The Discrete Fundamental Group of the Associahedron

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Abstract. The associahedron is an object that has been well studied and has numerous applications, particularly in the theory of operads, the study of non-crossing partitions, lattice theory and more recently in the study of cluster algebras. We approach the associahedron from the point of view of discrete homotopy theory, that is we consider 5-cycles in the 1-skeleton of the associahedron to be combinatorial holes, but 4-cycles to be contractible. We give a simple description of the equivalence classes of 5-cycles in the 1-skeleton and then identify a set of 5-cycles from which we may produce all other cycles. This set of 5-cycle equivalence classes turns out to be the generating set for the abelianization of the discrete fundamental group of the associahedron. In this paper we provide presentations for the discrete fundamental group and the abelianization of the discrete fundamental group. We also discuss applications to cluster algebras as well as generalizations to type B and D associahedra.

Résumé. L'associahèdre est un objet bien etudié que l'on retrouve dans plusieurs contextes. Par exemple, il est associé à la théorie des opérades, à l'étude des partitions non-croisées, à la théorie des treillis et plus récemment aux algèbres dámas. Nous étudions cet objet par le biais de la théorie des homotopies discretes. En bref cette théorie signifie qu'un cycle de longueur 5 (sur le squelette de l'associahèdre) est considéré comme étant le bord d'un trou combinatoire, alors qu'un cycle de longueur 4 peut être contracté sans problème. Les classes d'homotopies discrètes sont donc des classes d'équivalence de cycles de longueurs 5. Nous donnons une description simple de ces classes d'équivalence et identifions un ensemble de générateurs du groupe correspondant (abélien) d'homotopies discrètes. Nous d'ecrivons également les liens entre notre construction et les algèbres d'amas.

Keywords: associahedron, discrete fundamental group, conic arrangements

1 Introduction

Let $\mathcal{T}_n$ be the abstract simplicial complex on the set of all diagonals of a regular $(n+3)$-gon whose maximal simplices, T_i, correspond to triangulations of the regular $(n+3)$-gon. It is well known that if we (partially) order the simplices of $\mathcal{T}_n$ by reverse inclusion then we have a poset that is isomorphic to the face poset of the associahedron [14]. There is a wealth of recent literature focusing on the associahedron and its generalizations, [4, 6, 5, 9, 16, 12, 15]. Simion, in [14], gives an excellent description of the origins and early study of the associahedron. It is our intention to study the associahedron through the lens of the discrete homotopy theory, or A-theory, of Barcelo, Kramer, Laubenbacher and Weaver [1, 2]. This approach highlights some interesting combinatorial properties of the associahedron and provides a framework to study several of the generalizations of the associahedron in the same manner.

Our approach is not completely novel; we are motivated by the study of the discrete fundamental group of the permutahedron done by Barcelo and Smith. It had been shown previously by Babson [2] and independently by Björner that that the discrete fundamental group of the permutahedron is isomorphic to the classical homotopy group of the real complement of the k-equal arrangement, $M_{n,k}$. In [3], the authors provide a combinatorial method for calculating the abelianization of the discrete fundamental group of the permutahedron, which in turn gives a purely combinatorial method of calculating the Betti number of $M_{n,k}$. In our case, due to the structure of the associahedron we do not have a resulting subspace arrangement but we will give a link to what we call a *conic arrangement* as well as a connection to cluster algebras.

We present a short overview of the facts about A-theory needed here but for a more thorough understanding we refer the reader to [1, 2].

Recall that a triangulation of an $(n+3)$-gon contains n non crossing diagonals. Given two maximal simplices (triangulations) T_1, T_2 in $\mathcal{T}_n$, we say they are *near* if $|T_1 \cap T_2| = n-1$. We may also restate this as: T_1 and T_2 are near if they differ by a diagonal flip. A sequence of maximal simplices, $T_1 - T_2 - \cdots - T_k$ is called a *chain* if T_i, T_{i+1} are near for all $0 \leq i \leq k$ and a chain that starts and ends with the same simplex is called a *loop*. Note that in the general discrete homotopy theory, one can vary the definition of near by adjusting a parameter q. This parameter q is fixed in our case to be $n-2$.

