with two connected components each of which contains one distinguished vertex. For a subgraph H of G let us define

- $e_2(H)$ to be the number of edges in EH,
- $e_v(H)$ to be the number of edges connecting vertices in VH and $VG \setminus VH$, whose v-end lies in VH.
- $e_w(H)$ to be the number of edges connecting VH and $VG \setminus VH$, whose w-end lies in VH.
- $e_x(H)$ to be the number of edges in G connecting vertices in VH which are not in EH.

For the sake of notation set $a_{\nu,n} = |\mathsf{SG}^\nu(X_n)|$ for $\nu \in \{0, v, w, 2, x\}$, where v and w are interpreted as v_n and w_n in X_n. Obviously, $a_{\nu,n} \geq 1$ for $\nu \in \{v, w, 2, x\}$ and $n \geq 0$. It is not too difficult to prove that the following system of recursions holds:

$$a_{\nu,n+1} = [\nu = 0] s\, a_{0,n} + \sum_{H \in \mathsf{SG}^\nu(G)} a_{v,n}^{e_v(H)} a_{w,n}^{e_w(H)} a_{2,n}^{e_2(H)} a_{x,n}^{e_x(H)}. \tag{2}$$

Here we have used Iverson notation: $[\nu = 0]$ is equal to 1 if $\nu = 0$ and 0 otherwise. To see why this holds, simply notice that a connected subgraph on X_{n+1} induces either a connected subgraph, a subgraph of SG^x-type, or the empty set on each of the parts that are isomorphic to X_n. The graphs H describe all possible ways for this. The additional summand in the case $\nu = 0$ arises from the possibility that a connected subgraph can be contained in one of the parts (without its boundary vertices) only. Of course, such a connected subgraph cannot contain v_{n+1} or w_{n+1}.

The bound $a_{2,n+1} \geq a_{2,n}^s$ is a simple consequence of these recursions: Choose $H = G$ in the recursion (2) for $a_{2,n+1}$. Similarly, by choosing appropriate "extremal" subgraphs we obtain

$$a_{0,n+1} \geq a_{2,n}^{s-\deg(v)-\deg(w)}, \qquad a_{v,n+1} \geq a_{2,n}^{s-\deg(w)},$$
$$a_{w,n+1} \geq a_{2,n}^{s-\deg(v)}, \qquad a_{x,n+1} \geq a_{2,n}^{s-c},$$

where c the number of edges in a minimal v-w cut. As a consequence, $a_{2,n}$ grows at least doubly exponentially if G is not a path. Therefore, for $\nu \in \{0, v, w, x\}$, the quantity $a_{\nu,n}$ also grows at least doubly exponentially in this case.

The proof of Theorem 2 is split into several lemmata. In the following we assume that v, w and v_0, w_0 have degree 1 in G and X_0, respectively, and write v' and w' for the unique neighbors of v and w in G.

Lemma 9. *We have* $a_{\nu,n} \geq a_{2,n}$ *for* $\nu \in \{0, v, w\}$ *and*

$$\frac{a_{x,n+1}}{a_{2,n+1}} \geq 2 \cdot \frac{a_{x,n}}{a_{2,n}}.$$

Proof. The first inequality is plain due to the degree restriction: since v_n and w_n have degree 1 in X_n, we can obtain a connected subgraph that doesn't contain w_n from every connected subgraph that contains w_n by removing w_n and its incident edge. Hence, $a_{v,n} \geq a_{2,n}$, and likewise for $\nu = w$ and $\nu = 0$. In order to prove the second inequality, let us note that every connected subgraph H of G which contains v and w has to contain the edge vv' as well as the edge ww'. Therefore, we can construct two graphs $H_v, H_w \in \mathsf{SG}^x(G)$ from H by removing the edge vv' respectively ww'. Then

$$e_\nu(H_\nu) = e_\nu(H), \qquad e_2(H_\nu) = e_2(H) - 1, \qquad e_x(H_\nu) = e_x(H) + 1$$

for $\nu \in \{v, w\}$. Together with Equation (2), this shows that

$$a_{x,n+1} \geq 2a_{2,n+1} \cdot \frac{a_{x,n}}{a_{2,n}},$$

proving the statement. □

Assume that G is not a path, so that $a_{2,1} > 1$. Then we have already noticed that $a_{\nu,n}$ grows at least doubly exponentially for $\nu \in \{0, v, w, 2\}$ if $n \to \infty$:

