On the diameter of random planar graphs

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We show that the diameter $D(G_n)$ of a random (unembedded) labelled connected planar graph with n vertices is asymptotically almost surely of order $n^{1/4}$, in the sense that there exists a constant $c > 0$ such that $P(D(G_n) \in (n^{1/4-\epsilon}, n^{1/4+\epsilon})) \geq 1 - \exp(-n^{c\epsilon})$ for ϵ small enough and n large enough ($n \geq n_0(\epsilon)$). We prove similar statements for rooted 2-connected and 3-connected embedded (maps) and unembedded planar graphs.

Keywords: Random graph, planar graph, diameter, bivariate generating functions, singularity analysis

1 Introduction

The diameter of random *embedded* connected planar graphs (called planar maps) has attracted a lot of attention since the pioneering work by Chassaing and Schaeffer [6] on the radius $r(Q_n)$ of random quadrangulations with n vertices, where they show that $r(Q_n)$ rescaled by $n^{1/4}$ converges as $n \to \infty$ to an explicit (continuous) distribution related to the Brownian snake. This suggests that random maps of size n are to be rescaled by $n^{1/4}$ in order to converge; precise definitions of the convergence can be found in [14, 9], and the (spherical) topology of the limit is studied in [10, 16]; some general statements about the limiting profile and radius are obtained in [13, 15]. At the combinatorial level, the two-point function of random quadrangulations has surprisingly a simple exact expression, a beautiful result found in [4] that allows one to derive easily the limit distribution (rescaled by $n^{1/4}$) of the distance between two randomly chosen vertices in a random quadrangulation. In contrast, little is known about the profile of random *unembedded* connected planar graphs, even if it is strongly believed that the results should be similar as in the embedded case.

We have not been able to show a limit distribution for the profile (or radius, diameter) of a random connected planar graph rescaled by $n^{1/4}$; instead we have obtained large deviation results on the diameter that strongly support the belief that $n^{1/4}$ is the right scaling order. We say that a property A, defined for all values n of a parameter, holds asymptotically almost surely if

$$P(A) \to 1, \qquad \text{as } n \to \infty.$$

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In this case we write a.a.s. In this paper we need a certain rate of convergence of the probabilities. Suppose property A depends on a real number $\epsilon > 0$ (usually very small). Then we say that A holds a.a.s. with exponential rate if there is a constant $c > 0$, such that for every ϵ small enough there exist an integer $n_0(\epsilon)$ so that

$$P(\text{not } A) \leq e^{-n^{c\epsilon}} \qquad \text{for all } n \geq n_0(\epsilon). \tag{1}$$

The diameter of a graph (or map) G is denoted by $D(G)$. The main results proved in this paper are the following.

Theorem 1.1 *The diameter of a random connected labelled planar graph with n vertices is, a.a.s. with exponential rate, in the interval*

$$(n^{1/4-\epsilon}, n^{1/4+\epsilon}).$$

Theorem 1.2 *Let $1 < \mu < 3$. The diameter of a random connected labelled planar graph with n vertices and $\lfloor \mu n \rfloor$ edges is in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate.*

This contrasts with so-called "subcritical" graph families, such as trees, outerplanar graphs, series-parallel graphs, where the diameter is in the interval $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate. (see the remark just before the bibliography).

The basis of our proof is the result for planar maps (i.e., embedded planar graphs) mentioned above. Then we prove the result for 2-connected maps using the fact that a random map has a large 2-connected component a.a.s. A similar argument allows us to extend the result to 3-connected maps, which proves it also for 3-connected planar graphs, because they have a unique embedding in the sphere. We then reverse the previous arguments and go first to 2-connected and then connected planar graphs, but this is not straightforward. One difficulty is that the largest 3-connected component of a random 2-connected graph does not have the typical ratio between number of edges and number of vertices, and this is why we must study maps with a given ratio between edges and vertices. In addition, we must show that there is a 3-connected component of size $n^{1-\epsilon}$ a.a.s. with exponential rate, and similarly for blocks. Finally, we must show that the height of the tree associated to the decomposition of a 2-connected graph into 3-connected components is at most n^ϵ, and similarly for the tree of the decomposition of a connected graph into blocks.

2 Preliminaries

Let $f(z) = \sum_n f_n z^n$ be a series with nonnegative coefficients and let $x > 0$ be a value such that $f(x)$ converges (in particular x is at most the radius of convergence ρ). Recall the following elementary inequality: for $n \geq 0$ we have

$$f_n \leq f(x)x^{-n}. \tag{2}$$

(When minimized over x, this inequality is called *saddle-point bound*).

A bivariate version yields a lemma that will be used several times; it provides a simple criterion to ensure that the distribution of a parameter has exponentially fast decaying tail. First let us give some terminology. A *weighted combinatorial class* is a class of combintorial objects (such as graphs, trees, maps) $\mathcal{A} = \cup_n \mathcal{A}_n$ endowed with a weight-function $w : \mathcal{A} \mapsto \mathbb{R}_+$. The *weighted distribution* in size n is the unique distribution on $\mathcal{A}_n$ proportional to the weight: $P(\alpha) \propto w(\alpha)$ for every $\alpha \in \mathcal{A}_n$.