There is an equivalence relation, $\simeq_A$, that may be placed on the set of all loops based at T_0. A full description of this relation can be found in [1, 2].

Let $A_1^{n-2}(\mathcal{T}_n, T_0)$ be the set of equivalence classes of loops based at T_0 (the superscript $n-2$ is the parameter q mentioned previously). By Proposition 2.3 in [1], a group structure can be imposed on $A_1^{n-2}(\mathcal{T}_n, T_0)$ by adding the operation of concatenation of loops. We call $A_1^{n-2}(\mathcal{T}_n, T_0)$ with this group structure, the *discrete fundamental group* of $\mathcal{T}_n$. A grid between loops in $A_1^{n-2}(\mathcal{T}_n, T_0)$ can be thought of as analogous to a continuous deformation of one curve to another in classical homotopy theory. The structure of $A_1^{n-2}(\mathcal{T}_n, T_0)$ gives us information about $\mathcal{T}_n$ in the same way the classical homotopy group gives us information about a topological space.

Given the complex $\mathcal{T}_n$, we may also define a graph, $\Gamma^{n-2}(\mathcal{T}_n)$, where the vertex set of $\Gamma^{n-2}(\mathcal{T}_n)$ is in bijection with the set of maximal simplices of $\mathcal{T}_n$ and we put an edge between T_i and T_j if they are near. It is shown in [1] that closed walks based at T_0 in $\Gamma^{n-2}(\mathcal{T}_n)$ are in bijection with loops using elements from $\mathcal{T}_n$, and in fact two closed based walks in $\Gamma^{n-2}(\mathcal{T}_n)$ are homotopic if they differ by 3- and 4-cycles only. Thus we may think of $A_1^{n-2}(\mathcal{T}_n, T_0)$ as being the group of equivalence classes of closed based walks in $\Gamma^{n-2}(\mathcal{T}_n)$ with the obvious operation of concatenation and the identity and inverses just as in the previous description of $A_1^{n-2}(\mathcal{T}_n, T_0)$ in terms of loops. It is well known [14] that the graph $\Gamma^{n-2}(\mathcal{T}_n)$ is the 1-skeleton of the associahedron, hereafter referred to as Asc_n to reinforce the connection between $\mathcal{T}_n$ and the associahedron in the mind of the reader. Hence when we discuss $A_1^{n-2}(\mathcal{T}_n, T_0)$, we may think of elements in terms of walks in the 1-skeleton of the associahedron.

By Proposition 5.12 in [1], we know that $A_1^{n-2}(\mathcal{T}_n, T_0) \simeq \pi_1(X_\Gamma)$, where X_Γ is the topological space obtained by attaching a 2-cell to every 3- and 4-cycle of $\Gamma^{n-2}(\mathcal{T}_n)$. We refer to cycles in Asc_n that bound a 2-face of the associahedron as *basic cycles*. If we continue our analogy between discrete and classical homotopy theory, we can see now that a hole in $\mathcal{T}_n$ corresponds to a basic cycle in Asc_n of length ≥ 5. However, because X_Γ is not a graph, it is not guaranteed that $A_1^{n-2}(\mathcal{T}_n, T_0)$ is free and we show that there are in fact commutivity relations between the generators of $A_1^{n-2}(\mathcal{T}_n, T_0)$.

When we move on to the abelianization of $A_1^{n-2}(\mathcal{T}_n, T_0)$, we are considering the equivalence classes of holes, corresponding to 5-cycles in Asc_n, but we are able to show that although there are $\binom{n+3}{5}$ equivalence classes, we may recover all of the equivalence classes of 5-cycles using only a set of $\binom{n+2}{4}$ equivalence classes. This leads to the main result of Section 4.

Theorem 1.1 *The abelianization of $A_1^{n-2}(\mathcal{T}_n, T_0)$ is a free abelian group of rank $\binom{n+2}{4}$.*

Although the classical fundamental group of a convex polytope is always trivial, the discrete fundamental group is not, and seems to provide an indication of the complexity of the polytope as compared to the n-simplex. In [10], the authors provide an excellent view of the associahedron as a a truncation of the n-simplex and the permutahedron as a truncation of the associahedron. At each step in the truncation process, the number of generators of the abelianization of the discrete fundamental group increases, going from trivial in the case of the n-simplex to $\binom{n+2}{4}$ for the associahedron and $2^{n-3}(n^2 - 5n + 8) - 1$ for the permutahedron.

