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Quasipolynomial formulas for the Kronecker coefficients indexed by two two–row shapes (extended abstract)

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Abstract. We show that the Kronecker coefficients indexed by two two–row shapes are given by quadratic quasipolynomial formulas whose domains are the maximal cells of a fan. Simple calculations provide explicitly the quasipolynomial formulas and a description of the associated fan.

These new formulas are obtained from analogous formulas for the corresponding reduced Kronecker coefficients and a formula recovering the Kronecker coefficients from the reduced Kronecker coefficients.

As an application, we characterize all the Kronecker coefficients indexed by two two-row shapes that are equal to zero. This allowed us to disprove a conjecture of Mulmuley about the behavior of the stretching functions attached to the Kronecker coefficients.

Résumé. Nous démontrons que les coefficients de Kronecker indexés par deux partitions de longueur au plus 2 sont donnés par des formules quasipolynomiales quadratiques dont les domaines de validité sont les cellules maximales d'un éventail. Des calculs simples nous donnent une description explicite des formules quasipolynomiales et de l'éventail associé. Ces nouvelles formulas sont obtenues de formules analogues pour les coefficients de Kronecker réduits correspondants et au moyen d'une formule reconstruisant les coefficients de Kronecker à partir des coefficients de Kronecker réduits.

Une application est la caractérisation exacte de tous les coefficients de Kronecker non–nuls indexés par deux partitions de longueur au plus deux. Ceci nous a permis de réfuter une conjecture de Mulmuley au sujet des fonctions de dilatations associées aux coefficients de Kronecker.

Keywords: Kronecker coefficients, internal product of symmetric functions, Saturation properties, Representations of the symmetric group

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Introduction

A fundamental problem in algebraic combinatorics is the Clebsch-Gordan problem: given a linearly reductive group G, give a combinatorial description of the coefficients $m_{\mu\nu}^{\lambda}$ in the decomposition into irreducibles of the tensor product of two (finite-dimensional complex) irreducible representation $V_\mu(G)$ and $V_\nu(G)$:

$$V_\mu(G) \otimes V_\nu(G) \cong \bigoplus_{\lambda} m_{\mu\nu}^{\lambda} V_\lambda(G)$$

While this problem has been solved satisfactorily for the general linear group, $GL(n)$, the most elementary linear group, this is not the case for the symmetric group, S_n, the most fundamental finite group.

In the case of $GL(n)$, the coefficients $m_{\mu\nu}^{\lambda} = c_{\mu\nu}^{\lambda}$ are the well known Littlewood-Richardson coefficients. There exists several combinatorial descriptions for them. One of these descriptions was given by Berenstein and Zelevinsky (1992) that showed that $c_{\mu\nu}^{\lambda}$ counts the integral points in a well-defined family of polytopes. This initiated a series of works concerning the stretching functions associated to these coefficients that culminated with the proof by Knutson and Tao (1999) of the saturation conjecture. Finally, Rassart (2004) showed that the Littlewood-Richardson coefficients $c_{\mu\nu}^{\lambda}$ are given by polynomial functions of the parts of λ, μ and ν, on the maximal cells of a fan.

For the symmetric group S_n, the coefficients $m_{\mu\nu}^{\lambda} = g_{\mu\nu}^{\lambda}$ are called the Kronecker coefficients. Amazingly, there is no combinatorial description of these coefficients in general. Particular families have been investigated. In this paper the *Kronecker coefficients indexed by two two–row shapes* are considered. They are the coefficients $g_{\mu\nu}^{\lambda}$ such that both μ and ν have two rows. Formulas for them have already been given by Remmel and Whitehead (1994) and Rosas (2001). Recent works by Luque and Thibon (2003); Garsia et al. (2008); Brown et al. (2008) have revived the interest of obtaining better formulas for the Kronecker coefficients indexed by two two–row shapes as Hilbert series related to these coefficients have been linked to problems in quantum information theory.

New problems about the Clebsch–Gordan coefficients have been raised recently by the specialists of computational complexity. Narayanan (2006) showed that the computation of the Littlewood–Richardson coefficients is a #P–complete problem. Bürgisser and Ikenmeyer (2008) showed that the computation of the Kronecker coefficients is #P–hard. On the other hand, the saturation property implies that the non–vanishing of a Littlewood–Richardson coefficient can be decided in polynomial time (Mulmuley and Sohoni, 2005). Is it also the case for the Kronecker coefficients? This question lies at the heart of a detailed plan, *Geometric Complexity Theory*, that Mulmuley and Sohoni (2001) elaborated to prove that $P \neq NP$ over the complex numbers (an arithmetic, non–uniform version of $P \neq NP$). This lead Mulmuley (2007) to state a series of conjectures about the stretching functions associated to the Kronecker coefficients. The scarce information available about Kronecker coefficients made difficult even the experimental checking of these conjectures. By means of the formulas by Remmel and Whitehead (1994) and Rosas (2001) it was only possible to check them on large samples of Kronecker coefficients indexed by two two–row shapes (see Mulmuley, 2007).