$$a_{\nu,n} \geq C_\nu \alpha^{s^n} \tag{3}$$

for some $\alpha > 1$ and constants C_ν. Due to this observation it is possible to reduce the recursions given by (2) up to a very small error term: We say that a subgraph H of G is *important*, if every edge in G has at least one endpoint in H, and denote by $\mathsf{SG}_i^\nu(G)$ the important subgraphs of $\mathsf{SG}^\nu(G)$.

Lemma 10. *Assume that G is not a path, then for $\nu \in \{0, v, w, 2\}$ the following asymptotic formulæ hold:*

$$a_{\nu,n+1} = \Big(1 + O\big(\alpha^{-s^n}\big)\Big) \sum_{H \in \mathsf{SG}_i^\nu(G)} a_{v,n}^{e_v(H)} a_{w,n}^{e_w(H)} a_{2,n}^{e_2(H)} a_{x,n}^{e_x(H)}.$$

Proof. Since $a_{\nu,n}$ grows at least doubly exponentially only those summands in the Formulæ (2) can be of asymptotic importance which belong to a subgraph H, where $e_v(H) + e_w(H) + e_2(H) + e_x(H)$ is maximal. However, this equals $s - e_r(H)$, where $e_r(H)$ is the number of edges connecting vertices in $VG \setminus VH$. This means that every edge of G must have at least one endpoint in H, so that H must be an important subgraph. The error term is then implied by (3). □

We note that in Lemma 12 more explicit expressions for $a_{\nu,n+1}$ are derived.

Lemma 11. *Assume that G is not a path. Suppose that $\mathsf{h}_w(vv') \neq v$ or $\mathsf{h}_v(ww') \neq w$, then there exist constants c_ν for $\nu \in \{0, v, w\}$, so that*

$$a_{\nu,n} = c_\nu a_{2,n} + O\big(\alpha^{-s^n}\big)$$

holds. If $\mathsf{h}_w(vv') = v$ and $\mathsf{h}_v(ww') = w$, then a distinction on the parity of n is necessary: there are constants $c_{k,\nu}$ for $\nu \in \{0, v, w\}$ and $k \in \{0, 1\}$, so that

$$a_{\nu,n} = c_{k,\nu} a_{2,n} + O\big(\alpha^{-s^n}\big)$$

holds, where $k \in \{0, 1\}$ is the remainder of the division of n by 2.

Proof. Depending on the construction of X_{n+1}, there are four possible arrangements with respect to the replacement of the edges vv' and ww' by copies of X_n. We deal with two of these four cases, since they are all very similar. Hence assume that $\mathsf{h}_v(vv') = v$ and $\mathsf{h}_w(ww') = w$. Consider an important graph H in $\mathsf{SG}_i^v(G)$. This graph H must contain w' but not w. Hence we can construct an important graph $H_w \in \mathsf{SG}_i^2(G)$ by adding the vertex w and the edge ww'. The mapping $\mathsf{SG}_i^v(G) \to \mathsf{SG}_i^2(G)$, $H \mapsto H_w$ is in fact a one-to-one correspondence. Since

$$e_v(H_w) = e_v(H) - 1, \qquad e_w(H_w) = e_w(H),$$
$$e_2(H_w) = e_2(H) + 1, \qquad e_x(H_w) = e_x(H),$$

it follows that

$$a_{v,n+1} = a_{2,n+1} \cdot \left(\frac{a_{v,n}}{a_{2,n}} + O\big(\alpha^{-s^n}\big) \right)$$

or

$$\frac{a_{v,n+1}}{a_{2,n+1}} = \frac{a_{v,n}}{a_{2,n}} + O\big(\alpha^{-s^n}\big).$$

If ε_n denotes the error term on the right hand side, then

$$\frac{a_{v,n}}{a_{2,n}} = \frac{a_{v,0}}{a_{2,0}} + \sum_{k=0}^{n-1} \varepsilon_k,$$

which converges to

$$c_v = \frac{a_{v,0}}{a_{2,0}} + \sum_{k=0}^{\infty} \varepsilon_k,$$

and the error term is given by

$$\frac{a_{v,n}}{a_{2,n}} = c_v - \sum_{k=n}^{\infty} \varepsilon_k = c_v + O\big(\alpha^{-s^n}\big).$$