In Section 2 we establish a labeling scheme for edges of Asc_n and a set of words whose letters are the labels of edges in Asc_n. It is shown in [1] that loops based at T_0 are in one-to-one correspondence with closed walks in $\Gamma^{n-2}(\mathcal{T}_n)$ based at T_0 (we abuse notation here and use T_0 to refer to both a maximal simplex of $\mathcal{T}_n$ and a vertex of $\Gamma^{n-2}(\mathcal{T}_n)$). Thus we may think of $A_1^{n-2}(\mathcal{T}_n, T_0)$ both as the group on equivalence classes of loops and as a group on equivalence classes of based walks in Asc_n. This will allow us to work entirely with closed walks in Asc_n and words constructed from those walks.

In Section 3 we give a generating set for the group $A_1^{n-2}(\mathcal{T}_n, T_0)$ using a description of the classical fundamental group in terms of the cycles that bound 2-faces in the associahedron. This approach is informative in that it gives us a combinatorial description of the generating set for $A_1^{n-2}(\mathcal{T}_n, T_0)$.

Section 4 contains the proof of Theorem 1.1 as well as a series of lemmas needed to prove the result. We have omitted many of the proofs of these lemmas due to space considerations, however many of them follow easily from the definitions and ideas in Section 2. In proving Theorem 1.1 we give a simple combinatorial description of the generators.

Finally, we conclude in Section 5 with a description of applications and two directions for future study.

Due to space considerations we have omitted some details of proofs and background, however all of the material here appears in full detail in the first author's PhD thesis ([13]).

2 Properties of Asc_n

As noted in the introduction, the associahedron has been very well studied. For an overview of the basic properties and facts of this object we refer the reader to the list of references presented in the introduction. In this section we establish a labeling scheme for the edges of Asc_n. Using this new labeling scheme as an alphabet, we are able to translate a loop of simplices, or a walk in Asc_n, to a word. The use of words makes our proofs in the following sections more clear and gives an algebraic framework for our discussions of discrete homotopy theory.

We also look more closely at the equivalence relation $\simeq_A$, giving a shorter description as in [3]. Due to the lack of triangles in Asc_n, we may write any discrete homotopy between loops as a series of three fundamental changes to the loop; stretching at a simplex, inserting a new simplex, and commuting two simplices.

We begin with the edge labels for Asc_n. Fix a regular $(n+3)$-gon and label the vertices clockwise in order with $1, \dots, n+3$. Recall that an edge in Asc_n corresponds to changing one diagonal between two triangulations of an $(n+3)$-gon, or a diagonal flip. We may use this flip to label the edge in a distinct way.

Definition 2.1 *Let e be an edge in Asc_n which corresponds to changing the diagonal ac to the diagonal bd. Define the label set of e, $L(e)$ to be the set $\{a, b, c, d\}$, where a, b, c, d are elements of $\{1, \ldots, n+3\}$ corresponding to the vertices of the $(n+3)$-gon.*

Note that while the every edge has exactly one label set, many edges may share the same label set.

We also may derive the label for an edge by considering its corresponding simplex in $\mathcal{T}_n$. Edges in Asc_n correspond to a simplex S with $n-1$ diagonals, so we may take the $(n+3)$-gon and add all of the diagonals in S. When we have added all of the diagonals in S we have one region inside the $(n+3)$-gon which has not been triangulated. This region is a quadrilateral and the vertices that bound it are exactly the label set of the edge corresponding to S. This method of determining the label set is very easy to understand with an illustration, so we have provided the graph Asc_2 with the triangulated 5-gon corresponding to each vertex, and each edge labeled with our scheme in Figure 1. Observe that each diagonal flip occurs inside a fixed quadrilateral and we may read off the edge label from the vertices of that quadrilateral.

