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A Combinatorial Formula for Orthogonal Idempotents in the 0-Hecke Algebra of S_N

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Abstract. Building on the work of P.N. Norton, we give combinatorial formulae for two maximal decompositions of the identity into orthogonal idempotents in the 0-Hecke algebra of the symmetric group, $\mathbb{C}H_0(S_N)$. This construction is compatible with the branching from $H_0(S_{N-1})$ to $H_0(S_N)$.

Résumé. En s'appuyant sur le travail de P.N. Norton, nous donnons des formules combinatoires pour deux décompositions maximales de l'identité en idempotents orthogonaux dans l'algèbre de Hecke $H_0(S_N)$ du groupe symétrique à $q = 0$. Ces constructions sont compatibles avec le branchement de $H_0(S_{N-1})$ à $H_0(S_N)$.

Keywords: Iwahori-Hecke algebra, idempotents, semigroups, combinatorics, representation theory

1 Introduction

The 0-Hecke algebra $\mathbb{C}H_0(S_N)$ for the symmetric group S_N can be obtained as the Iwahori-Hecke algebra of the symmetric group $H_q(S_N)$ at $q = 0$. It can also be constructed as the algebra of the monoid generated by anti-sorting operators on permutations of N.

P.N. Norton described the full representation theory of $\mathbb{C}H_0(S_N)$ in Norton (1979): In brief, there is a collection of 2^{N-1} simple representations indexed by subsets of the usual generating set for the symmetric group, and an additional collection of 2^{N-1} projective indecomposable modules. Norton gave a construction for some elements generating these projective modules, but these elements were neither orthogonal nor idempotent. While it was known that an orthogonal collection of idempotents to generate the indecomposable modules exists, there was no known formula for these elements.

Herein, we describe an explicit construction for two different families of orthogonal idempotents in $\mathbb{C}H_0(S_N)$, one for each of the two orientations of the Dynkin diagram for S_N. The construction proceeds by creating a collection of 2^{N-1} *demipotent* elements, which we call *diagram demipotents*, each indexed by a copy of the Dynkin diagram with signs attached to each node. These elements are demipotent in the sense that for each element X, there exists some number $k \leq N-1$ such that X^j is idempotent for all $j \geq k$. The collection of idempotents thus obtained provides a maximal orthogonal decomposition of the identity.

An important feature of the 0-Hecke algebra is that it is the monoid algebra of a $\mathcal{J}$-trivial monoid. As a result, its representation theory is highly combinatorial. This paper is part of an ongoing effort with

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Florent Hivert, Anne Schilling, and Nicolas Thiéry to characterize the representation theory of general J-trivial monoids, continuing the work of Norton (1979), Hivert and Thiéry (2009), Hivert et al. (2009). The fundamentals of the representation theory of semigroups can be found in Ganyushkin et al. (2009). All proofs of statements in this paper will appear in Denton et al. (2010).

The diagram demipotents obey a branching rule which compares well to the situation in Okounkov and Vershik (1996) in their 'New Approach to the Representation Theory of the Symmetric Group.' In their construction, the branching rule for S_N is given primary importance, and yields a canonical basis for the irreducible modules for S_N which pull back to bases for irreducible modules for S_{N-M}.

Okounkov and Vershik further make extensive use of a maximal commutative algebra generated by the Jucys-Murphy elements. In the 0-Hecke algebra, their construction does not directly apply, because the deformation of Jucys-Murphy elements (which span a maximal commutative subalgebra of $\mathbb{C}S_N$) to the 0-Hecke algebra no longer commute. Instead, the idempotents obtained from the diagram demipotents play the role of the Jucys-Murphy elements, generating a commutative subalgebra of $\mathbb{C}H_0(S_N)$ and giving a natural decomposition into indecomposable modules, while the branching diagram describes the multiplicities of the irreducible modules.