Analogous arguments show that

$$\frac{a_{w,n}}{a_{2,n}} = c_w + O\big(\alpha^{-s^n}\big)$$

and

$$\frac{a_{0,n}}{a_{2,n}} = c_0 + O\big(\alpha^{-s^n}\big),$$

which implies the statement for this arrangement. If $\mathsf{h}_v(vv') = v$, $\mathsf{h}_w(ww') = w'$ or $\mathsf{h}_v(vv') = v'$, $\mathsf{h}_w(ww') = w$, we obtain the same result by essentially the same method, but if $\mathsf{h}_v(vv') = v'$ and $\mathsf{h}_w(ww') = w'$, which is equivalent to $\mathsf{h}_w(vv') = v$ and $\mathsf{h}_v(ww') = w$, things are slightly different. Again, there is a one-to-one correspondence $H \mapsto H_w$ between $\mathsf{SG}_i^v(G)$ and $\mathsf{SG}_i^2(G)$ (constructed as before), but we have

$$\begin{aligned} e_v(H_w) &= e_v(H), & e_w(H_w) &= e_w(H) - 1,\\ e_2(H_w) &= e_2(H) + 1, & e_x(H_w) &= e_x(H). \end{aligned}$$

Therefore, we get

$$a_{v,n+1} = a_{2,n+1} \cdot \left(\frac{a_{w,n}}{a_{2,n}} + O\big(\alpha^{-s^n}\big)\right)$$

and analogously

$$a_{w,n+1} = a_{2,n+1} \cdot \left(\frac{a_{v,n}}{a_{2,n}} + O\big(\alpha^{-s^n}\big)\right).$$

Combining the two, we obtain

$$a_{v,n+2} = a_{2,n+2} \cdot \left(\frac{a_{v,n}}{a_{2,n}} + O\big(\alpha^{-s^n}\big)\right),$$

and following the above arguments we find that $a_{v,n}$ converges to some value $c_{0,v}$ for even n and some value $c_{1,v}$ for odd n, which makes the distinction of the two cases necessary. The rest of the proof is analogous to the first case again. $\square$

As before we write $\tau(G)$ for the number of spanning trees in G and $\theta(G)$ for the number of thickets in G with respect to v and w. Furthermore, recall that $\delta = |VG| - 1$ is the number of edges in a spanning tree of G. A vertex of degree 1 is called a leaf and we write ℓ_v to denote the number of leaves (except v, w) of G, which are not v-ends. Let t_v be equal to v, if w is the w-end of the edge ww', or equal to w otherwise. We use ℓ_w and t_w analogous to ℓ_v and t_v, respectively. For simplicity let $\ell = \ell_v + \ell_w$ be the number of leaves (except v, w) in G.

Lemma 12. *If G is not a path, then, for $\nu \in \{v, w\}$,*

$$\begin{aligned} a_{0,n+1} &= \tau(G)\, a_{t_v,n} a_{t_w,n} a_{2,n}^{\delta-\ell-2} (a_{v,n} + a_{2,n})^{\ell_v} (a_{w,n} + a_{2,n})^{\ell_w} a_{x,n}^{s-\delta} (1 + O(2^{-n})),\\ a_{\nu,n+1} &= \tau(G)\, a_{t_\nu,n} a_{2,n}^{\delta-\ell-1} (a_{v,n} + a_{2,n})^{\ell_v} (a_{w,n} + a_{2,n})^{\ell_w} a_{x,n}^{s-\delta} (1 + O(2^{-n})),\\ a_{2,n+1} &= \tau(G)\, a_{2,n}^{\delta-\ell} (a_{v,n} + a_{2,n})^{\ell_v} (a_{w,n} + a_{2,n})^{\ell_w} a_{x,n}^{s-\delta} (1 + O(2^{-n})),\\ a_{x,n+1} &= \theta(G)\, a_{2,n}^{\delta-\ell-1} (a_{v,n} + a_{2,n})^{\ell_v} (a_{w,n} + a_{2,n})^{\ell_w} a_{x,n}^{s-\delta+1} (1 + O(2^{-n})). \end{aligned}$$