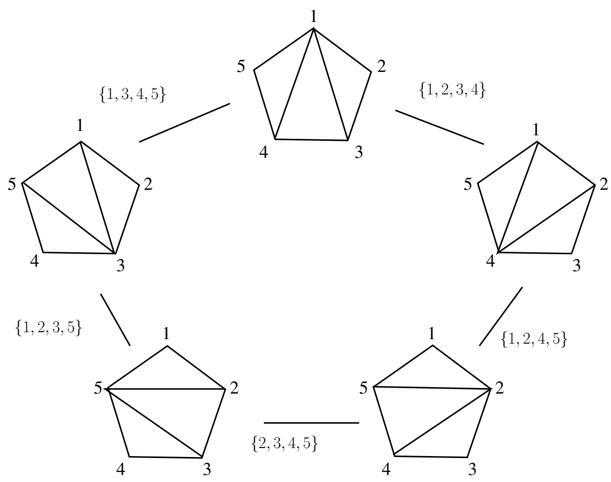

Fig. 1: The graph Asc_2 with vertices as triangulations of a regular 5-gon and edges labeled.

Just as we have label sets for the edges of Asc_n, we also introduce the notion of basic cycle label sets. The basic cycle label is a natural extension of the edge label obtained in a very similar manner.

Definition 2.2 *Let C be a basic cycle in Asc_n and let e and f be two edges on C, with $L(e) \neq L(f)$. Define the label set of C, $L(C)$ to be the set $L(e) \cup L(f)$.*

We also have an intuitive way to see the label set of a basic cycle given its corresponding partial triangulation of an $(n+3)$-gon. As in the case of edge label sets, we consider the simplex S that corresponds to a basic cycle. This simplex has $(n-2)$ diagonals and so when we add these diagonals to the $(n+3)$-gon we have a partial triangulation. Each missing diagonal gives us a quadrilateral region inside the $(n+3)$-gon. The boundary vertices of these two regions give us the label set of the cycle. An illustration of the regions corresponding to a basic 4-cycle and a basic 5-cycle can be seen in Figure 2.

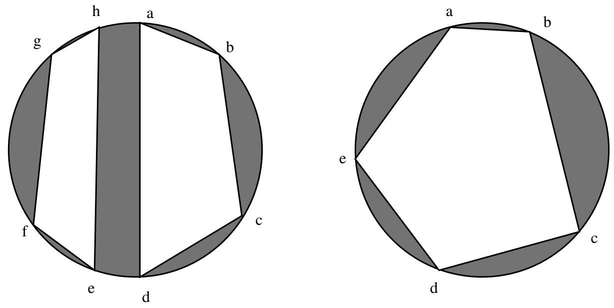

Fig. 2: Regions inside a regular $(n+3)$-gon corresponding to a basic 4-cycle and basic 5-cycle respectively. Shading indicates a region is triangulated.

In the case that the two regions have interiors that overlap it must be the case that they share three boundary vertices and hence their intersection is a pentagon. As seen in Figure 1, there are five ways to triangulate a pentagon and so we obtain a corresponding basic 5-cycle in Asc_n. If the regions do not have intersecting interiors then it is easy to see we may triangulate them each in two different ways, giving us the four vertices of a basic 4-cycle.

If a basic cycle is a 4-cycle then the opposite edges on the cycle must have the same label set, since we are performing two separate flips in sequence and then performing the exact same flips in the same sequence again, in effect undoing them. If a basic cycle is a 5-cycle, then the edges of the cycle must have distinct label sets. This can be observed in Figure 1. In fact, given a basic 5-cycle we know even more about the label sets of the edges.

Proposition 2.3 *At least one of the edges of a basic 5-cycle has a label set that does not contain the element 1.*

Proof: Let C be a basic 5-cycle with label set $L(C)$. Then if $e_1, \dots, e_5$ are the edges of C, the label sets $L(e_1), \dots, L(e_5)$ are the five subsets of $L(C)$ of size 4. If $1 \in L(C)$ then one subset of $L(C)$ of size 4 must not contain 1 (If $1 \notin L(C)$ it is clear no subset of size 4 will contain 1). □

Now that we have established labels for the edges of Asc_n, we may think of elements of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ in terms of words on the alphabet of edge labels. Due to the fact that the elements of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ are closed walks based at T_0, we have a very easy way to write down the corresponding words.