Lemma 2.1 *Let $\mathcal{A} = \cup_n \mathcal{A}_n$ be a weighted combinatorial class, and let $A(z) = \sum_{\alpha\in\mathcal{A}} w(\alpha) z^{|\alpha|}$ be the corresponding weighted generating function. Let χ be a parameter for objects in $\mathcal{A}$ with associated bivariate generating function $A(z,u) = \sum_{\alpha\in\mathcal{A}} w(\alpha) z^{|\alpha|} u^{\chi(\alpha)}$, where z marks size and u marks χ. Let $\rho > 0$ be the dominant singularity of $A(z) = A(z,1)$. Assume there is a polynomial $p(n)$ such that*

$$A_n = [z^n]A(z) \geq \frac{1}{p(n)}\,\rho^{-n} \qquad \text{for } n \text{ large enough.}$$

Assume also that there exists $u_0 > 1$ such that $A(z,u_0)$ has dominant singularity ρ and $A(\rho,u_0)$ is finite.

Then, there exists constants $C > 0$ and $c > 0$ such that, for R_n taken at random under the weighted distribution in size n

$$P(\chi(R_n) \geq k) \leq C\,p(n)\,e^{-ck},$$

for every n and k positive.

Proof: We have $P(\chi(R_n) = k) = [z^n u^k]A(z,u)/[z^n]A(z)$. A bivariate version of (2) ensures that $[z^n u^k]A(z,u) \leq A(\rho,u_0)\rho^{-n}u_0^{-k} = O(\rho^{-n}e^{-ck})$ where $c = \log(u_0)$. Hence $P(\chi(R_n) = k) = O(p(n)e^{-ck})$, so $P(\chi(R_n) \geq k) = O(p(n)e^{-ck})$. □

3 Quadrangulations and maps

We recall here the definitions of maps. A *planar map* (shortly called a map here) is a connected unlabelled graph embedded in the plane up to isotopic deformation. Loops and multiple edges are allowed. A *rooted map* is a map where an edge incident to the outer face is marked and oriented so as to have the outer face on its left; the *root-vertex* is the origin of the root; the outer face is also called the *root face*. A *quadrangulation* is a map where all faces have degree 4.

3.1 Quadrangulations

We recall Schaeffer's bijection (itself an adaptation of an earlier bijection by Cori and Vauquelin [7]) between labelled trees and quadrangulations. A *rooted plane tree* is a rooted map with a unique face. A *labelled tree* is a rooted plane tree with a integer label $\ell(v) \in \mathbb{Z}$ on each vertex v so that the labels of the extremities of each edge $e = (v,v')$ satisfy $|\ell(v) - \ell(v')| \leq 1$, and such that the root vertex has label 0. A useful observation is that labelled trees are in bijection with rooted plane trees where a subset of the edges is oriented arbitrarily (for the onto mapping, one orients an edge with labels $(i,i+1)$ toward the vertex with label $i+1$ and one leaves an edge of type (i,i) unoriented). Thus the number of labelled trees with n edges is $3^n C_n$ with $C_n := (2n)!/n!/(n+1)!$ the nth Catalan number. A *signed* labelled tree is a pair (τ,σ) where τ is a labelled tree and σ is an element of $\{-1,+1\}$.

Theorem 3.1 (Schaeffer [17], Chassaing, Schaeffer [6]) *Signed labelled trees with n vertices are in bijection with rooted quadrangulations with n vertices and a secondary pointed vertex v_0. Each vertex v of a labelled tree corresponds to a non-pointed vertex ($\neq v_0$) in the associated quadrangulation Q, and $\ell(v) - \ell_{min} + 1$ gives the distance from v to v_0 in Q, where ℓ_{min} is the minimum label in the tree.*

From this bijection, it is easy to show large deviation results for the diameter of a quadrangulation (the basic idea, originating in [6], is that the typical depth k of a vertex in the tree is $n^{1/2}$, and the typical discrepancy of the labels along a branch is $k^{1/2} = n^{1/4}$). The main result we use, from [8], is the property that (under general conditions) the height of a random tree of size n from a given family is in $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate.

Let $\mathcal{T}$ be a family of trees, with $y = y(z)$ the associated generating function (according to the number of nodes) and ρ the radius of convergence of y. The family $\mathcal{T}$ is called *admissible* if $y = y(z)$ satisfies an equation of the form

$$y = F(z, y), \tag{3}$$

with $F(z, y)$ a bivariate function with nonnegative coefficients, nonlinear in y, analytic around $(0, 0)$, satisfying $F(0, y) = 0$, and such that F is analytic at (ρ, τ) where $\tau := y(\rho)$ is the value of y at its dominant singularity (observe that $\tau < \infty$ since $F(z, y)$ is non-linear in y). We have:

Lemma 3.2 (Flajolet et al. Theorem 3.1 in [8]) *Let $\mathcal{T}$ be an admissible family of rooted trees, let T_n be a tree chosen uniformly at random in $\mathcal{T}_n$, and let $\xi(T_n)$ denote the height of T_n. Then $\xi(T_n) \in (n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate* [(i)].