The present article obtains a new description for the Kronecker coefficients indexed by two two–row shapes, given by quasi–polynomial functions on the chambers of fans, resembling the description of Rassart (2004) for the Littlewood–Richardson coefficients. It is efficient enough to check Mulmuley's conjectures for all Kronecker coefficients indexed by two two–row shapes (and, actually, disprove them by providing explicit counter–examples). We start our investigation by looking at Murnaghan's *reduced Kronecker coefficients* $\bar{g}_{\alpha\beta}^{\gamma}$ (Murnaghan, 1938), a related family of coefficients indexed by triples of partitions, which are stable values of stationary sequences of Kronecker coefficients. Our first result expresses the Kronecker coefficients in terms of the reduced Kronecker coefficients (Theorem 3). Exploiting the work of Rosas (2001) we are able to show that the reduced Kronecker polynomials related to the two-row family count integral points in a polygon of $\mathbb{R}^2$. From this we describe an explicit piecewise quasipolynomial formula for these reduced Kronecker coefficients. The pieces are the 26 maximal cells of a fan. Last, using our formula that recovers the Kronecker coefficients from the reduced Kronecker coefficients, we obtain, with the help of the Maple package *convex* by Franz (2006), explicit piecewise quasipolynomial formulas for the Kronecker coefficients indexed by two two-row shapes. It is given by 74 quadratic quasipolynomials whose domains are the maximal cells of a fan.

As an application, we list all Kronecker coefficients indexed by two two-row shapes that are equal to zero. This made possible the discovery of counter–examples to Mulmuley's conjectures (Briand et al., 2008). In short, the advantage of our results is that for the first time we can completely study a complete nontrivial family of the Kronecker coefficients.

The detailed proofs will be presented in a full version (Briand et al., In preparation) of this extended abstract.

1 Piecewise Quasipolynomials

We now give a more detailed description of the main result. A quasipolynomial is a function on $\mathbb{Z}^n$ given by polynomial formulas, whose domains are the cosets of a full rank sublattice of $\mathbb{Z}^n$. Remarkable examples of (univariate) quasipolynomials are the Ehrhart functions of polytopes of $\mathbb{R}^k$ with rational vertices, that count the integral points in the dilations of the polytope (see Stanley, 1997, chap. 4).

We will obtain a description for the Kronecker coefficients indexed by two two–row shapes as a function of the following kind.

Definition 1 *A* vector partition–like function *is a function ϕ on $\mathbb{Z}^n$ fulfilling the following: (i) There exists a convex rational polyhedral cone C such that ϕ is zero outside C. (ii) Inside C, the function ϕ is given by quasipolynomial formulas whose domains are (the sets of integral points of) the maximal (closed) cells of a fan $\mathfrak{F}$.*

If C and $\mathfrak{F}$ are as above and Q is the family of quasipolynomial formulas, indexed by the maximal cells of $\mathfrak{F}$, we say that the triple $(C, \mathfrak{F}, Q)$ is a presentation of ϕ as a vector partition–like function.

Remark 1 *A sum of vector partition–like functions ϕ_1, ϕ_2 is not necessarily vector partition–like. It is, however, the case when the functions admit presentations $(C, \mathfrak{F}, Q)$ and $(C', \mathfrak{F}', Q')$ with the same cone: $C = C'$.*

Examples of vector partition–like functions are the vector partition functions, whose corresponding fans are the *chamber complexes* (see Sturmfels, 1995; Brion and Vergne, 1997).

Vector partition–like functions also arise as functions counting integral solutions to some systems of linear inequalities depending on parameters. Precisely, consider a system of inequalities of the form

$$u_i(x) + c_i(h) \geq 0, \qquad i = 1, \ldots, N \tag{1}$$

where the functions u_i and c_i are integral, homogeneous linear forms on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. The unknown is x and the parameter is h. Assume that for any $h \in \mathbb{R}^n$ the set of solutions x of the system is bounded. Let $h \mapsto \phi(h)$ be the function that counts the *integral* solutions x of the system. This function ϕ

is vector partition–like. This follows from the reduction of this function to a vector partition function (see Brion and Vergne, 1997). Here the cone C in Definition 1 is the set of values of the parameter h making the system feasible.

Let ℓ be a positive integer. The function $(\lambda, \mu, \nu) \mapsto c^{\lambda}_{\mu,\nu}$ from triples of partitions with at most ℓ parts to Littlewood–Richardson coefficients is vector partition–like. This is because this function counts the integral solutions of a system of inequalities depending on parameters (the parts of the partitions) of the form (1). Indeed, such a system can be derived from the Littlewood–Richardson rule (see Mulmuley and Sohoni, 2005). Alternatively, one can use the system defining Knutson and Tao's Hive polytopes (see the exposition by Buch, 2000).

It is natural to ask if similar results also hold for the Kronecker coefficients. Let ℓ_1 and ℓ_2 be positive integers. If μ and ν are partitions of length at most ℓ_1 and ℓ_2 respectively then $g^{\lambda}_{\mu,\nu}$ can be nonzero only if λ has at most $\ell_1\ell_2$ parts. The analogous function to consider is thus $G_{\ell_1,\ell_2} : (\lambda, \mu, \nu) \mapsto g^{\lambda}_{\mu,\nu}$ defined on triples of partitions with at most $\ell_1\ell_2$, ℓ_1 and ℓ_2 parts respectively. No interpretation of the functions G_{ℓ_1,ℓ_2} as counting integral solutions to systems of inequalities of the form (1) is known. Nevertheless, very close results were obtained by Mulmuley (2007): (i) The functions G_{ℓ_1,ℓ_2} fulfill the conditions in Definition 1 with $\mathfrak{F}$ a complex of polyhedral cones instead of a fan. (ii) For any λ, μ, ν, the *stretching function* $N \in \mathbb{N} \mapsto g^{N^{\lambda}}_{N\mu,N\nu}$ is a univariate quasipolynomial. Here $N\lambda$ stands for the partitions obtained from λ by multiplying all parts by N. Combining these two results, one gets that the functions G_{ℓ_1,ℓ_2} fulfill the conditions in the definition of vector partition–like with "maximal closed cells" replaced with "open cells" in (ii).