Definition 2.4 *Let $W = e_1 \dots e_n$ be a closed walk in Asc_n based at vertex T_0. The word for the walk W is $w = L(e_1) \cdots L(e_n)$.*

Although we still consider elements of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ primarily as closed walks based at T_0, it will be useful to operate on the words corresponding to the walks. To do this we must establish how the relation $\simeq_A$ affects words. We make use of a result from [3] that because Asc_n is triangle free, there are only 3 possible changes we may make to walks and words that will preserve $\simeq_A$. We list the three changes (T1)-(T3) briefly and refer the reader to the reference above for further information due to space considerations.

- **(T1) Stretch.** We may stretch a loop by repeating a vertex one or more times.

$$\ell = T_0 - \cdots - T_i - \cdots - T_0 \simeq T_0 - \cdots - T_i - T_i - \cdots - T_0$$

In Asc_n we have not traversed any new edges so the walk stays the same. We may think of this operation as holding at a vertex. This operation also does not change a word since there is no new edge label added.

- **(T2) Insertion.** This change consists of inserting a new simplex in a loop. Suppose we have already stretched at T_i and suppose T_j and T_i are near. Then

$$\ell = T_0 - \cdots - T_i - T_i - \cdots - T_0 \simeq T_0 - \cdots - T_i - T_j - T_i - \cdots - T_0.$$

In Asc_n this change corresponds to traversing an edge e from T_i to a new vertex T_j, then traversing e in the opposite direction to return to the original walk. In the word corresponding to the walk we have added the letter $L(e)$ twice.

- **(T3) Switch.** Let $\ell = T_0 - \cdots - T_{i-1} - T_i - T_{i+1} - T_{i+1} - \cdots - T_0$ be a loop and let T_j be near to both T_{i-1} and T_{i+1}. Then we may switch T_j for T_i and have

$$\begin{aligned}\ell = T_0 - \cdots - T_{i-1} - T_i {-} T_{i+1} - T_{i+1} - \cdots - T_0 \\ \simeq T_0 - \cdots - T_{i-1} - T_{i-1} - T_j - T_{i+1} - \cdots - T_0.\end{aligned}$$

In Asc_n we have a 4-cycle $T_{i-1}, T_i, T_{i+1}, T_j$ with the edges e_1, e_2, e_3, e_4 respectively, and we change the walk from traversing edges e_1, e_2 to edges e_4, e_3. Recall that the opposite edges in a 4-cycle have the same label set, hence $L(e_1) = L(e_3)$ $L(e_2) = L(e_4)$. Thus in the word corresponding to the walk, we have commuted the letters $L(e_1)$ and $L(e_2)$.

Remark 2.5 *The change (T3) tells us that two letters commute if their associated edge label sets are adjacent on some 4-cycle in Asc_n. Recall that the label set of an edge e, $L(e)$, gives the boundary vertices of a quadrilateral region inside an $(n+3)$-gon. Thus, given a letter $L(e)$, we may commute it with any letter $L(f)$ as long as the region inside an $(n+3)$-gon bounded by the elements of $L(e)$ does not intersect the region bounded by the elements of $L(f)$. This implies that $|L(e) \cup L(f)| \leq 2$, but it is not a sufficient condition for $L(e)$ and $L(f)$ to commute as letters.*

Another important fact about (T2)-(T3) concerns their effect on the parity of letters in a word w.

Remark 2.6 *The changes (T2)-(T3) preserve the parity of letters in a word.*

It should also be noted that we may apply the relation $\simeq_A$ to paths in Asc_n that have the same start and end point. This is equivalent to comparing two chains of simplices that have the same start and end simplices. All of the changes (T1)-(T3) make sense in this case and the underlying idea of having a grid between the two chains works the same. We use this type of comparison of paths in Section 4.

3 A Description of $A_1^{n-2}(\mathfrak{T}_n, T_0)$

We now give a description of $A_1^{n-2}(\mathfrak{T}_n, T_0)$. Though we are primarily concerned in this abstract with the abelianization of $A_1^{n-2}(\mathfrak{T}_n, T_0)$, we feel that exposing the structure of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ is still a rewarding exercise in and of itself. We omit the majority of the proofs due to space considerations but provide a sketch of our main result as it gives us insight into the generators of $A_1^{n-2}(\mathfrak{T}_n, T_0)$..