Proof. By Lemma 11 the quantities $a_{\nu,n}$ for $\nu \in \{0, v, w, 2\}$ are of the same asymptotic order. Lemma 9 implies $a_{x,n} \geq c\, 2^n\, a_{2,n}$ for some constant c. Consequently, only those summands in the Equations (2) are of interest which belong to an important subgraph H, where $e_x(H)$ is maximal (all the other summands are smaller by an exponential factor). Certainly, H must not contain any circles, since we could remove edges in this case to increase $e_x(H)$ and decrease $e_2(H)$ instead. Furthermore, $VG \setminus VH$ contains leaves only. Otherwise, assume that there is a vertex u in $VG \setminus VH$ that is not a leaf. Since the subgraph H is important, all neighbors of this vertex have to be in VH, and there are at least two of them, since u is not a leaf. Now we can add u and an edge between u and one of its neighbors to H to obtain a new important subgraph H', and $e_x(H') > e_x(H)$ (there is at least one edge between u and one of the vertices in VH' that doesn't belong to EH' and therefore contributes to $e_x(H')$, but not to $e_x(H)$).

Now consider for instance the recurrence for $a_{2,n+1}$ in (2). Each important subgraph H in the sum for which $e_x(H)$ is maximized is obtained by taking a spanning tree and possibly removing some of the leaves (since it must be connected and cycle-free and contain all vertices except possibly for the leaves). There are $\tau(G)$ choices for the spanning tree, and for each of the leaves we may choose whether to include it or not. Edges whose one endpoint is a leaf contribute a factor $a_{v,n}$ (if the leaf is not included and a w-end), a factor $a_{w,n}$ (if the leaf is not included and a v-end) or a factor $a_{2,n}$ (if the leaf is included). The remaining edges contribute a factor of $a_{2,n}$ if they are contained in H (there are always precisely $\delta - \ell$ such edges) and $a_{x,n}$ otherwise (there are precisely $s - \delta$ such edges). Combining everything, we obtain the equation for $a_{2,n+1}$. The other three equations follow analogously. □

Corollary 13. *If G is not a path, then there is a constant c_x, so that*

$$a_{x,n} = c_x \rho^n a_{2,n}(1 + O(2^{-n})).$$

Proof. Divide the last two equations in the previous lemma to find

$$\frac{a_{x,n+1}}{a_{2,n+1}} = \frac{\theta(G)}{\tau(G)} \cdot \frac{a_{x,n}}{a_{2,n}}(1 + O(2^{-n})) = \rho \cdot \frac{a_{x,n}}{a_{2,n}}(1 + O(2^{-n})).$$

Now we can apply the same technique as in Lemma 8 to show that

$$\frac{a_{x,n}}{a_{2,n}} = c_x \rho^n (1 + O(2^{-n}))$$

for some constant c_x, which concludes the proof of this lemma. □

Proof of Theorem 2. Suppose that $\mathsf{h}_w(vv') \neq v$ or $\mathsf{h}_v(ww') \neq w$. Using Lemma 11, Lemma 12, and Corollary 13 we obtain

$$a_{2,n+1} = C\rho^{n(s-\delta)} a_{2,n}^s (1 + O(2^{-n}))$$

for some constant C. The methods employed in the proof of Lemma 8 now yield the statement. The case $\mathsf{h}_w(vv') = v$ and $\mathsf{h}_v(ww') = w$ is similar. □

We conjecture that a result similar to Theorem 2 holds, when the assumption that v, w are leaves is replaced by $\rho > 1$. The following example indicates the differences and problems.

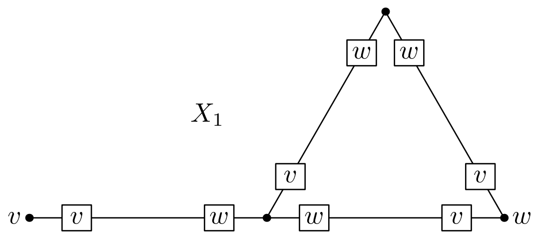

Fig. 6: The first stage in the sequence of Austria graphs. Boxed labels indicate v-ends and w-ends of edges.