Proposition 3.3 *The diameter of a random rooted quadrangulation with n vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$.*

Proof: The range $\Delta := \ell_{max} - \ell_{min} + 1$ in a labelled tree T gives the radius of the associated rooted pointed quadrangulation Q with respect to the pointed vertex. Hence $D(Q)/2 \leq \Delta \leq D(Q)$. So we just have to show that Δ is in $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate. Since the label either increases by 1, stays equal, or decreases by 1 along each edge (going away from the root), the series $T(z)$ of labelled trees counted according to vertices satisfies

$$T(z) = \frac{z}{1 - 3T(z)}.$$

The equation is clearly admissible (the singularity is $1/12$ and $T(1/12) = 1/6$), hence by Lemma 3.2 the height is in $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate. So in a random labelled tree there is a.a.s. with exponential rate a branch B of length $k = n^{1/2-\epsilon}$. The labels along B form a random walk with increments $+1$, 0, -1 each with probability $1/3$. Classically the maximum of such a walk is at least $k^{1/2-\epsilon}$ (which is $\geq n^{1/4-2\epsilon}$) a.a.s. with exponential rate. Hence the label of the vertex v on B at which the maximum occurs is at least the label of the root-vertex plus $n^{1/4-2\epsilon}$, so $\ell_{max} \geq n^{1/4-2\epsilon}$ a.a.s. with exponential rate. Since $l_{min} \leq 0$, this proves the lower bound (one can replace 2ϵ by ϵ up to dividing by 2 the constant c in (1)).

For the upper bound (already proved in [6]), since the height is at most $n^{1/2+\epsilon}$ a.a.s. with exponential rate, the same is true for the depth k of a random vertex v in a random labelled tree of size n. Along the path from the root-vertex to v, the random walk of the labels has maximum at most $k^{1/2+\epsilon}$. Hence $|\ell(v)| \leq n^{(1/2+\epsilon)^2}$ a.a.s. with exponential rate, so the same easily holds for the property $|\ell(v)| \leq n^{1/4+\epsilon}$. Hence (multiplying by n keeps the probability of failure exponentially small), the property $\{\forall v,\ |\ell(v)| \leq n^{1/4+\epsilon}\}$ is true a.a.s. with exponential rate. This completes the proof. $\square$

We also need a weighted version of the previous theorem. Recall that a rooted quadrangulation Q has a unique bicoloration of its vertices in black and white such that the origin of the root is black and each

(i) In [8] they prove the result on an example and say that all the arguments in the proof hold for any system of the form $y = z\phi(y)$. We have checked that actually all arguments still hold in the general case of an admissible system of the form $y = F(z, y)$.

edge connects a black with a white vertex. Call it the *canonical bicoloration* of Q. Given $x > 0$, a rooted quadrangulation with v black vertices is weighted with parameter x if we assign to it weight x^v. The next theorem generalizes Proposition 3.3 to the weighted case.

Theorem 3.4 *Let $0 < a < b$. The diameter of a random quadrangulation weighted by x is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

The proof, which is omited, consists of the following steps: write a two-line system to specify the generating function of weighted labelled trees, transform the system into a one-line equation (gathering the nodes in monochromatic components), and use a weighted version of Lemma 3.2 to show that the height (defined here as the maximum number of edges with label-increment along a path from the root) in the weighted case is still in $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate. This easily yields the theorem (similarly as in Proposition 3.3) using the fact that the typical discrepancy of labels along a path of length ℓ is $\sqrt{\ell}$.

3.2 Maps

We recall the classical bijection between rooted quadrangulations with n faces (and thus $n + 2$ vertices) and rooted maps with n edges. Starting from Q endowed with its canonical bicoloration, add in each face a new edge connecting the two (diagonally opposed) black vertices. Return the rooted map M formed by the newly added edges and the black vertices, rooted at the edge corresponding to the root-face of Q, and with same root-vertex as Q. Conversely, to obtain Q from M, add a new white vertex v_f inside each face f of M (even the outer face) and add new edges from v_f to every corner around f; then delete all edges from M, and take as root-edge of Q the one corresponding to the incidence root-vertex/outer-face in M. Clearly, under this bijection, vertices of a map correspond to black vertices of the associated quadrangulation, and faces correspond to white vertices.

Map families are here weighted at their vertices, i.e., for a given parameter $x > 0$, a map with v vertices has weight x^v.

Theorem 3.5 *Let $0 < a < b$. The diameter of a random rooted map with n edges and weight x at the vertices is in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

Proof: The first important observation is that the bijection transports the weighted (weight $x > 0$ for each black vertex) distribution on rooted quadrangulations with $n+2$ vertices to the weighted (weight x for each vertex) distribution on rooted maps with n edges. We start with the proof for $x = 1$ (uniform distribution). Let M be a random rooted map with n edges and let Q be the associated rooted quadrangulation (with $n + 2$ vertices). Every path $b_1 b_2 \dots b_k$ in M yields a path $b_1 w_1 b_2 \dots w_{k-1} b_k$ in Q, calling w_i the white vertex corresponding to the face on the left of (b_i, b_{i+1}). Hence $D(Q) \leq 2D(M)$. This shows that the diameter of M is at least $n^{1/4-\epsilon}$ a.a.s. with exponential rate. To prove the upper bound, let $x = b_1 w_1 b_2 w_2 \dots b_k = y$ be a path in Q, where the b_i are black vertices and the w_i are white. Let f_i be the size of the face in M corresponding to b_i. Then we can find a path in M between x and y of length $k + f_1 + \cdots + f_k$. Therefore, calling $\Delta(M)$ the maximal face-degree in M, we have

$$D(M) \leq D(Q) \cdot \Delta(M). \tag{4}$$

Let $A(z, u)$ be the series counting rooted maps where z marks the number of edges and u marks the root-face degree. Using the quadratic method, Tutte [18] has found an explicit expression for $A(z, u)$ ensuring that

- $[z^n]A(z,1) = \Theta(n^{-5/2}12^n)$,
- for u in an open interval around 1, $z \mapsto A(z,u)$ has radius of convergence $1/12$ and $A(1/12,u)$ is finite.