In [7] the authors show that the classical fundamental group of the 1-skeleton of the associahedron is generated by all of the basic 4- and 5-cycles, pinned down to a base point. This result follows from two theorems in Massey ([11]). We note that the theorems in Massey allow us to choose the paths from the base point to a basic cycle and we make use of this to choose paths such that any two cycles with the same label will have corresponding loops that are homotopic.

Given a basic cycle label class, we fix a representative C of that class by fixing a partial triangulation such that all of the diagonals outside of the embedded pentaton are connected to the smallest labeled vertex in the region of the $(n+3)$-gon that they are in. We then fix a path P from the base vertex v_0 to the basic cycle C. Now, for every other cycle C' with the same label we fix a path PQ to the cycle, such that P is the path from v_0 to C and Q is a path from C to C'. Such a path exists by Lemma 4.1. Also, we may use Theorem 4.3 to conclude that any two loops that use basic cycles with the same label set are homotopic.

Theorem 3.1 *A generating set of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ is given by $\{PCP^{-1}\}$, where C ranges over all fixed representatives of label set equivalence classes such that 1 is in the label set. P is as described above. There are $\binom{n+2}{4}$ such loops.*

Proof: This result follows from the description of the classical homotopy group of Asc_n given above, the relationship between $\pi_1(Asc_n)$ and $A_1^{n-2}(\mathfrak{T}_n, T_0)$, and Theorem 4.3. Full details are provided in [13]. $\square$

We note that we have not given a nice description of the relations between the generators here. Doing so is much more complicated and loses some of the elegance of this description, however in [13] we do give a full presentation of $A_1^{n-2}(\mathfrak{T}_n, T_0)$.

4 The abelianization of $A_1^{n-2}(\mathfrak{T}_n, T_0)$

Just as in [3], in order to find $A_1^{n-2}(\mathfrak{T}_n, T_0)^{ab}$ we must count the equivalence classes of basic cycles under the relation $\simeq_A$. Recall that Asc_n has only basic 4- and 5-cycles, and that under $\simeq_A$ 4-cycles are contractible, so our goal may be reduced to counting the equivalence classes of basic 5-cycles in Asc_n. In the case of Asc_3, which is shown in Figure 3, we can see that there are six basic 5-cycles, however we know from a simple computation that we may write the outside basic 5-cycle as a product of those inside.

It does not suffice to count the classes of basic 5-cycles; we must also provide a minimal generating set. It turns out that there is a very simple combinatorial description of the equivalence classes of basic 5-cycles using the cycle labels introduced in Section 2, and that a minimal generating set for $A_1^{n-2}(\mathfrak{T}_n, T_0)^{ab}$ contains only the equivalence classes of basic 5-cycles whose label set contain 1.

We start with some results needed to prove Theorem 1.1.

Lemma 4.1 *Let $C = e_1, \dots, e_5$ and $C' = e'_1, \dots, e'_5$ be two basic 5-cycles with $L(C) = L(C')$. Then there is a series of edges $p_1, \dots, p_k$ in Asc_n between e_i and e'_i such that $L(p_j)$ (taken as letters in a word) commute with $L(e_i)$ and $L(e'_i)$ for every i and j.*

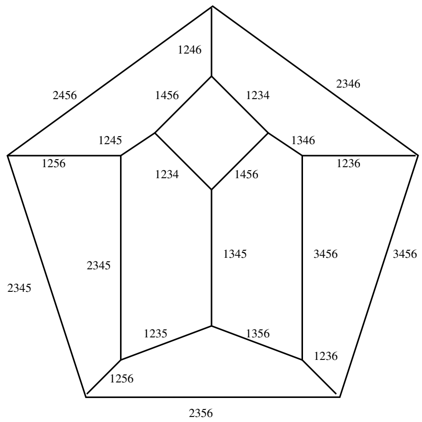

Fig. 3: Asc_3 with edge labels.