Example. Let us consider the number of connected subgraphs in the sequence of Austria graphs, see Example 2. Using Formula (2) and Figure 6 we obtain:

$$\begin{aligned}
a_{0,n+1} &= 4a_{0,n} + a_{w,n}^2 + a_{v,n}a_{w,n}^2 + a_{w,n}^3 a_{2,n},\\
a_{v,n+1} &= a_{v,n} + a_{v,n}a_{w,n}a_{2,n} + a_{w,n}^2 a_{2,n}^2,\\
a_{w,n+1} &= a_{v,n}^2 + a_{v,n}a_{w,n}a_{2,n} + a_{v,n}^2 a_{w,n}a_{2,n} + a_{w,n}a_{2,n}^3 + 3a_{w,n}a_{2,n}^2 a_{x,n},\\
a_{2,n+1} &= a_{v,n}^2 a_{2,n}^2 + a_{2,n}^4 + 3a_{2,n}^3 a_{x,n},\\
a_{x,n+1} &= a_{v,n}^3 + a_{v,n}^2 a_{w,n}a_{2,n} + 2a_{v,n}^2 a_{2,n}a_{x,n} + a_{2,n}^3 a_{x,n} + 5a_{2,n}^2 a_{x,n}^2.
\end{aligned}$$

Since all the numbers $a_{\nu,n}$ grow at least doubly exponentially, say $a_{\nu,n} \geq C_\nu \zeta^{4^n}$ (for some constants C_ν and $\zeta > 1$), the terms of total degree 4 are much larger than the others. So we have to study the following:

$$\begin{aligned}
a_{0,n+1} &= a_{w,n}^3 a_{2,n}\bigl(1 + O(\zeta^{-4^n})\bigr),\\
a_{v,n+1} &= a_{w,n}^2 a_{2,n}^2\bigl(1 + O(\zeta^{-4^n})\bigr),\\
a_{w,n+1} &= a_{w,n}(a_{v,n}^2 a_{2,n} + a_{2,n}^3 + 3a_{2,n}^2 a_{x,n})\bigl(1 + O(\zeta^{-4^n})\bigr),\\
a_{2,n+1} &= a_{2,n}(a_{v,n}^2 a_{2,n} + a_{2,n}^3 + 3a_{2,n}^2 a_{x,n}),\\
a_{x,n+1} &= (a_{v,n}^2 a_{w,n}a_{2,n} + 2a_{v,n}^2 a_{2,n}a_{x,n} + a_{2,n}^3 a_{x,n} + 5a_{2,n}^2 a_{x,n}^2)\bigl(1 + O(\zeta^{-4^n})\bigr).
\end{aligned}$$

It follows that

$$\frac{a_{0,n+1}}{a_{v,n+1}} = \frac{a_{w,n}}{a_{2,n}} \cdot \bigl(1 + O(\zeta^{-4^n})\bigr), \qquad \frac{a_{w,n+1}}{a_{2,n+1}} = \frac{a_{w,n}}{a_{2,n}} \cdot \bigl(1 + O(\zeta^{-4^n})\bigr),$$

so that

$$a_{0,n} = c\, a_{v,n}\bigl(1 + O(\zeta^{-4^n})\bigr) \qquad \text{and} \qquad a_{w,n} = c\, a_{2,n}\bigl(1 + O(\zeta^{-4^n})\bigr)$$

for some constant c. By removing the edge incident to v_n from a connected subgraph in $\mathsf{SG}^2(X_n)$ a subgraph in $SG^x(X_n)$ is obtained. Thus $a_{x,n} \geq a_{2,n}$. Furthermore, using the recursions we get

$$\frac{a_{x,n+1}}{a_{2,n+1}} \geq \frac{2a_{v,n}^2 a_{2,n}a_{x,n} + a_{2,n}^3 a_{x,n} + 5a_{2,n}^2 a_{x,n}^2}{a_{v,n}^2 a_{2,n}^2 + a_{2,n}^4 + 3a_{2,n}^3 a_{x,n}} = \frac{a_{x,n}}{a_{2,n}} \cdot \left(1 + \frac{a_{v,n}^2 + 2a_{2,n}a_{x,n}}{a_{v,n}^2 + a_{2,n}^2 + 3a_{2,n}a_{x,n}}\right).$$