Hence, by Lemma 2.1, the root-face degree $\delta(M)$ in a random rooted planar map with n edges satisfies $P(\delta(M) \geq k) = O(e^{-ck})$ for some $c > 0$. The probability distribution is the same if the map is bi-rooted (two root-edges possibly equal, the root-face being the face incident to the primary root). When exchanging the secondary root with the primary root, the root-face can be seen as a face f taken at random under distribution $P(f) = \deg(f)/(2n)$. Thus $\delta(M)$ is distributed as the degree of the (random) face f. Hence $P(\Delta(M) \geq k) \leq \frac{2n}{k}P(\delta(M) \geq k)$, so $P(\Delta(M) \geq k) = O(e^{-ck/2})$. Since the diameter of $D(Q)$ is at most $n^{1/4+\epsilon/2}$ and since $\Delta(M)$ is at most $n^{\epsilon/2}$ a.a.s. with exponential rate, we conclude from (4) that the diameter of M is at most $n^{1/4+\epsilon}$ a.a.s. with exponential rate.

The proof for arbitrary $x \in [a,b]$ is similar. Let $A(x,z,u)$ be the series counting rooted maps where x marks non-root vertices, z marks edges, and u marks the root-face degree. Let ρ_x be the radius of convergence of $z \mapsto A(x,z,1)$. As an easy extension of Tutte's result, see [3], one finds an expression of $A(x,z,u)$ ensuring that

- $[z^n]A(x,z,1) \sim c_x \rho_x^{-n} n^{-5/2}$ with c_x a positive constant that evolves continuously in x,
- there exists $u_0 > 1$ such that for each $x \in [a,b]$, $A(x,\rho(x),u_0)$ is a finite constant that evolves continuously in x.

Consequently all arguments used in the case $x = 1$ hold in the same way. □

3.3 2-connected maps

Here it is convenient to include the empty map in the families $\mathcal{M} = \cup_n \mathcal{M}_n$ of rooted maps and $\mathcal{C} = \cup_n \mathcal{C}_n$ of rooted 2-connected maps. As described by Tutte in [18], a rooted map M is obtained by taking a rooted 2-connected map C, called the *core* of M, and then inserting in each corner i of C an arbitrary rooted map M_i. The maps M_i are called the *pieces* of M. Denoting by $M(x,z)$ ($C(x,z)$, resp.) the series of rooted connected (2-connected, resp.) maps according to non-root vertices and edges, this decomposition yields

$$M(x,z) = C(x, H(x,z)), \quad \text{where } H(x,z) = zM(x,z)^2, \tag{5}$$

since a core with k edges has $2k$ corners where to insert rooted maps.

An important property of the composition scheme is to preserve the uniform distribution, as well as the (vertex-)weighted distribution. Precisely, let M be a rooted map with n edges and weight x at the vertices. Let C be the core of M, call k its size, and let $M_1, \ldots, M_{2k}$ be the pieces of M, call $n_1, \ldots, n_{2k}$ their sizes. Then, conditioned to have size k, C is a random rooted 2-connected map with k edges and weight x at vertices, and conditioned to have size n_i the ith piece M_i is a random rooted map with n_i edges and weight x at vertices.

Lemma 3.6 *Let $0 < a < b$, and let $x \in [a,b]$. Let $\rho^{(x)}$ be the radius of convergence of $z \mapsto M(x,z)$. Following [1], define*

$$\alpha^{(x)} = \frac{H(x,\rho^{(x)})}{\rho^{(x)} H_z(x,\rho^{(x)})}.$$

Let $n \geq 0$, and let M be a random rooted map with n edges and weight x at vertices. Let $X_n = |C|$ be the size of the core of M, and let $M_1, \ldots, M_{2|C|}$ be the pieces of M. Then

$$P\big(X_n = \lfloor \alpha^{(x)} n \rfloor,\ \max(|M_i|) \leq n^{3/4}\big) = \Theta(n^{-2/3})$$

uniformly over $x \in [a, b]$.

The proof is omitted. In [1] the authors derive the limit distribution of X_n and they show that $P(X_n = \lfloor \alpha^{(x)} n \rfloor) = \Theta(n^{-2/3})$. So Lemma 3.6 says that the asymptotic order of $P(X_n = \lfloor \alpha^{(x)} n \rfloor)$ is the same under the additional condition that all pieces are of size at most $n^{3/4}$ (one could actually ask $n^{2/3+\delta}$ for any $\delta > 0$). A closely related result proved in [11] is that, for any fixed $\delta > 0$, there is a.a.s. no piece of size larger than $n^{2/3+\delta}$ provided the core has size larger than $n^{2/3+\delta}$.

Theorem 3.7 *For $0 < a < b$, the diameter of a random rooted 2-connected map with n edges and weight x at vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

Proof: Again we treat only the case $x = 1$ in details. Let M be a rooted map with n edges taken uniformly at random, C the core of M and $(M_i)_{i \in [1..2|C|]}$ the pieces of M. Note that

$$D(C) \leq D(M) \leq D(C) + 2\max(D(M_i)),$$

the first inequality is trivial, the second one follows from the property that a diametral path in M either stays in a same piece, or it starts in a certain piece, travels along edges of C, and then finishes in another piece.