The simplest non–trivial case is $G_{2,2}$, describing the Kronecker coefficients indexed by two two–row shapes. Even this case is somehow difficult. In this work we prove the following:

Theorem 1 *The function*

$$G_{2,2} : \ (\lambda_1, \ldots, \lambda_4, \mu_1, \mu_2, \nu_1, \nu_2) \in \mathbb{Z}^8 \mapsto g^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)}$$

is vector partition–like.

Remark 2 *A Kronecker coefficient $g^{\lambda}_{\mu,\nu}$ can be nonzero only if its three indexing partitions have the same weight. This and the formula $g^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)} = g^{(\lambda_1-1,\lambda_2-1,\lambda_3-1,\lambda_4-1)}_{(\mu_1-2,\mu_2-2)(\nu_1-2,\nu_2-2)}$ reduce the study of $G_{2,2}$ to the study of the function*

$$(n, \gamma_1, \gamma_2, r, s) \mapsto g^{(n-\gamma_1-\gamma_2,\gamma_1,\gamma_2)}_{(n-r,r)(n-s,s)}$$

2 Murnaghan's Theorem and reduced Kronecker coefficients

In this section we introduce Murnaghan's reduced Kronecker coefficients $\overline{g}^{\gamma}_{\alpha,\beta}$. They are integers indexed by triples of partitions closely related to the Kronecker coefficients. The Kronecker coefficients indexed by two two–row shapes will be re–obtained from the reduced Kronecker coefficients indexed by two one–row shapes (Section 3) which will be easy to describe (Theorem 4 and Section 4).

The Jacobi–Trudi formula expresses the Schur functions as determinants in the complete sums h_k. When λ has at most k parts, it asserts that:

$$s_\lambda = \det(h_{j-i+\lambda_i})_{i,j=1,\ldots,k}$$

(where $h_k = 0$ when $k < 0$, $h_0 = 1$ and $\lambda_i = 0$ for i greater than the length of λ.)

This formula can also be applied in the case when λ is not a partition, i.e. is not nondecreasing. The functions s_λ obtained are either 0, or Schur functions up to a sign.

Let n be an integer and λ a partition. Then $|\lambda|$ stands for the sum of the parts of λ and for any integer n, we denote with $(n-|\lambda|, \lambda)$ the sequence $(n-|\lambda|, \lambda_1, \lambda_2, \ldots)$. This is a partition if and only if $n \geq |\lambda|+\lambda_1$. Last $\overline{\lambda}$ stands for the partition $(\lambda_2, \lambda_3, \ldots)$, which is obtained by removing the first part of λ.

Theorem 2 (Murnaghan (1938, 1955)) *There exists a family of nonnegative integers $(\overline{g}_{\alpha,\beta}^{\gamma})$ indexed by triples of partitions (α, β, γ) such that, for fixed partitions α and β, only finitely many terms $\overline{g}_{\alpha,\beta}^{\gamma}$ are non–zero, and for all $n \geq 0$,*

$$s_{(n-|\alpha|,\alpha)} * s_{(n-|\beta|,\beta)} = \sum_{\gamma} \overline{g}_{\alpha,\beta}^{\gamma} s_{(n-|\gamma|,\gamma)}$$

Following Klyachko (2004), we call the coefficients $\overline{g}_{\alpha,\beta}^{\gamma}$ the *reduced Kronecker coefficients*. They are called *extended Littlewood–Richardson numbers* in Kirillov (2004) because of the following property, observed first in Murnaghan (1955) and proved in Littlewood (1958): if α, β and γ are three partitions such that $|\gamma| = |\alpha| + |\beta|$ then $\overline{g}_{\alpha\beta}^{\gamma} = c_{\alpha\beta}^{\gamma}$.

Remark 3 *It follows from Murnaghan's Theorem that for fixed partitions α, β, γ, the sequence of Kronecker coefficients $g_{(n-|\alpha|,\alpha),(n-|\beta|,\beta)}^{(n-|\gamma|,\gamma)}$ (n big enough so that all three indices are partitions) is stationary with limit $\overline{g}_{\alpha\beta}^{\gamma}$.*

3 From reduced to non–reduced Kronecker coefficients

In this section we give a formula that allows us to recover the Kronecker coefficients from the reduced Kronecker coefficients, and we apply it for the Kronecker coefficients indexed by two two–row shapes.