Since

$$2(a_{v,n}^2 + 2a_{2,n}a_{x,n}) \geq 2a_{v,n}^2 + a_{2,n}^2 + 3a_{2,n}a_{x,n} \geq a_{v,n}^2 + a_{2,n}^2 + 3a_{2,n}a_{x,n}$$

the inequality

$$\frac{a_{x,n+1}}{a_{2,n+1}} \geq \frac{3}{2} \cdot \frac{a_{x,n}}{a_{2,n}}.$$

follows. Using this estimate we obtain

$$a_{x,n} \geq c' \big(\tfrac{3}{2}\big)^n a_{2,n} \qquad \text{and} \qquad a_{w,n} \geq c'' \big(\tfrac{3}{2}\big)^n a_{v,n},$$

for some constants c' and c'', since

$$\frac{a_{w,n+1}}{a_{v,n+1}} \geq \frac{3a_{w,n}a_{2,n}^2 a_{x,n}}{a_{w,n}^2 a_{2,n}^2} \cdot \big(1 + O\big(\zeta^{-4^n}\big)\big) = \frac{3a_{x,n}}{a_{w,n}} \cdot \big(1 + O\big(\zeta^{-4^n}\big)\big).$$

As a consequence we can further simplify the recursions:

$$\begin{aligned}
a_{0,n+1} &= a_{w,n}^3 a_{2,n} \big(1 + O\big(\zeta^{-4^n}\big)\big),\\
a_{v,n+1} &= a_{w,n}^2 a_{2,n}^2 \big(1 + O\big(\zeta^{-4^n}\big)\big),\\
a_{w,n+1} &= 3a_{w,n} a_{2,n}^2 a_{x,n} \big(1 + O\big(\big(\tfrac{2}{3}\big)^n\big)\big),\\
a_{2,n+1} &= 3a_{2,n}^3 a_{x,n} \big(1 + O\big(\big(\tfrac{2}{3}\big)^n\big)\big),\\
a_{x,n+1} &= 5a_{2,n}^2 a_{x,n}^2 \big(1 + O\big(\big(\tfrac{2}{3}\big)^n\big)\big).
\end{aligned}$$

Using the last two relations we get

$$a_{x,n} = c_x \big(\tfrac{5}{3}\big)^n a_{2,n} \big(1 + O\big(\big(\tfrac{2}{3}\big)^n\big)\big)$$

for some c_x. Now the methods of Lemma 8 yield

$$a_{\nu,n} = C_\nu \big(\tfrac{5}{3}\big)^{(2/3-k_\nu)n} \beta^{4^n} \big(1 + O\big(\big(\tfrac{2}{3}\big)^n\big),$$

where

$$k_\nu = \begin{cases} 0 & \text{if } \nu = x,\\ 1 & \text{if } \nu \in \{2, w\},\\ 2 & \text{if } \nu \in \{v, 0\}. \end{cases}$$

The approximate numerical value of β is 1.77280776, whereas $(C_0, C_v, C_w, C_2, C_x)$ is approximately equal to

$$(1.13215234, 0.793383233, 0.950775521, 0.666279023, 0.632751624).$$

For comparison with Theorem 2 we recall that $\rho = \frac{5}{3}$ and that the exponent of ρ in this theorem is given by

$$-\frac{s-\delta}{s-1} = -\frac{1}{3}.$$

Although the number of connected subgraphs is given by the same asymptotic form as before, the quantities $a_{\nu,n}$ show a different behavior.

We notice that the asymptotics derived in the previous example always holds, when $\deg(v_n) = 1$ and $\deg(w_n) = c > 1$ or vice versa. The argument for this follows the line of the proof of Theorem 2 with suitable modifications. The final example demonstrates that quite a different asymptotic behavior is possible if the resistance scaling factor ρ is less than 1.