The Okounkov-Vershik construction is well-known to extend to group algebras of general finite Coxeter groups (Ram (1997)). It remains to be seen whether our construction for orthogonal idempotents generalizes beyond type A. However, the existence of a process for type A gives hope that the Okounkov-Vershik process might extend to more general 0-Hecke algebras of Coxeter groups.

Section 2 establishes notation and describes the relevant background necessary for the rest of the paper. For further background information on the properties of the symmetric group, one can refer to the books of Humphreys (1990) and Stanley (1997). Section 3 gives the construction of the diagram demipotents. Section 4 describes the branching rule the diagram demipotents obey, and also establishes the Sibling Rivalry Lemma, which is useful in proving the main results, in Theorem 4.8. Section 5 establishes bounds on the power to which the diagram demipotents must be raised to obtain an idempotent. Finally, remaining questions are discussed in Section 6.

2 Background and Notation

Let S_N be the symmetric group defined by the generators s_i for $i \in I = \{1, \dots, N-1\}$ with the usual relations:

- Reflection: $s_i^2 = 1$,
- Commutation: $s_i s_j = s_j s_i$ for $|i - j| > 1$,
- Braid relation: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

The relations between distinct generators are encoded in the *Dynkin diagram* for S_N, which is a graph with one node for each generator s_i, and an edge between the pairs of nodes corresponding to generators s_i and s_{i+1} for each i. Here, an edge encodes the braid relation, and generators whose nodes are not connected by an edge commute. (See figure 3.)

Definition 2.1 *The* **0-Hecke monoid** $H_0(S_N)$ *is generated by the collection* π_i *for* i *in the set* $I = \{1, \dots, N-1\}$ *with relations:*

- *Idempotence:* $\pi_i^2 = \pi_i$,

- *Commutation:* $\pi_i \pi_j = \pi_j \pi_i$ *for* $|i-j| > 1$,
- *Braid Relation:* $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$.

The 0-Hecke monoid can be realized combinatorially as the collection of anti-sorting operators on permutations of N. For any permutation σ, $\pi_i \sigma = \sigma$ if $i+1$ comes before i in the one-line notation for σ, and $\pi_i \sigma = s_i \sigma$ otherwise.

Additionally, $\sigma \pi_i = \sigma s_i$ if the ith entry of σ is less than the $i+1$th entry, and $\sigma \pi_i = \sigma$ otherwise. (The left action of π_i is on *values*, and the right action is on *positions*.)

Definition 2.2 *The* 0**-Hecke algebra** $\mathbb{C}H_0(S_N)$ *is the monoid algebra of the* 0*-Hecke monoid.*

Words for S_N and $H_0(S_N)$ Elements. The set $I = \{1, \dots, N-1\}$ is called the *index set* for the Dynkin diagram. A *word* is a sequence $(i_1, \dots, i_k)$ of elements of the index set. To any word w we can associate a permutation $s_w = s_{i_1} \dots s_{i_k}$ and an element of the 0-Hecke monoid $\pi_w = \pi_{i_1} \cdots \pi_{i_k}$. A word w is *reduced* if its length is minimal amongst words with permutation s_w. The *length* of a permutation σ is equal to the length of a reduced word for σ.

Elements of the 0-Hecke monoid are indexed by permutations: Any reduced word $s = s_{i_1} \dots s_{i_k}$ for a permutation σ gives a reduced word in the 0-Hecke monoid, $\pi_{i_1} \cdots \pi_{i_k}$. Furthermore, given two reduced words w and v for a permutation σ, then w is related to v by a sequence of braid and commutation relations. These relations still hold in the 0-Hecke monoid, so $\pi_w = \pi_v$.

From this, we can see that the 0-Hecke monoid has $N!$ elements, and that the 0-Hecke algebra has dimension $N!$ as a vector space. Additionally, the length of a permutation is the same as the length of the associated $H_0(S_N)$ element.