Proof: This result is obtained by flipping diagonals outside of the pentagon region given by the label set of both C and C' in the partially triangulated $(n+3)$-gon. We have omitted the full details but they are available in [13]. $\square$

Lemma 4.2 *Given two basic 5-cycles $C = e_1, \ldots, e_5$ and $C' = e'_1, \ldots, e'_5$ in Asc_n, if $L(C) \neq L(C')$ then, there is at most one pair, e_i, e'_j such that $L(e_i) = L(e'_j)$.*

Proof: This result follows very easily from Definition 2.2. Full details are available in [13]. $\square$

We are now ready to provide a necessary and sufficient condition for two basic 5-cycles to be equivalent under $\simeq_A$.

Theorem 4.3 *Let C and C' be basic 5-cycles in Asc_n. Then $L(C) = L(C)'$ if and only if $C \simeq_A C'$.*

Proof: We keep this proof in its entirety as we feel that it provides a method to visualize the homotopies between basic 5-cycles in Asc_n.

We start by showing that if $L(C) = L(C')$, then $C \simeq_A C'$. By Lemma 4.1 we know there is a sequence of edges between C and C' whose associated letters commute with the letters of C and C'. Let C have associated word $w = L(e_1)L(e_2)L(e_3)L(e_4)L(e_5)$. Using changes (T2) and (T3) we can inductively construct a new word w' which is equivalent to w and has associated 5-cycle C'.

Suppose the sequence of edges is length 1 with associated letter $L(x)$. We change w as follows:

$$\begin{aligned} L(e_1)L(e_2)L(e_3)L(e_4)L(e_5) &\simeq_A L(e_1)L(e_2)L(e_3)L(x)L(x)L(e_4)L(e_5) && (T2) && (1)\\ &\simeq_A L(e_1)L(e_2)L(x)L(e_3)L(e_4)L(x)L(e_5) && (T3) && (2)\\ &\simeq_A L(e_1)L(x)L(e_2)L(e_3)L(e_4)L(e_5)L(x) && (T3) && (3)\\ &\simeq_A L(x)L(e_1)L(e_2)L(e_3)L(e_4)L(e_5)L(x) && (T3). && (4)\end{aligned}$$

Now suppose the sequence is of length k with associated letters $L(x_1), \ldots, L(x_k)$ and assume that the hypothesis holds for a sequence of length $k-1$. Then we use the hypothesis to insert letters $L(x_1), \ldots, L(x_{k-1})$ and commute them so we have a word

$$(L(x_1) \cdots L(x_{k-1}))L(e_1)L(e_2)L(e_3)L(e_4)L(e_5)(L(x_1) \cdots L(x_{k-1}))^{-1} \simeq_A w$$

Using the same argument above, we can then insert $L(x_k)$ and obtain a new equivalent word $w' = (L(x_1) \cdots L(x_k))L(e_1)L(e_2)L(e_3)L(e_4)L(e_5)(x_1 \cdots x_k)^{-1}$. This new word corresponds to a path that goes around C' and is equivalent to the path around C.

What we are doing is forming a net of basic 4-cycles between the two basic 5-cycles with the same label set. At each step on the path we have a new basic 5-cycle with the same label set and the homotopy relation can be read off immediately.

For the other direction we proceed by contradiction. Assume that we have two 5-cycles, C and C' that do not have the same label set.

Let $w = L(e_1)L(e_2)L(e_3)L(e_4)L(e_5)$ and $w' = L(e_1')L(e_2')L(e_3')L(e_4')L(e_5')$ be the words associated to C and C'. By Lemma 4.2, if $L(C) \neq L(C')$ then they share at most one edge label, and hence their words share at most one letter. However, by Remark 2.6, we are unable to change the parity of letters by using (T1), (T2) and (T3), so w has a letter of odd parity that can only have even parity in w'. This implies we cannot change w to w' using (T1), (T2) and (T3), which contradicts $C \simeq_A C'$.

□

Now that we have established that the equivalence classes of basic 5-cycles are in bijection with the label sets we may count the equivalence classes easily. We obtain a label set for a basic 5-cycle by choosing five vertices on the $(n+3)$-gon to form a pentagon region and then we may triangulate all regions outside that pentagon to arrive at a specific cycle with that label set. There are $\binom{n+3}{5}$ label sets for basic 5-cycles and thus $\binom{n+3}{5}$ equivalence classes of basic 5-cycles. If we stipulate that a label set must contain the element 1, then we have $\binom{n+2}{4}$ equivalence classes of 5-cycles with 1 in their label set, and now we show that in fact any equivalence class of basic 5-cycles without 1 in its label set can be written as a product of those that do have 1 in their label set.