For any infinite sequence $u = (u_1, u_2, \ldots)$ and any positive integer i we denote with $u^{\dagger i}$ the sequence obtained from u by incrementing by 1 its $i-1$ first terms and removing its i–th term, that is: $u^{\dagger i} = (u_1{+}1, u_2{+}1, \ldots, u_{i-1}{+}1, u_{i+1}, u_{i+2} \ldots)$. Partitions are identified with infinite sequences by appending trailing zeros. Under this identification, if λ is a partition then so is $\lambda^{\dagger i}$ for all i.

Theorem 3 *Let ℓ_1, ℓ_2 and n be positive integers. Let λ, μ, ν be partitions of n such that μ has length at most ℓ_1 and ν has length at most ℓ_2. Then:*

$$g_{\mu\nu}^{\lambda} = \sum_{i=1}^{\ell_1\ell_2} (-1)^{i+1} \overline{g}_{\overline{\mu},\overline{\nu}}^{\lambda^{\dagger i}} \tag{2}$$

For $\ell_1 = \ell_2 = 2$, Formula (2) applies as follows:

$$g_{(n-r,r)(n-s,s)}^{(\lambda_1,\lambda_2,\lambda_3)} = \overline{g}_{(r)(s)}^{(\lambda_2,\lambda_3)} - \overline{g}_{(r)(s)}^{(\lambda_1+1,\lambda_3)} + \overline{g}_{(r)(s)}^{(\lambda_1+1,\lambda_2+1)} \tag{3}$$

where $n = |\lambda|$, because the last expected summand $\overline{g}_{(r)(s)}^{(\lambda_1+1,\lambda_2+1,\lambda_3+1)}$ is always zero.

The reduced Kronecker coefficients that appear in this formula are all of the form $\overline{g}_{(r)(s)}^{(\gamma_1,\gamma_2)}$. These coefficients admit the following description, derived in Briand et al. (2008) from the description for the Kronecker coefficients indexed by two two–row shapes provided by Rosas (2001). An equivalent description for the reduced Kronecker coefficients indexed by two one–row shapes is given by Thibon (1991).

Theorem 4 (Briand et al. (2008)) *Let r, s and $\gamma_1 \geq \gamma_2$ be nonnegative integers and $h = (r, s, \gamma_1, \gamma_2)$. The reduced Kronecker coefficient $\overline{g}_{(r)(s)}^{(\gamma_1,\gamma_2)}$ counts the integral solutions to the system of inequalities $u_i(X, Y) + c_i(h) \geq 0$ for $i = 0, \ldots, 6$, where:*

$$\begin{array}{lcl} u_0(v) + c_0(h) &=& X - s \\ u_1(v) + c_1(h) &=& X - r \\ u_2(v) + c_2(h) &=& X + Y - r - s + \gamma_1 \\ u_3(v) + c_3(h) &=& Y \end{array} \qquad \begin{array}{lcl} u_4(v) + c_4(h) &=& Y - X + |\gamma| \\ u_5(v) + c_5(h) &=& -X - Y + r + s - \gamma_2 \\ u_6(v) + c_6(h) &=& X - Y - \gamma_1 \end{array} \tag{4}$$

In particular, the function $R : (r, s, \gamma_1, \gamma_2) \in \mathbb{Z}^4 \mapsto \overline{g}_{(r)(s)}^{(\gamma_1,\gamma_2)}$ is vector partition–like.

Theorem 4 and Formula (3) provide a piecewise quasipolynomial description for $G_{2,2}$ (see Remark 2). But the corresponding domains of quasipolynomiality obtained are neither closed, nor cones. The remainder of this work is devoted to correct this and obtain, still from Theorem 4 and Formula (3) a vector partition–like presentation for $G_{2,2}$.

The main tools are the Lemma 1, below, and an explicit vector partition–like presentation for the function R (section 4) showing that the lemma applies.

Let F_0, F_1, F_2 be the linear maps from $\mathbb{R}^5$ to $\mathbb{R}^4$ that send $(n, r, s, \gamma_1, \gamma_2)$ to $(r, s, \gamma_1, \gamma_2)$, $(r, s, n - \gamma_1 - \gamma_2, \gamma_2)$, $(r, s, n - \gamma_1 - \gamma_2, \gamma_1)$ respectively. Let T_1 and T_2 be the translations in $\mathbb{R}^4$ of vector $v_1 = (0, 0, 1, 0)$ and $v_2 = (0, 0, 1, 1)$ respectively.

Let Δ (resp. Δ') be the cone of $\mathbb{R}^5$ (resp. of $\mathbb{R}^4$) generated by all $(n, r, s, \gamma_1, \gamma_2) \in \mathbb{Z}^5$ (resp. all $(r, s, \gamma_1, \gamma_2) \in \mathbb{Z}^4$) such that the Kronecker coefficient $g_{(n-r,r)(n-s,s)}^{(n-\gamma_1-\gamma_2,\gamma_1,\gamma_2)}$ (resp. the reduced Kronecker coefficient $\overline{g}_{(r)(s)}^{(\gamma_1,\gamma_2)}$) is defined and positive. The explicit description of Δ is provided by Bravyi (2004) (see also the general approach by Klyachko (2004)). The cone Δ' is the image of Δ under F_0.