Since $|C| = \lfloor n/3 \rfloor$ has polynomially small probability (order $\Theta(n^{-2/3})$ as shown in [1]) and $D(M) > n^{1/4+\epsilon}$ has exponentially small probability (meaning smaller than $\exp(-n^{c\epsilon})$ for some $c > 0$), the event $D(M) > n^{1/4+\epsilon}$ knowing $|C| = \lfloor n/3 \rfloor$ has exponentially small probability. Since $D(M) \geq D(C)$, this yields the a.a.s. upper bound on $D(C)$.

For the lower bound, Lemma 3.6 ensures that the event $\mathcal{E} := \{|C| = \lfloor n/3 \rfloor,\ \max(|M_i|) \leq n^{3/4}\}$ occurs with polynomially small probabilty (of order $\Theta(n^{-2/3})$). We claim that the event $\max(|M_i|) \leq n^{3/4}$ implies a.a.s. in n that $\max(D(M_i)) \leq n^{1/5}$. (Proof: when $n_i := |M_i| \leq n^{1/5}$, $D(M_i) \leq n^{1/5}$ trivially. Moreover for $\delta > 0$ small enough, $P(D(M_i) > n_i^{1/4+\delta}) \leq \exp(-n_i^{c\delta})$ for some $c > 0$, so when $n^{1/5} \leq n_i \leq n^{3/4}$, $P(D(M_i) > n^{3/4(1/4+\delta)}) \leq \exp(-n^{c\delta/5})$, and we can take δ small enough so that $3/4(1/4 + \delta) \leq 1/5$. So $D(M_i) > n^{1/5}$ has exponentially small probability in n, and the same holds for $\max(D(M_i))$.) Hence the event $\mathcal{E}' := \{|C| = \lfloor n/3 \rfloor,\ \max(D(M_i) \leq n^{1/5}\}$ occurs with polynomially small probability. In that case, since $D(C) \geq D(M) - 2\max(D(M_i))$ and since the event $D(M) < n^{1/4} - \epsilon$ occurs with exponentially small probability, we conclude that the event $D(C) < n^{1/4-\epsilon} - 2n^{1/5}$ occurs with exponentially small probability. As $n^{1/5} << n^{1/4-\epsilon}$ for ϵ small enough, this finally gives the a.a.s. lower bound on $D(C)$.

The case of arbitrary $x > 0$ is done by the same arguments, the uniformity in $x \in [a, b]$ of the bounds following from the uniformity in x in Theorem 3.5 and Lemma 3.6. □

3.4 3-connected maps

In a similar way as in Section 3.3 (where one goes from connected to 2-connected maps) there is a decomposition of 2-connected maps in terms of 3-connected components that allows to transfer the diameter concentration property from 2-connected to 3-connected maps. In this section it is convenient to exclude the loop-map from the family of 2-connected maps, so all 2-connected maps are loopless.

As shown by Tutte [18], a rooted 2-connected map C is either a series or parallel composition of 2-connected maps, or it is obtained from a rooted 3-connected map T where each non-root edge e is possibly substituted by a rooted 2-connected map C_e (identifying the extremities of e with the extremities of the root of C_e). In that case T is called the *3-connected core* of C and the components C_e are called the *pieces* of C. Call $C(x,z)$ ($\widehat{C}(x,z)$) the series counting rooted 2-connected maps (rooted 2-connected maps with a 3-connected core, resp.) according to vertices not incident to the root (variable x) and edges (variable z). Call $T(x,z)$ the series counting rooted 3-connected maps according to vertices not incident to the root (variable x) and edges (variable z). Then

$$\widehat{C}(x,z) = T(x, C(x,z)). \tag{6}$$

Accordingly (similarly as in Section 3.3), for a random rooted 2-connected map with n edges, weight x at vertices, and conditioned to have a 3-connected core T of size k, T is a random rooted 3-connected map with k edges and weight x at vertices; and each piece C_e conditioned to have a given size n_e is a random rooted 2-connected map with n_e edges and weight x at vertices.

Calling f_e the degree of the root face of C_e, we have

$$D(T) \leq D(C) \leq D(T) \cdot \max_e(f_e) + 2\max_e(D(C_e)). \tag{7}$$

The first inequality is trivial. The second one follows from the fact that a diametral path P in C starts in a piece, ends in a piece, and in between it passes by adjacent vertices $v_1, \ldots, v_k$ of T such that for $1 \leq i < k$, v_i and v_{i+1} are connected in T by an edge e and P travels in the piece C_e to reach v_{i+1} from v_i (since P is geodesic, its length in C_e is bounded by the distance from v_i to v_{i+1}, which is clearly bounded by f_e).