We first consider Asc_3 (Figure 3). It is easy to show (though we do not do so here due to space considerations) that we can write the outside 5-cycle of Asc_3 as a product homotopic to the product the inner 5-cycles. Having done this for Asc_3, we show that there is an isomorphic copy of Asc_3 in Asc_n for $n > 3$ and so we may reduce to that case for all n.

Due to the structure of Asc_n, we can find an isomorphic copy of Asc_3 where the outside basic 5-cycle can have any label set that does not include 1. The basic 5-cycles inside still have 1 in their label sets and in this way we may write any basic 5-cycle without 1 in its label set as a product of basic 5-cycles that have 1 in their label sets.

Lemma 4.4 *Any basic 5-cycle C in Asc_n such that $1 \notin L(C)$ can be written as a product of basic 5-cycles whose label set do contain 1.*

Proof: The result follows easily however since we may create a copy of Asc_3 inside Asc_n with the desired labels by choosing an appropriate hexagon region inside the $(n+3)$-gon and triangulating it in all possible ways. Full details available in [13]. □

We now know that any basic 5-cycle whose label set does not contain 1 may be written as a product of those that do contain 1, so we need at most $\binom{n+2}{4}$ classes of basic 5-cycles to generate $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$. In fact, we cannot reduce the number of generators below $\binom{n+2}{4}$.

Lemma 4.5 *The $\binom{n+2}{4}$ equivalence classes of basic 5-cycles whose label set contain 1 is a minimal generating set for $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$.*

Proof: This result follows similarly to the latter half of the proof of Theorem 4.3. □

We have a minimal generating set for $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$ and we can see that there are no relations between the generators outside of commutivity. Theorem 1.1 follows; that is, $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$ is free abelian and of rank $\binom{n+2}{4}$.

5 Applications and Future Directions

We have provided a study of the discrete fundamental group of the complex $\mathcal{T}_n$, and now we give a sketch of the applications of this study as well as two areas for future research.

We first consider an application to cluster algebras. It is well known that the complex $\mathcal{T}_n$ is a cluster complex and its exchange graph is Asc_n [8]. In the same paper, it is also noted that the first szygy module of the cluster algebra is generated by all of the edges of Asc_n. It is easy then to see that $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$ is giving us a quotient of the second szygy module of the cluster algebra. The basic 5-cycles that generate this module correspond exactly to occurences of the pentagon recurrence noted in [8] and in taking a quotient by the 4-cycles we are removing any basic cycles that do not correspond to this pentagon recurrence. The equivalence classes of $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$ correspond exactly to equivalences in the pentagon reccurence as well. That is, two cycles are homotopic if and only if they give the same recurrence. By studying $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$, we are able to identify all the ways the pentagon recurrence occurs in the cluster algebra and classify them combinatorially.

A second application involves what we will define as a *conic arrangement*. Let $\mathcal{F}$ be the normal fan of the associahedron and $\mathcal{C} \subset \mathcal{F}$ be the set of cones corresponding to the 2-faces bounded by basic 5-cycles. Then, in a similar fashion to [3], we are able to show a link between $A_1^{n-2}(\mathcal{T}_n, T_0)$ and the classical fundamental group of the topological space obtained by removing all of the cones in $\mathcal{C}$ from $\mathbb{R}^n$.

Finally, we are led to two natural expansions of our study of $A_1^{n-2}(\mathcal{T}_n, T_0)$. First, we note that the associahedron is a graph associahedra. In [5], [12] and [16] the authors give methods of constructing and realizing graph associahedra as well as many of their properties. Our initial investigations suggest that many of the results obtained in the study of $A_1^{n-2}(\mathcal{T}_n, T_0)$ may be applied to this generalization.

A second expansion brings us back to cluster algebras. In [13], we study the discrete fundamental group of the cluster complexes of type B and D. These complexes have combinatorial descriptions similar to those of the associahedron and provide similar insight into the syzygy modules of the corresponding cluster algebras.

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