Example. We consider the graphs X_n defined in Example 3. Since the graphs X_n are symmetric with respect to v_n and w_n, the quantities $a_{v,n}$ and $a_{w,n}$ are the same: Thus set $a_{1,n} = a_{v,n} = a_{w,n}$. Then we get

$$\begin{aligned}
a_{0,n+1} &= 6a_{0,n} + 3a_{1,n}^2, \\
a_{1,n+1} &= a_{1,n}^3 + 3a_{1,n}^3 a_{2,n} + 3a_{1,n}^3 a_{2,n}^2 + a_{1,n}^3 a_{2,n}^3 = a_{1,n}^3(1 + a_{2,n})^3, \\
a_{2,n+1} &= 3a_{1,n}^4 a_{2,n}^2 + 3a_{1,n}^2 a_{2,n}^4 + a_{2,n}^6 + 12a_{1,n}^2 a_{2,n}^3 a_{x,n} + 12a_{2,n}^4 a_{x,n}^2 + 6a_{2,n}^5 a_{x,n}, \\
a_{x,n+1} &= a_{1,n}^6 + 6a_{1,n}^4 a_{2,n} a_{x,n} + 12a_{1,n}^2 a_{2,n}^2 a_{x,n}^2 + 8a_{2,n}^3 a_{x,n}^3 = (a_{1,n}^2 + 2a_{2,n}a_{x,n})^3.
\end{aligned}$$

As usual the numbers $a_{\nu,n}$ ($\nu \in \{0, 1, 2, x\}$) grow at least doubly exponentially, say $a_{\nu,n} \geq c_\nu \zeta^{6^n}$, it follows that

$$\frac{a_{2,n+1}}{a_{1,n+1}} \geq \frac{3a_{1,n}^2 a_{2,n}^4}{a_{1,n}^3(1 + a_{2,n})^3} \geq 3 \cdot \frac{a_{2,n}}{a_{1,n}} \cdot \big(1 + O(\zeta^{-6^n})\big)$$

We have $a_{1,n} \leq a_{x,n}$ by induction: $a_{1,n+1} \leq 8a_{1,n}^3 a_{2,n}^3 \leq 8a_{2,n}^3 a_{x,n}^3 \leq a_{x,n+1}$. As a consequence it follows that

$$\begin{aligned}
\frac{a_{x,n+1}}{a_{1,n+1}} &= \frac{(a_{1,n}^2 + 2a_{2,n}a_{x,n})^3}{a_{1,n}^3(1 + a_{2,n})^3} = \left(O(3^{-n}) + 2 \cdot \frac{a_{x,n}}{a_{1,n}}\right)^3 \big(1 + O(\zeta^{-6^n})\big) \\
&= 8\left(\frac{a_{x,n}}{a_{1,n}}\right)^3 (1 + O(3^{-n})).
\end{aligned}$$

Using the techniques of Lemma 6 we obtain

$$a_{x,n} = 2^{-3/2}\beta^{3^n} a_{1,n}(1 + O(3^{-n}))$$

for some constant $\beta > 1$. Finally this implies that

$$\frac{a_{2,n+1}}{a_{x,n+1}} \geq \frac{12a_{2,n}^4 a_{x,n}^2}{(a_{1,n}^2 + 2a_{2,n}a_{x,n})^3} = \frac{3}{2} \cdot \frac{a_{2,n}}{a_{x,n}} \big(1 + O(\beta^{-3^n})\big).$$

Thus we can reduce the system of recursions to

$$\begin{aligned}
a_{0,n+1} &= 3a_{1,n}^2\big(1 + O(\zeta^{-6^n})\big), \\
a_{1,n+1} &= a_{1,n}^3 a_{2,n}^3\big(1 + O(\zeta^{-6^n})\big), \\
a_{2,n+1} &= a_{2,n}^6\big(1 + O\big(\big(\tfrac{2}{3}\big)^n\big)\big), \\
a_{x,n+1} &= 8a_{2,n}^3 a_{x,n}^3\big(1 + O(\beta^{-3^n})\big).
\end{aligned}$$