Theorem 3.8 *Let $0 < a < b$. The diameter of a random 3-connected map with n edges with weight x at the vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a,b]$.*

Proof: Let us give the sketch of proof when $x = 1$. The upper bound again follows from $D(T) \leq D(C)$ and the fact [1] that the probability of having a 3-connected core of size $\lfloor n/3 \rfloor$ is polynomially small whereas the probability that $D(C) > n^{1/4+\epsilon}$ is exponentially small. For the lower bound we look at the second inequality in (7). Similarly as in the proof of Theorem 3.5 one can prove (using Lemma 2.1) that $P(f_e \geq k) \leq \exp(-ck)$ for some $c > 0$, so $\max_e(f_e) \leq n^\epsilon$ a.a.s. with exponential rate. Moreover, similarly as in Lemma 3.6, one can show that the probability of the event $\mathcal{E} := \{|T| = \lfloor n/3 \rfloor,\ \max(|C_e|) \leq n^{3/4}\}$ is $\Theta(n^{-2/3})$. Since the f_e are small and $D(C) \geq n^{1/4-\epsilon}$ a.a.s. with exponential rate, Equation (7) easily implies that, conditioned to have $\mathcal{E}$, $D(T) \geq n^{1/4-\epsilon}$ a.a.s. with exponential rate. The same arguments hold in the weighted case. □

4 Planar graphs

4.1 3-connected planar graphs

For the moment we need 3-connected graphs labelled at the edges (this is enough to avoid symmetries). The number of edges is now m, and n is reserved for the number of vertices. By Whitney's theorem 3-connected graphs have a unique embedding on the sphere (up to reflexion). Hence from the last theorem on 3-connected maps we obtain directly the following:

Theorem 4.1 *Let $0 < a < b$. The diameter of a random 3-connected planar graph with m edges with weight x at the vertices is, a.a.s. with exponential rate, in the interval $(m^{1/4-\epsilon}, m^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

4.2 Networks

Before handling 2-connected planar graphs we treat the closely related family of (planar) *networks*. A *network* is a connected simple planar graph with two marked vertices called the poles, such that adding an edge between the poles, called the root-edge, makes the graph 2-connected. At first it is convenient to consider the networks as labelled at the edges.

Theorem 4.2 *Let $0 < a < b$. The diameter of a random network with m edges with weight x at the vertices is, a.a.s. with exponential rate, in the interval $(m^{1/4-\epsilon}, m^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

Proof: We provide only a sketch of proof in the case $x = 1$, and will provide a more detailed proof for the transition from 2-connected to connected planar graphs, which uses similar arguments but allows for a simpler presentation. The arguments are the following. First there is a classical decomposition of a network N into (edge-rooted) 3-connected planar components that are assembled together using series-parallel operations. Each 3-connected component is uniformly distributed when conditioned on a fixed number of edges. Using the bound (2) (see Lemma 4.5 later) one shows that there is a 3-connected component T of size $k \geq n^{1-\epsilon}$ a.a.s. with exponential rate. This provides the lower bound since $D(T) \leq D(N)$ and since $D(T) \geq k^{1/4-\epsilon}$ a.a.s. with exponential rate.

For the upper bound one considers first the tree τ whose nodes are the 3-connected components (there are also nodes for the series and the parallel compositions), and shows using Lemma 2.1 (and using the fact that the composition scheme from 3-connected planar graphs to networks is critical) that the diameter of τ is at most n^ϵ a.a.s. with exponential rate. Then one has to show that two vertices (actually it is more convenient to work with edges) on a same component H of τ are at distance at most $n^{1/4+\epsilon}$ a.a.s. with exponential rate. Say that H is 3-connected (the case of a series or parallel composition node of τ is of smaller contribution to the diameter). One knows from Tutte's monograph [19] that the edges of N (including the additional root-edge) can be partitioned into networks N_e attached by their poles at each edge $e \in H$. Call d_e the geodesic distance in N_e between the poles (i.e., the extremities of e). Using again Lemma 2.1, one shows that under the uniform distribution on networks with a marked 3-connected component, the maximum of d_e over $e \in H$ is at most n^ϵ a.a.s. with exponential rate. Moreover by Theorem 4.1 $D(H) \leq n^{1/4+\epsilon}$ a.a.s. with exponential rate; since the d_e are small, this remains true even when each edge $e \in H$ has contribution d_e (instead of 1) to the distance between two vertices on H, which corresponds to the geodesic distance in N. □

Lemma 4.3 *Let $1 < a < b < 3$. For $N_{n,m}$ a network with n vertices and m labelled edges taken uniformly at random, $D(N_{n,m}) \in (n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate, uniformly in $\frac{m}{n} \in [a, b]$.*

Proof: Let $\mu \in [a, b]$, let $m \geq 0$, and define $n := \lfloor \mu m \rfloor$. For $x > 0$, let X_m be the number of vertices of a random network $N_m^{(x)}$ with m edges and vertices weighted by x. As shown in [2] there exists $x_\mu > 0$ such that, for $x = x_\mu$, $P(X_m = n) = \Theta(m^{-1/2})$, uniformly over $\mu \in [a, b]$. In addition x_μ evolves continuously increasingly with μ so it maps $[a, b]$ to a compact interval. Therefore, Theorem 4.2 implies that $D(N_m^{(x)}) \in [m^{1/4-\epsilon}, m^{1/4+\epsilon}]$ a.a.s. with exponential rate uniformly over

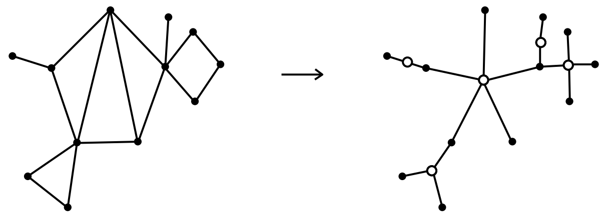

Fig. 1: The block-decomposition of a connected planar graph. In the associated tree (incidences between 2-connected blocks and vertices) there is a white vertex for each block.