For $x \in \mathbb{Z}^5$ set $\chi_\Delta(x) = 1$ if $x \in \Delta$ and $\chi_\Delta(x) = 0$ otherwise. Then we can rewrite Formula 3 as follows:

$$G(x) = R \circ F_0(x) - \chi_\Delta(x) \cdot R \circ T_1 \circ F_1(x) + \chi_\Delta(x) \cdot R \circ T_2 \circ F_2(x)$$

where $G(x) = G(n, r, s, \gamma_1, \gamma_2) = g_{(n-r,r)(n-s,s)}^{(n-\gamma_1-\gamma_2,\gamma_1,\gamma_2)}$ when $(n - r, r)$, $(n - s, s)$, $(n - \gamma_1 - \gamma_2, \gamma_1, \gamma_2)$ are partitions, and $G(n, r, s, \gamma_1, \gamma_2) = 0$ otherwise.

After Remark 1, Theorem 1 will be proved if we show that all three vector partition–like functions $R \circ F_0$, $\chi_\Delta \cdot R \circ T_1 \circ F_1$ and $\chi_\Delta \cdot R \circ T_2 \circ F_2$ admit presentations with the same cone: $(\Delta, \mathfrak{F}_0, Q_0)$, $(\Delta, \mathfrak{F}_1, Q_1)$ and $(\Delta, \mathfrak{F}_2, Q_2)$.

That $R \circ F_0$ admits a presentation $(\Delta, \mathfrak{F}_0, Q_0)$ is immediate because $F_0^{-1}(\Delta') = \Delta$. To show that $\chi_\Delta \cdot R \circ T_1 \circ F_1$ and $\chi_\Delta \cdot R \circ T_2 \circ F_2$ also admit presentations with cone Δ we will need to apply two times Lemma 1 below, with $p = 5$, $q = 4$, $C = \Delta$, $C' = \Delta'$, $\phi = R$ and $F = F_i$, $v = v_i$ for $i = 1, 2$.

Given subsets A, B of $\mathbb{R}^q$ we denote with $A + B$ the set $\{a + b \mid a \in A, b \in B\}$. Given $v \in \mathbb{R}^q$ and I subset of $\mathbb{R}$ we denote with $I\,v$ the set $\{xv \mid x \in I\}$.

Lemma 1 *Let ϕ be a vector partition–like function on $\mathbb{Z}^q$ with presentation $(C', \mathfrak{F}', Q)$. Let C be a convex rational polyhedral cone of $\mathbb{R}^p$ and F an integral linear map from $\mathbb{R}^p$ onto $\mathbb{R}^q$. Let $v \in \mathbb{Z}^q$ and T be the translation of $\mathbb{R}^q$ of vector v. Let $\mathfrak{F}$ be the fan subdividing $C \cap F^{-1}(C')$, whose cells are all sets of the form $C \cap F^{-1}(\sigma')$ for σ' cell of $\mathfrak{F}'$.*

Assume that the cone $C \cap F^{-1}(C')$ is full–dimensional in $\mathbb{R}^p$. Assume also that:

(a) *Whenever H is a hyperplane separating two adjacent maximal cells σ'_1, σ'_2 of $\mathfrak{F}'$ such that $F(C)$ is not included in $H + \mathbb{R}_+ v$, the following holds: The quasipolynomials $Q_{\sigma'_1}$ and $Q_{\sigma'_2}$ coincide on the integral points of the strip $H +]0; 1]\, v$.*

(b) *Whenever H is a hyperplane containing a facet of C', such that $\mathbb{R}_+\, v\; +\; F(C)$ is not contained in the half–plane $H + C'$, the following holds: For all maximal cells σ' of $\mathfrak{F}'$ having a facet contained in H, the quasipolynomial $Q_{\sigma'}$ vanishes on the integral points of the strip $H +]0; 1]\, v$.*

Then

(i) *The function $\phi \circ T \circ F$ is zero on the integral points of the closure of $C \setminus F^{-1}(C')$.*

(ii) *If $C \cap F^{-1}(\sigma')$ is a maximal cell of $\mathfrak{F}$ (where σ' is a maximal cell of $\mathfrak{F}'$) then $\phi \circ T \circ F$ and $Q_\sigma \circ T \circ F$ coincide on its integral points.*

Applying the lemma as indicated requires a precise description of a presentation $(\Delta', \mathfrak{F}_R, Q_R)$ of R. The next section provides such a description.

4 Formulas for the reduced Kronecker coefficients indexed by two one–row shapes

Let u_i and c_i, for $i = 0, 1, \ldots, 6$ be the integral linear forms defined in (4). After Brion and Vergne (1997), the function ψ that associates to $y \in \mathbb{Z}^7$ the number of integral solutions of the system $u_i(X, Y) + y_i \geq 0$, $i = 0, \ldots, 6$ is a vector partition function. In particular, it admits a very well–described vector partition–like presentation $(C_\psi, \mathcal{F}_\psi, Q_\psi)$. The corresponding fan is the *chamber complex* of ψ, see Brion and Vergne (1997); Sturmfels (1995).