We can obtain a *parabolic sub-object* (group, monoid, algebra) by considering the object whose generators are indexed by a subset $J \subset I$, retaining the relations of the original object. The Dynkin diagram of the corresponding object is obtained by deleting the relevant nodes and connecting edges from the original Dynkin diagram. Every parabolic subgroup of S_N contains a unique longest element, being an element whose length is maximal amongst all elements of the subgroup. We will denote the longest element in the parabolic sub-monoid of $H_0(S_N)$ with generators indexed by $J \subset I$ by w_J^+, and use $\hat{J}$ to denote the complement of J in I. For example, in $H_0(S_8)$ with $J = \{1, 2, 6\}$, then $w_J^+ = \pi_{1216}$, and $w_{\hat{J}}^+ = \pi_{3453437}$.

Definition 2.3 *An element* x *of a semigroup or algebra is* demipotent *if there exists some* k *such that* $x^\omega := x^k = x^{k+1}$. *A semigroup is* aperiodic *if every element is demipotent.*

The 0-Hecke monoid is aperiodic. Namely, for any element $x \in H_0(S_N)$, let:

$$J(x) = \{i \in I \mid \text{ s.t. } i \text{ appears in some reduced word for } x\}.$$

This set is well defined because if i appears in some reduced word for x, then it appears in every reduced word for x. Then $x^\omega = w_{J(x)}^+$.

The Algebra Automorphism Ψ of $\mathbb{C}H_0(S_N)$. $\mathbb{C}H_0(S_N)$ is alternatively generated as an algebra by elements $\pi_i^- := (1 - \pi_i)$, which satisfy the same relations as the π_i generators. There is a unique automorphism Ψ of $\mathbb{C}H_0(S_N)$ defined by sending $\pi_i \to (1 - \pi_i)$.

For any longest element w_J^+, the image $\Psi(w_J^+)$ is a longest element in the $(1 - \pi_i)$ generators; this element is denoted w_J^-.

The Dynkin diagram Automorphism of $\mathbb{C}H_0(S_N)$. A Dynkin diagram automorphism is a graph automorphism of the underlying graph. For the Dynkin diagram of S_N, there is exactly one non-trivial automorphism, sending the node i to $N-i$.

This diagram automorphism induces an automorphism of the symmetric group, sending the generator $s_i \to s_{N-i}$ and extending multiplicatively. Similarly, there is an automorphism of the 0-Hecke monoid sending the generator $\pi_i \to \pi_{N-i}$ and extending multiplicatively.

Bruhat Order. The *(left) weak order* on the set of permutations is defined by the relation $\sigma \leq \tau$ if there exist reduced words v, w such that $\sigma = s_v, \tau = s_w$, and v is a prefix of w in the sense that $w = v_1, v_2, \ldots, v_j, w_j+1, \ldots, w_k$. The right weak order is defined analogously, where v must appear as a suffix of w.

The left weak order also exists on the set of 0-Hecke monoid elements, with exactly the same definition. Indeed, $s_v \leq s_w$ if and only if $\pi_v \leq \pi_w$.

For a permutation σ, we say that i is a *(left) descent* of σ if $s_i\sigma \leq \sigma$. We can define a descent in the same way for any element π_w of the 0-Hecke monoid. We write $D_L(\sigma)$ and $D_L(\pi_w)$ for the set of all descents of σ and m respectively. Right descents are defined analogously, and are denoted $D_R(\sigma)$ and $D_R(\pi_w)$, respectively.

It is well known that i is a left descent of σ if and only if there exists a reduced word w for σ with $w_1 = i$. As a consequence, if $D_L(\pi_w) = J$, then $w_J^+\pi_w = \pi_w$. Likewise, i is a right descent if and only if there exists a reduced word for σ ending in i, and if $D_R(\pi_w) = J$, then $\pi_w w_J^+ = \pi_w$.

Bruhat order is defined by the relation $\sigma \leq \tau$ if there exist reduced words v and w such that $s_v = \sigma$ and $s_w = \tau$ and v appears as a subword of w. For example, 13 appears as a subword of 123, so $s_{12} \leq s_{123}$ in strong Bruhat order.