$\mu \in [a,b]$. Since $P(X_m = n) = \Theta(m^{-1/2})$, uniformly over $\mu \in [a,b]$, we conclude that the event $D(N_m^{(x)}) \in [m^{1/4-\epsilon}, m^{1/4+\epsilon}]$ knowing that $X_m = n$ holds a.a.s. with exponential rate uniformly over $\mu \in [a,b]$, which concludes the proof (note that the distribution of $N_m^{(x)}$ knowing that $X_m = n$ is the uniform distribution on networks with m edges and n vertices). □

An important remark is that networks with n vertices and m edges can be labelled either at vertices or at edges, and the uniform distribution in one case corresponds to the uniform distribution in the second case. Hence the result of Lemma 4.3 holds for random networks with n vertices and m edges and labelled at vertices.

4.3 2-connected planar graphs

It is proved in [2] that for a random network N_n with n vertices the ratio $r = \#edges/\#vertices$ is concentrated around a certain $\mu \approx 2.2$, implying that for $\delta > 0$ $P(r \notin [\mu-\delta, \mu+\delta])$ is exponentially small. Hence $D(N_n) \in (n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate. The same holds for the diameter of a random 2-connected planar graph B_n with n vertices (indeed 2-connected planar graphs are a subset of networks, the ratios of the cardinalities being of order n). We obtain:

Theorem 4.4 *The diameter of a random 2-connected planar graph with n vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$.*

4.4 Connected planar graphs

We prove here from Theorem 4.4 that a random connected planar graph with n vertices has diameter in $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate. We use the well known decomposition of a connected planar graph C into 2-connected blocks such that the incidences of the blocks with the vertices form a tree, see Figure 1. An important point is that if C is chosen uniformly at random among connected planar graphs with n vertices, then each block B of C is uniformly distributed when conditioned to have a given size. Formulated on pointed graphs, the block-decomposition ensures that a pointed planar graph is obtained as follows: take a collection of 2-connected pointed planar graphs, and merge their pointed vertices into a single vertex; then attach at each non-marked vertex v in these blocks a pointed connected planar graph C_v. Calling $C(z)$ ($B(z)$) the series counting pointed connected (2-connected, resp.) planar graphs, this yields the equation

$$F(z) = z\exp(B'(F(z))), \quad \text{where } F(z) = zC'(z). \tag{8}$$

Note that the inverse of $F(z)$ is the function $\phi(u) = u\exp(-g(u))$, where $g(u) := B'(u)$. Call ρ the radius of convergence of $C(z)$ and R the radius of convergence of $B(u)$.

Lemma 4.5 *A random connected planar graph with n vertices has a block of size at least $n^{1-\epsilon}$ a.a.s. with exponential rate.*

Proof: Call $b_i := [u^i]g(u)$, $g_k(u) := \sum_{i\leq k} b_i u^i$, and call $F_k(z)$ the series of pointed connected planar graphs where all blocks have size at most k. Note that the probability of a random connected planar graphs with n vertices to have all its blocks of size at most k is $[z^n]F_k(z)/[z^n]F(z)$. Clearly

$$F_k(z) = z\exp(g_k(F_k(z)),$$

hence the inverse of $F_k(z)$ is $\phi_k(u) := u\exp(-g_k(u))$. Call ρ_k the singularity of $F_k(z)$. Since $\phi_k(u)$ is analytic everywhere, the singularity at ρ_k is caused by a branch point, i.e., $\rho_k = \phi_k(R_k)$, where R_k is the unique $u > 0$ such that $\phi_k'(u) = 0$: $\phi_k'(u) > 0$ for $0 < u < R_k$ and $\phi'(u) < 0$ for $u > R_k$. According to (2), $[z^n]F_k(z) \leq F_k(x)x^{-n}$ for $x < \rho_k$, or equivalently, writting $u = F_k(x)$,

$$[z^n]F_k(z) \leq u\phi_k(u)^{-n} \ \text{ for all } u \text{ s.t. } \phi_k'(u) > 0. \tag{9}$$

Define $u_k := R\cdot(1 + 1/(k\log k))$. Observing that $(u_k/R)^k \sim 1$ one easily shows that $g_k(u_k) \to g(R)$, $g_k'(u_k) \to g'(R)$, hence $\phi_k'(u_k) \to \phi'(R)$. It is shown in [12] that $a := \phi'(R)$ is strictly positive (i.e., the singularity of $F(z)$ is not due to a branch point), so for k large enough, $\phi_k'(u_k) \geq a/2 > 0$, i.e., the bound (9) can be used, giving

$$[z^n]F_k(z) \leq 2\phi_k(u_k)^{-n} \ \text{ for } k \text{ large enough and any } n \geq 0.$$

Moreover

$$\phi_k(u_k) - \rho = \big(\phi_k(u_k) - \phi_k(R)\big) + \big(\phi_k(R) - \phi(R)\big) \sim a\cdot(u_k - R) + O(k^{-3/2}) \sim \frac{a}{k\log k},$$

where $\phi_k(R) - \phi(R) = O(k^{-3/2})$ is due to $b_i = \Theta(R^{-i}i^{-5/2})$ which is shown in [12] (so $g(R) - g_k(R) = O(k^{-3/2})$). Hence for k large enough and any $n \geq 0$:

$$[z^n]F_k(z) \leq 2\Big(\rho + \frac{a}{2k\log k}\Big)^{-n}.$$

Hence, for $k = n^{1-\epsilon}$, $[z^n]F_k(z) = \Theta(\rho^{-n}\exp(-n^{\epsilon/2}))$. Finally, according to [12], $[z^n]F(z) = \Theta(\rho^{-n}n^{-5/2})$, so $[z^n]F_k(z)/[z^n]F(z) = O(\exp(-n^{\epsilon/3}))$. □

Lemma 4.5 directly implies that a random connected planar graph with n vertices has diameter at least $n^{1/4-\epsilon}$. Indeed it has a block of size $k \geq n^{1-\epsilon}$ a.a.s. with exponential rate and since the block is uniformly distributed in size k, it has diameter at least $k^{1/4-\epsilon}$ a.a.s. with exponential rate.