Remember (Theorem 4) that R is the function that associates the reduced Kronecker coefficient $\overline{g}_{(r)(s)}^{(\gamma_1,\gamma_2)}$ to $(r, s, \gamma_1, \gamma_2) \in \mathbb{Z}^4$. Then $R = \psi \circ c$, where c is the linear map from $\mathbb{R}^4$ to $\mathbb{R}^7$ that maps $h = (r, s, \gamma_1, \gamma_2)$ to $(c_0(h), c_1(h), \ldots, c_6(h))$. Therefore, one obtains a very explicit vector partition–like presentation $(c^{-1}(C_\psi), \mathfrak{F}_R, Q_R)$ for R by taking for $\mathfrak{F}_R$ the inverse image of $\mathfrak{F}_\psi$ under c, and for Q_R the family of functions $Q_{R,c^{-1}(\sigma)} = Q_{\psi,\sigma} \circ c$ for σ maximal cell of $\mathfrak{F}_\psi$. We present this description.

Let $h \in \mathbb{R}^7$. Denote with $\Pi(h)$ the set of real solutions of the system (4). For $i = 0, 1, \ldots, 6$, let $L_i(h)$ be the line with equation $a_i X + b_i Y + c_i(h) = 0$ where $u_i(X, Y) = a_i X + b_i Y$.

For any three elements i, j, k of $\{0, 1, \ldots, 6\}$ define:

$$f_{ijk}(h) = -\begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i(h) & c_j(h) & c_k(h) \end{vmatrix} \tag{5}$$

Define also $f_{25} = \gamma_1 - \gamma_2$ and $f_{46} = \gamma_2$. The linear form f_{25} (resp. f_{46}) is proportional to f_{25k} for all $k \neq 2, 5$ (resp.: to f_{46k} for all $k \neq 4, 6$) and its vanishing is the condition for the two parallel lines L_2 and L_5 (resp. L_4 and L_6) to coincide.

- **The cone** $c^{-1}(C_\psi)$ is equal to the cone Δ' introduced in Section 3. It is defined by the system of linear inequalities:

$$f_{145} \leq 0, \quad f_{045} \leq 0, \quad f_{356} \leq 0, \quad f_{035} \leq 0, \quad f_{135} \leq 0, \quad f_{25} \geq 0, \quad f_{46} \geq 0.$$

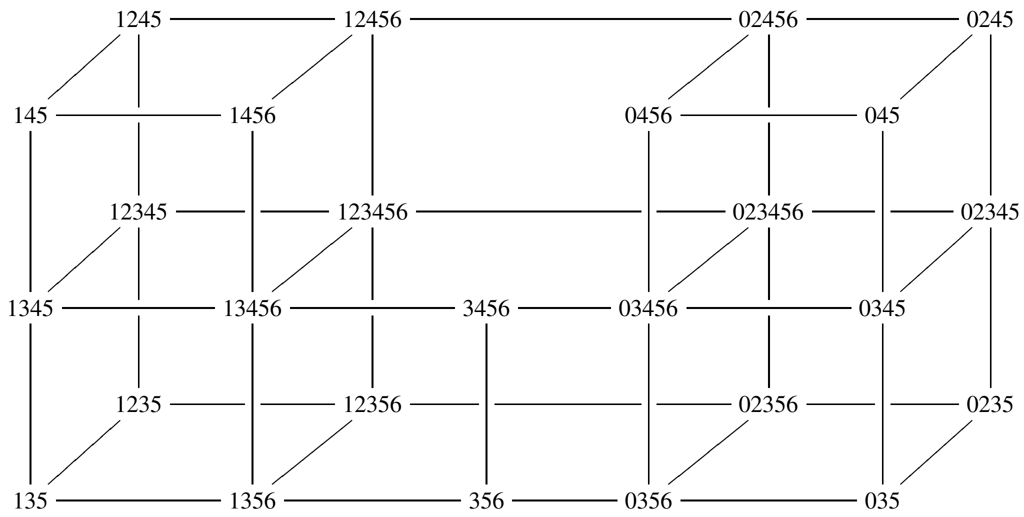

Fig. 1: The graph $\mathcal{G}$.

- **The fan $\mathfrak{F}_R$:** Let S be the locus of parameters h such that three lines $L_i(h)$, $L_j(h)$, $L_k(h)$ meet in $\Pi(h)$. The fan $\mathfrak{F}_R$ is the fan whose chambers (maximal open cells) are the connected components of $\Delta' \setminus S$. In each chamber σ the set of indices i such that $L_i(h)$ supports a side of $\Pi(h)$ is constant. Denote this set with $\mathsf{Sides}(\sigma)$. This set $\mathsf{Sides}(\sigma)$ determines σ. Therefore we denote a chamber σ with σ_I when $\mathsf{Sides}(\sigma) = I$, e.g. σ_{1245} for the chamber σ such that $\mathsf{Sides}(\sigma) = \{1, 2, 4, 5\}$. There are 26 chambers σ_I in $\mathfrak{F}_R$. The corresponding indices $I = \mathsf{Sides}(\sigma_I)$ are the vertices of the graph $\mathcal{G}$ in Figure 1. Adjacency in $\mathcal{G}$ represents adjacency in $\mathfrak{F}_R$: chambers σ_I and σ_J are adjacent (i.e. their closures have a common facet) if and only if I and J are adjacent vertices in $\mathcal{G}$. Observe that when σ_I and σ_J are adjacent then:
  - either I and J are obtained from each other by exchanging 0 and 1. Then σ_I and σ_J are separated by the hyperplane of equation $r = s$. There is $r > s$ on σ_I if $1 \in I$.
  - or one of the sets is obtained from the other by inserting a unique element. Say $J = I \cup \{j\}$ with $j \notin I$. If the elements of J are $p_1 < p_2 < \cdots < p_t$ say that the successor of p_q is p_{q+1}, for $q = 1, \ldots, t-1$, and that the successor of p_t is p_1. This defines a cyclic order on J. Let i and k be the predecessor and successor of j in this cyclic order. Then σ_I and σ_J are separated by the hyperplane of equation $f_{ijk} = 0$, and $f_{ijk} > 0$ on σ_I.
- **The quasipolynomial formulas on each maximal cell:** For simplicity we set $q_I = Q_{R,\overline{\sigma_I}}$. This is the quasipolynomial formula for R valid on the cell $\overline{\sigma_I}$ (the topological closure of the chamber σ_I). Rather than displaying explicit expressions for all quasi–polynomials q_I, it is enough to present one of them (we choose q_{135}) and display all differences $q_I - q_J$ for σ_I and σ_J adjacent. All quasi–polynomials q_I can be recovered easily from this information by chasing on the graph $\mathcal{G}$ (Figure 1),