Representation Theory The representation theory of $\mathbb{C}H_0(S_N)$ was described in Norton (1979) and expanded to generic finite Coxeter groups in Carter (1986). A more general approach to the representation theory can be taken by approaching the 0-Hecke algebra as a semigroup algebra, as per Ganyushkin et al. (2009). The principal results are reproduced here for ease of reference.

For any subset $J \subset I$, let λ_J denote the one-dimensional representation of H defined by the action of the generators:

$$\lambda_J(\pi_i) = \begin{cases} 0 & \text{if } i \in J, \\ -1 & \text{if } i \notin J. \end{cases}$$

The λ_J are 2^{N-1} non-isomorphic representations, all one-dimensional and thus simple. In fact, these are all of the simple representations of $\mathbb{C}H_0(S_N)$.

The nilpotent radical $\mathcal{N}$ in $\mathbb{C}H_0(S_N)$ is spanned by elements of the form $x - w_{J(x)}^+$, where x is an element of the monoid $H_0(S_N)$, and $w_{J(x)}^+$ is the longest element in the parabolic submonoid whose generators are exactly the generators in any given reduced word for x. This element $w_{J(x)}^+$ is idempotent. If y is already idempotent, then $y = w_{J(y)}^+$, and so $y - w_{J(y)}^+ = 0$ contributes nothing to $\mathcal{N}$. However, all other elements $x - w_{J(x)}^+$ for x not idempotent are linearly independent, and thus give a basis of $\mathcal{N}$.

Norton further showed that

$$\mathbb{C}H_0(S_N) = \bigoplus_{J \subset I} H_0(S_N) w_J^- w_{\hat{J}}^+$$

is a direct sum decomposition of $\mathbb{C}H_0(S_N)$ into indecomposable left ideals.

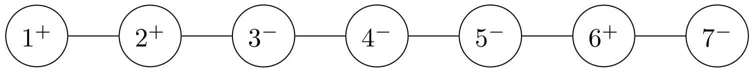

Fig. 1: A signed Dynkin diagram for S_8.

Theorem 2.4 (Norton, 1979) *Let $\{p_J | J \subset I\}$ be a set of mutually orthogonal primitive idempotents with $p_J \in \mathbb{C}H_0(S_N)w_J^- w_{\hat{J}}^+$ for all $J \subset I$ such that $\sum_{J\subset I} p_J = 1$.*

Then $\mathbb{C}H_0(S_N)w_J^- w_{\hat{J}}^+ = \mathbb{C}H_0(S_N)p_J$, and if $\mathcal{N}$ is the nilpotent radical of $\mathbb{C}H_0(S_N)$, $\mathcal{N}w_J^- w_{\hat{J}}^+ = \mathcal{N}p_J$ is the unique maximal left ideal of $\mathbb{C}H_0(S_N)p_J$, and $\mathbb{C}H_0(S_N)p_J/\mathcal{N}p_J$ affords the representation λ_J.

Finally, the commutative algebra $\mathbb{C}H_0(S_N)/\mathcal{N} = \bigoplus_{J\subset I} \mathbb{C}H_0(S_N)p_J/\mathcal{N}p_J = \mathbb{C}^{2^{N-1}}$.

The proof of this theorem is non-constructive, and does not give a formula for the idempotents.

3 Diagram Demipotents

The elements π_i and $(1 - \pi_i)$ are idempotent. There are actually 2^{N-1} idempotents in $H_0(S_N)$, namely the elements w_J^+ for any $J \subset I$. These idempotents are clearly not orthogonal, though.

The goal of this paper is to give a formula for a collection of *orthogonal* idempotents in $\mathbb{C}H_0(S_N)$.

Definition 3.1 *A* signed diagram *is a Dynkin diagram in which each vertex is labeled with a $+$ or $-$.*

Figure 3 depicts a signed diagram for type A_7, corresponding to $H_0(S_8)$. For brevity, a diagram can be written as just a string of signs. For example, the signed diagram in the Figure is written $++---+-$.