Let us now prove the upper bound, which relies on the following lemma:

Lemma 4.6 *The block-decomposition tree τ of a random connected planar graph with n vertices has diameter at most n^{ϵ} a.a.s. with exponential rate.*

Proof: Define a bi-pointed graph as a graph with two marked vertices (a primary and a secondary) that are distinct and unlabelled (and do not contribute to the size). The series counting bi-pointed connected (2-connected, resp.) planar graphs is $z \mapsto C''(z)$ ($w \mapsto B''(w)$, resp.). Deriving (8) w.r.t. z one obtains:

$$z^2 C''(z) = \frac{w^2 B''(w)}{1 - wB''(w)}, \text{ where } w = zC'(z).$$

This equation has an easy combinatorial interpretation: there is a chain of (bi-pointed) blocks to go from the first pointed to the second pointed vertex, and at each vertex of the chain of blocks one might attach a pointed connected planar graph. Call $C''(z, u)$ the bivariate refinement of $C''(z)$ where u marks the number χ of blocks in the chain of blocks. Then

$$z^2 C''(z, u) = \frac{uw^2 B''(w)}{1 - uwB''(w)}, \text{ where } w = zC'(z).$$

It is shown in [12] that $w = R$ when $z = \rho$ and that $C''(\rho)$ is finite. Hence $B''(R)$ has to be strictly smaller than 1, which also implies that $C''(\rho, u)$ remains finite for u slightly larger than 1. Hence, by Lemma 2.1, $P(\chi \geq k) \leq \exp(-ck)$ for some $c > 0$, hence $\chi \leq n^\epsilon$ a.a.s. with exponential rate. Since a connected planar graph C with n vertices has $n(n-1)/2$ pairs of distinct vertices, $\max(\chi)$ over all pairs of distinct vertices of C is also smaller than n^ϵ a.a.s. with exponential rate. This concludes the proof, since one easily shows that $\max(\chi)$ is equal to one plus the diameter of τ. □

Lemma 4.6 easily implies that the diameter of a random connected planar graph C with n vertices is at most $n^{1/4+\epsilon}$ a.a.s. with exponential rate. Indeed, calling τ the block-decomposition tree of C and B_i the blocks of C, one has $D(C) \leq D(\tau) \cdot \max_i D(B_i)$. Lemma 4.6 ensures that $D(\tau) \leq n^\epsilon$ a.a.s. with exponential rate. Moreover Theorem 4.4 easily implies that a random 2-connected planar graph of size $k \leq n$ has diameter at most $n^{1/4+\epsilon}$ a.a.s. with exponential rate, whatever $k \leq n$ is (proof by splitting in two cases: $k \leq n^{1/4}$ and $n^{1/4} \leq k \leq n$). Hence, since each of the blocks has size at most n, $\max_i D(B_i) \leq n^{1/4+\epsilon}$ a.a.s. with exponential rate. Therefore we have completed the proof of Theorem 1.1.

To show Theorem 1.2, one needs to extend the statements of Theorem 4.4 and Lemmas 4.5, 4.6 to the case of a random graph of size n with weight $y > 0$ on each edge. Then, one uses the fact (proved in [12]) that for each $\mu \in (1, 3)$ there exists $y > 0$ such that a random planar graph with n edges and weight y on edges has probability $\Theta(n^{-1/2})$ to have $\lfloor \mu n \rfloor$ edges.

We conclude with a remark on so-called "subcritical" graph families, these are the families where the system

$$y = z \exp(B'(y)) =: F(z, y) \tag{10}$$

to specify pointed connected from pointed 2-connected graphs in the family is admissible, i.e., $F(z, y)$ is analytic at (ρ, τ) where ρ is the radius of convergence of $y = y(z)$ and $\tau = y(\rho)$.

Define the *block-distance* of a vertex v in a vertex-pointed connected graph G as the minimal number of blocks one can use to travel from the pointed vertex to v; and define the *block-height* of G as the maximum of the block-distance over all vertices of G. Now observe that the block-height plays the same role in the system (10) as the usual height of a tree in Equation (3). Hence by Lemma 3.2 (or a weighted version of it), the block-height h of a random pointed connected graph G with n vertices from a subcritical family is in $[n^{1/2-\epsilon}, n^{1/2+\epsilon}]$ a.a.s. with exponential rate. Clearly $D(G) \geq h - 1$ since the distance between two vertices is at least the block-distance minus 1. Hence $D(G) \geq n^{1/2-\epsilon}$ a.a.s. with exponential rate. For the upper bound, note that $D(G) \leq h \cdot \max_i(|B_i|)]$, where the B_i's are the blocks of G. Using Lemma 2.1 and subcriticallity one easily shows that $\max_i(|B_i|) \leq n^{\epsilon}$ a.a.s. with exponential rate. This implies that $D(G) \leq n^{1/2+\epsilon}$ a.a.s. with exponential rate.

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