ijk	$q_I(h) - q_J(h)$	**Values** δ **s. t.** $q_I = q_J$ **on** $f_{ijk} = \delta$
$613, 123, 134$ $603, 023, 034$	$\frac{1}{2} f_{ijk}(h)\,(f_{ijk}(h) - 1)$	$0, 1$
234	$\frac{1}{4}\,(f_{ijk}(h))^2 + \begin{cases} 0 & \text{if } f_{ijk}(h) \equiv 0 \mod 2 \\ -1/4 & \text{else.} \end{cases}$	$-1, 0, 1$
$345, 124, 561$ $024, 560$	$\frac{1}{4} f_{ijk}(h)\,(f_{ijk}(h) - 2) + \begin{cases} 0 & \text{if } f_{ijk}(h) \equiv 0 \mod 2 \\ 1/4 & \text{else} \end{cases}$	$0, 1, 2$

Tab. 1: The differences $q_I - q_J$ for σ_I and σ_J adjacent chambers of $\mathfrak{F}$.

e.g.

$$q_{1456} = (q_{1456} - q_{13456}) + (q_{13456} - q_{1356}) + (q_{1356} - q_{135}) + q_{135}$$

There is:

$$q_{135}(r, s, \gamma_1, \gamma_2) = \frac{1}{2}\,(s - \gamma_2 + 1)\,(s - \gamma_2 + 2)$$

Let σ_I and σ_J be two adjacent chambers of $\mathfrak{F}$.

- If I and J are obtained from each other by exchanging 0 and 1 then $q_I = q_J$.
- If $J = I \cup \{j\}$ with $j \notin I$ then $q_I - q_J$ depend only of j and its predecessor i and successor k in J, and is as indicated in Table 1.

If σ_I and σ_J are adjacent, the quasi–polynomials q_I and q_J coincide not only on the affine hyperplane spanned by the facet $\overline{\sigma_I} \cap \overline{\sigma_J}$ but also on close parallel hyperplanes.

Proposition 1 *Let σ_I and σ_J be two adjacent chambers of $\mathfrak{F}$ such that $J = I \cup \{j\}$ with $j \notin I$. Let i and k be the predecessor and successor, respectively, of j in J.*

Then $q_I - q_J$ coincide on the affine hyperplanes $f_{ijk} = \delta$ for the values of δ given by the third column in Table 1.

Similarly, if the hyperplane H supports a facet of a maximal cell $\overline{\sigma_I}$, and this facet is contained in the border of Δ', then q_I vanishes on affine hyperplanes close and parallel to H.

Proposition 2 *Let σ_I be a chamber of $\mathfrak{F}$ and τ an external facet of $\overline{\sigma_I}$ (i.e. a facet contained in the border of Δ'). The hyperplane supporting τ admits as equation $f = 0$ where f is one of the linear forms f_{145}, f_{045}, f_{356}, f_{035}, f_{135}, f_{25}, f_{46}.*

The set of values $\delta \in \mathbb{Z}$ such that f vanishes identically on the affine hyperplane of equation $f = \delta$ is provided by Table 2.

It is immediate that $R \circ F_0$ has a vector partition–like presentations $(\Delta, \mathfrak{F}_0, Q_0)$. Propositions 1 and 2 are used to apply Lemma 1 and show that $\chi_\Delta \cdot R \circ T_1 \circ F_1$ and $\chi_\Delta \cdot R \circ T_2 \circ F_2$ have vector partition–like presentations $(\Delta, \mathfrak{F}_1, Q_1)$ and $(\Delta, \mathfrak{F}_2, Q_2)$. After Remark 1, this proves Theorem 1 and provides a way to compute a vector partition–like presentation for G and $G_{2,2}$.

Form f	**Chambers having a facet supported by $f = 0$**	**Values δ such that q_I vanishes identically on $f = \delta$**
$f_{46} = \gamma_2$	$3456, 1456, 0456$	-1
$f_{25} = \gamma_1 - \gamma_2$	$1245, 0245, 1235, 0235$	-1
$f_{145} = r - s - \gamma_1$	145	$1, 2, 3$
$f_{045} = s - r - \gamma_1$	045	$1, 2, 3$
$f_{356} = \lvert\gamma\rvert - r - s$	356	$1, 2, 3$
$f_{035} = \gamma_2 - r$	035	$1, 2$
$f_{135} = \gamma_2 - s$	135	$1, 2$

Tab. 2: The linear forms defining the facets of Δ'.