We now construct a *diagram demipotent* corresponding to each signed diagram. Let P be a composition of the index set I obtained from a signed diagram D by grouping together sets of adjacent pluses and minuses. For the diagram in Figure 3, we would have $P = \{\{1,2\},\{3,4,5\},\{6,7\}\}$. Let P_k denote the kth subset in P. For each P_k, let $w_{P_k}^{sgn(k)}$ be the longest element of the parabolic sub-monoid associated to the index set P_k, constructed with the generators π_i if $sgn(k) = +$ and constructed with the $(1-\pi_i)$ generators if $sgn(k) = -$.

Definition 3.2 *Let D be a signed diagram with associated composition $P = P_1 \cup \cdots \cup P_m$. Set:*

$$
\begin{aligned}
L_D &= w_{P_1}^{sgn(1)} w_{P_2}^{sgn(2)} \cdots w_{P_m}^{sgn(m)}, \text{ and} \\
R_D &= w_{P_m}^{sgn(m)} w_{P_{m-1}}^{sgn(m-1)} \cdots w_{P_1}^{sgn(1)}.
\end{aligned}
$$

The diagram demipotent *C_D associated to the signed diagram D is then $L_D R_D$. The* opposite diagram demipotent *C'_D is $R_D L_D$.*

Thus, the diagram demipotent for the diagram in Figure 3 is $\pi_{121}^+ \pi_{345343}^- \pi_6^+ \pi_7^- \pi_6^+ \pi_{345343}^- \pi_{121}^+$.

It is not immediately obvious that these elements are demipotent; this is a direct result of Lemma 4.4.

For $N = 1$, there is only the empty diagram, and the diagram demipotent is just the identity.

For $N = 2$, there are two diagrams, $+$ and $-$, and the two diagram demipotents are π_1 and $1 - \pi_1$ respectively. Notice that these form a decomposition of the identity, as $\pi_i + (1 - \pi_i) = 1$.

For $N = 3$, we have the following list of diagram demipotents. The first column gives the diagram, the second gives the element written as a product, and the third expands the element as a sum. For brevity, words in the π_i or π_i^- generators are written as strings in the subscripts. Thus, $\pi_1\pi_2$ is abbreviated to π_{12}.

D	C_D	Expanded Demipotent
$++$	π_{121}	π_{121}
$+-$	$\pi_1\pi_2^-\pi_1$	$\pi_1 - \pi_{121}$
$-+$	$\pi_1^-\pi_2\pi_1^-$	$\pi_2 - \pi_{12} - \pi_{21} + \pi_{121}$
$--$	π_{121}^-	$1 - \pi_1 - \pi_2 + \pi_{12} + \pi_{21} - \pi_{121}$

Observations.

- The idempotent π_{121}^- is an alternating sum over the monoid. This is a general phenomenon: By Norton (1979), w_J^- is the length-alternating signed sum over the elements of the parabolic sub-monoid with generators indexed by J.
- The shortest element in each expanded sum is an idempotent in the monoid with π_i generators; this is also a general phenomenon. The shortest term is just the product of longest elements in nonadjacent parabolic sub-monoids, and is thus idempotent. Then the shortest term of C_D is π_J^+, where J is the set of nodes in D marked with a $+$. Each diagram yields a different leading term, so we can immediately see that the 2^{N-1} idempotents in the monoid appear as a leading term for exactly one of the diagram demipotents, and that they are linearly independent.
- For many purposes, one only needs to explicitly compute half of the list of diagram demipotents; the other half can be obtained via the automorphism Ψ. A given diagram demipotent x is orthogonal to $\Psi(x)$, since one has left and right π_1 descents, and the other has left and right π_1^- descents, and $\pi_1\pi_1^- = 0$.
- The diagram demipotents C^D and C^E for $D \neq E$ do not necessarily commute. Non-commuting demipotents first arise with $N = 6$. However, the idempotents obtained from the demipotents are orthogonal and do commute.
- It should also be noted that these demipotents (and the resulting idempotents) are not in the projective modules constructed by Norton, but generate projective modules isomorphic to Norton's.
- The diagram demipotents C_D listed here are not fixed under the automorphism induced by the Dynkin diagram automorphism. In particular, the 'opposite' diagram demipotents $C'_D = R_D L_D$ really are different elements of the algebra, and yield an equally valid but different set of orthogonal idempotents. For purposes of comparison, the diagram demipotents for the reversed Dynkin diagram are listed below for $N = 3$.