From this we easily obtain

$$\begin{aligned}
a_{0,n} &= 3\beta_1^{6^n/3}\beta_2^{2\cdot 3^n/3}\big(1+O\big((\tfrac{2}{3})^n\big)\big),\\
a_{1,n} &= \beta_1^{6^n}\beta_2^{3^n}\big(1+O\big((\tfrac{2}{3})^n\big)\big),\\
a_{2,n} &= \beta_1^{6^n}\big(1+O\big((\tfrac{2}{3})^n\big)\big),\\
a_{x,n} &= 2^{-3/2}\beta_1^{6^n}\beta_3^{3^n}\big(1+O\big((\tfrac{2}{3})^n\big)\big),
\end{aligned}$$

where $\beta_1 > 1$ and $\beta_2, \beta_3 \in (0,1)$. Since $a_{1,n} \le a_{x,n}$ it follows that $\beta_2 < \beta_3$. Numerical values of $\beta_1, \beta_2, \beta_3$ are given by

$$\beta_1 \approx 1.96936033, \qquad \beta_2 \approx 0.43546557, \qquad \beta_3 \approx 0.92526029.$$

We remark that this type of asymptotics always holds when $a_{\nu,n} = o(a_{2,n})$ for $\nu \in \{0, v, w, x\}$. In this case the reasoning used in the example above applies, so that

$$a_{v,n} \approx \beta_1^{s^n}\beta_v^{d_w^n}, \qquad a_{w,n} \approx \beta_1^{s^n}\beta_w^{d_v^n}, \qquad a_{2,n} \approx \beta_1^{s^n}, \qquad a_{x,n} \approx C\beta_1^{s^n}\beta_x^{c^n}, \tag{4}$$

where $\beta_1 > 1$, $\beta_v, \beta_w, \beta_x \in (0,1)$, $C > 0$, $d_v = \deg(v)$, $d_w = \deg(w)$, and c is the number of edges in a minimal v-w cut. The behavior of $a_{0,n}$ is possibly different due to the fact that $G \setminus \{v, w\}$ may be disconnected.

Finally, we conjecture that the number of connected subgraphs exhibit a phase transition: when $\rho > 1$, then the asymptotics of Theorem 2 holds, whereas the asymptotics is given by (4) if $\rho < 1$.

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References

[1] M. T. Barlow. Diffusions on fractals. In *Lectures on probability theory and statistics (Saint-Flour, 1995)*, pages 1–121. Springer Verlag, Berlin, 1998.

[2] B. Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[3] S.-C. Chang and L.-C. Chen. Spanning forests on the Sierpinski gasket. preprint, arXiv:math-ph/0612083v1, 2006.

[4] S.-C. Chang and L.-C. Chen. Dimer coverings on the sierpinski gasket with possible vacancies on the outmost vertices. *J. Stat. Phys.*, 131(4):631–650, 2008. arXiv:0711.0573v1.

[5] S.-C. Chang and L.-C. Chen. Dimer-monomer model on the Sierpinski gasket. *Physica A: Stat. Mech. Appl.*, 387(7):1551–1566, 2008. arXiv:cond-mat/0702071v1.

[6] S.-C. Chang, L.-C. Chen, and W.-S. Yang. Spanning trees on the Sierpinski gasket. *J. Stat. Phys.*, 126(3):649–667, 2007. arXiv:cond-mat/0609453v1.

[7] P. G. Doyle and J. L. Snell. *Random walks and electric networks*, volume 22 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 1984. Also available at `arXiv.org` as `math.PR/0001057`.

[8] J. Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[9] G. R. Kirchhoff. Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird. *Ann. Phys. Chem.*, 72(4):497–508, 1847. Gesammelte Abhandlungen, Leipzig, 1882.

[10] B. Krön. Growth of self-similar graphs. *J. Graph Theory*, 45(3):224–239, 2004.

[11] L. A. Malozemov. The difference Laplacian Δ on the modified Koch curve. *Russian J. Math. Phys.*, 1(4):495–509, 1993.

[12] V. Metz. The short-cut test. *J. Funct. Anal.*, 220(1):118–156, 2005.

[13] A. D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In *Surveys in combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 173–226. Cambridge Univ. Press, Cambridge, 2005.

[14] E. Teufl and S. Wagner. The number of spanning trees of finite sierpinski graphs. In *Fourth Colloquium on Mathematics and Computer Science*, volume AG of *DMTCS Proceedings*, pages 411–414, 2006.

[15] E. Teufl and S. Wagner. Enumeration problems for classes of self-similar graphs. *J. Combin. Theory Ser. A*, 114(7):1254–1277, 2007.

[16] E. Teufl and S. Wagner. The number of spanning trees in self-similar graphs. preprint, 2008.