5 Formulas for the Kronecker coefficients indexed by two two–row shapes

Once the presentations $(\Delta, \mathfrak{F}_0, Q_0)$, $(\Delta, \mathfrak{F}_1, Q_1)$, $(\Delta, \mathfrak{F}_2, Q_2)$ for $R \circ F_0$, $\chi_\Delta \cdot R \circ T_1 \circ F_1$ and $\chi_\Delta \cdot R \circ T_2 \circ F_2$ have been determined, an explicit presentation $(\Delta, \mathfrak{F}_3, Q_3)$ for G is obtained: The cells of $\mathfrak{F}_3$ are the intersection $\sigma_0 \cap \sigma_1 \cap \sigma_2$ for σ_i a cell of $\mathfrak{F}_i$, $i \in \{0, 1, 2\}$. If $\sigma_0 \cap \sigma_1 \cap \sigma_2$ is a maximal cell of $\mathfrak{F}_3$ then the corresponding quasipolynomial formula for G is $Q_{0,\sigma_0} - Q_{1,\sigma_1} + Q_{2,\sigma_2}$. We computed the description for $\mathfrak{F}_3$ by using the Maple Package CONVEX by Franz (2006): it has 177 maximal cells. It turns out that on some of them G is given by the same quasipolynomial formulas, and that they can be glued together to form the maximal cells of a new fan $\mathfrak{F}_K$. In the new presentation $(\Delta, \mathfrak{F}_K, P)$ obtained for G the fan $\mathfrak{F}_K$ has only 74 maximal cells.

All 74 quasipolynomial formulas P_σ have the following form:

$$P_\sigma = 1/4\; Q_\sigma + 1/2\; L_\sigma + M_\sigma/4 \tag{6}$$

where Q_σ and L_σ are integral homogeneous polynomials in $(n, r, s, \gamma_1, \gamma_2)$ respectively quadratic and linear. The function M_σ takes integral values, fulfills $M_\sigma(0)/4 = 1$ and is constant on each coset of $\mathbb{Z}^5$ modulo the sublattice defined by $r + s \equiv n \equiv \gamma_1 \equiv \gamma_2 \equiv 0 \mod 2$.

Moreover, for all maximal cells σ, the functions Q_σ, L_σ are nonnegative on σ. This also holds for M_σ, for all cells σ except four. This makes specially easy studying the *support* of the Kronecker coefficients indexed by two two–row shapes. This is the set of all triples (λ, μ, ν) such that $g^\lambda_{\mu,\nu} > 0$ and μ and ν have at most two parts.

We obtain the following result. Let $(n, r, s, \gamma_1, \gamma_2) \in \Delta$. Then $g^{(n-\gamma_1-\gamma_2,\gamma_1,\gamma_2)}_{(n-r,r)(n-s,s)}$ is zero if and only if at least one of the following five systems of conditions is fulfilled:

$$\left\{\begin{array}{l} n = 2\,s = 2\,r \\ \gamma_1 \text{ or } \gamma_2 \text{ odd.} \end{array}\right. \qquad \left\{\begin{array}{l} n = \max(2\,r, 2\,s, |\gamma| + \gamma_1) \\ \gamma_2 = 0 \\ r + s + \gamma_1 \text{ odd.} \end{array}\right. \qquad \left\{\begin{array}{l} n = max(2\,r, 2\,s) \\ |r - s| = 1 \\ \min(2\,r, 2\,s) \geq |\gamma| + \gamma_1 \\ \gamma_1 \text{ or } \gamma_2 \text{ even.} \end{array}\right.$$

$$\left\{\begin{array}{l} n = \max(2\,r, 2\,s) \\ \gamma_1 = \gamma_2 \\ r + s + \gamma_1 \text{ odd.} \end{array}\right. \qquad \left\{\begin{array}{l} n = |\gamma| + \gamma_1 = \max(2\,r, 2\,s) \\ r + s + \gamma_1 \text{ odd.} \end{array}\right. \tag{7}$$

This exhaustive description led us to a family of counterexamples for SH, a saturation conjecture formulated by Mulmuley (2007). The *stretching functions* $\widetilde{g}^{\lambda}_{\mu,\nu} : N \mapsto g^{N\lambda}_{N\mu,N\nu}$ attached to the Kronecker coefficients are quasipolynomials (Mulmuley, 2007). This means that for any fixed λ, μ, ν there exist an integer k and polynomials $p_1, p_2, \ldots, p_k$ such that for any $N \geq 1$, $\widetilde{g}^{\lambda}_{\mu,\nu}(N) = p_i(N)$ when $N \equiv i \mod k$. Mulmuley's SH conjecture stated that for any such description, $g^{\lambda}_{\mu,\nu} = 0 \Leftrightarrow F_1 = 0$. The rightmost system of conditions in (7) above provides a family of counterexamples to this conjecture (Briand et al., 2008). The discovery of these counterexamples led Mulmuley (2008) to propose a weaker form of the conjecture SH, still strong enough for the aims of Geometric Complexity Theory.

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