D	C'_D	Expanded Demipotent
$++$	π_{212}	π_{212}
$+-$	$\pi_2\pi_1^-\pi_2$	$\pi_2 - \pi_{212}$
$-+$	$\pi_2^-\pi_1\pi_2^-$	$\pi_1 - \pi_{12} - \pi_{21} + \pi_{212}$
$--$	π_{212}^-	$1 - \pi_1 - \pi_2 + \pi_{12} + \pi_{21} - \pi_{212}$

For $N \leq 4$, the diagram demipotents are actually idempotent and orthogonal. For larger N, raising the diagram demipotent to a sufficiently large power yields an idempotent (see below 4.8); in other words, the diagram demipotents are demipotent. The power that an diagram demipotent must be raised to in order to obtain an actual idempotent is called its *nilpotence degree*.

For $N = 5$, two of the diagram demipotents need to be squared to obtain an idempotent. For $N = 6$, eight elements must be squared. For $N = 7$, there are four elements that must be cubed, and many others must be squared. Some pretty good upper bounds on the nilpotence degree of the diagram demipotents are given in Section 5. As a preview, for $N > 4$ the nilpotence degree is always $\leq N - 3$, and conditions on the diagram can often greatly reduce this bound.

As an alternative to raising the demipotent to some power, we can express the idempotents as a product of diagram demipotents for smaller diagrams. Let D_k be the signed diagram obtained by taking only the first k nodes of D. Then, as we will see, the idempotents can also be expressed as the product $C_{D_1}C_{D_2}C_{D_3}\cdots C_{D_{N-1}=D}$.

The following is an adaptation of a standard lemma for Coxeter groups to the 0-Hecke algebra, which yields triangularity of the diagram demipotents with respect to the weak order.

Lemma 3.3 *Let m be a standard basis element of the 0-Hecke algebra in the π_i basis. Then for any $i \in D_L(m)$, $\pi_i m = m$, and for any $i \notin D_L(m)$ then $\pi_i m$ is lower than m in left weak order.*

Corollary 3.4 (Diagram Demipotent Triangularity) *Let C_D be a diagram demipotent and m an element of the 0-Hecke monoid in the π_i generators. Then $C_D m = \lambda m + x$, where x is an element of H spanned by monoid elements lower in Bruhat order than m, and $\lambda \in \{0, 1\}$. Furthermore, $\lambda = 1$ if and only if $Des(m)$ is exactly the set of nodes in D marked with pluses.*

Theorem 3.5 *Each diagram demipotent is the sum of a non-zero idempotent part and a nilpotent part. That is, all eigenvalues of a diagram demipotent are either 1 or 0.*

4 Branching

There is a very convenient branching of the diagram demipotents for $H_0(S_N)$ into diagram demipotents for $H_0(S_{N+1})$.

Lemma 4.1 *Let $J = \{i, i+1, \ldots, N-1\}$ Then $w_J^+ \pi_N w_J^+$ is the longest element in the generators i through N. Likewise, $w_J^+ \pi_{i-1} w_J^+$ is the longest element in the generators $i-1$ through $N-1$. Similar statements hold for $w_J^- \pi_N^- w_J^-$ and $w_J^- \pi_{i-1}^- w_J^-$.*

The proof of this lemma relies on the formation of the longest words in the symmetric group; one can find an expression for the longest element in the generators $i-1$ through $N-1$ as a subword of the product $w_J^+ \pi_{i-1} w_J^+$. The result then follows immediately.

Recall that each diagram demipotent C_D is the product of two elements L_D and R_D. For a signed diagram D, let $D+$ indicate the diagram with an extra $+$ adjoined at the end. Define $D-$ analogously.

Corollary 4.2 *Let $C_D = L_D R_D$ be the diagram demipotent associated to the signed diagram D for S_N. Then $C_{D+} = L_D \pi_N R_D$ and $C_{D-} = L_D \pi_N^- R_D$. In particular, $C_{D+} + C_{D-} = C_D$.*

Corollary 4.3 *The sum of all diagram demipotents for $H_0(S_N)$ is the identity.*

Next we have a key lemma for proving many of the remaining results in this paper:

Lemma 4.4 (Sibling Rivalry) *Sibling diagram demipotents commute and are orthogonal: $C_{D-}C_{D+} = C_{D+}C_{D-} = 0$. Equivalently, $C_D C_{D+} = C_{D+}C_D = C_{D+}^2$ and $C_D C_{D-} = C_{D-}C_D = C_{D-}^2$.*

The proof uses induction on the tree of diagram demipotents, checking four different cases depending on the last two entries of the diagram D. In particular, it is directly checked that $C_{D+++}C_{D++} = C^2_{D+++}$, and $C_{D+-+}C_{D+-} = C^2_{D+-+}$; all other cases and statements follow from symmetry or application of the automorphism Ψ. The first of these calculations, $C_{D+++}C_{D++} = C^2_{D+++}$, is quite instructive.

Corollary 4.5 *The diagram demipotents C_D are demipotent.*

This follows immediately by induction: if $C^k_D = C^{k+1}_D$, then $C_{D+}C^k_D = C_{D+}C^{k+1}_D$, and by sibling rivalry, $C^{k+1}_{D+} = C^{k+2}_{D+}$.

Now we can say a bit more about the structure of the diagram demipotents.

Proposition 4.6 *Let $p = C_D, x = C_{D+}, y = C_{D-}$, so $p = x + y$ and $xy = 0$. Let v be an element of H. Furthermore, let p, x, and y have abstract Jordan decomposition $p = p_i + p_n$, $x = x_i + x_n$, $y = y_i + y_n$, with $p_ip_n = p_np_i$ and $p_i^2 = p_i$, $p_n^k = 0$ for some k, and similar relations for the Jordan decompositions of x and y.*

Then we have the following relations:

1. *If there exists k such that $p^kv = 0$, then $x^{k+1}v = y^{k+1}v = 0$.*
2. *If $pv = v$, then $x(x-1)v = 0$*
3. *If $(x-1)^kv = 0$, then $(x-1)v = 0$*
4. *If $pv = v$ and $x^kv = 0$ for some k, then $yv = v$.*
5. *If $xv = v$, then $yv = 0$ and $pv = v$.*
6. *Let u_i^x be a basis of the 1-space of x, so that $xu_i^x = u_i^x$, $yu_i^x = 0$ and $pu_i^x = v$, and u_j^y a basis of the 1-space of y. Then the collection $\{u_i^x, u_j^y\}$ is a basis for the 1-space of p.*
7. *$p_i = x_i + y_i$, $p_n = x_n + y_n$, $x_iy_i = 0$.*

The proof follows mainly from simple algebraic manipulations.

Corollary 4.7 *There exists a linear basis v_D^j of $\mathbb{C}H_0(S_N)$, indexed by a signed diagram D and some numbers j, such that the idempotent I_D obtained from the abstract Jordan decomposition of C_D fixes every v_D^j. For every signed diagram $E \neq D$, the idempotent I_E kills v_D^j.*

The proof of the corollary further shows that this basis respects the branching from $H_0(S_{N-1})$ to $H_0(S_N)$. In particular, finding this linear basis for $H_0(S_N)$ allows the easy recovery of the bases for the indecomposable modules for any $M < N$.

We now state the main result. For D a signed diagram, let D_i be the signed sub-diagram consisting of the first i entries of D.

Theorem 4.8 *Each diagram demipotent C_D (see Definition 3.2) for $H_0(S_N)$ is demipotent, and yields an idempotent $I_D = C_{D_1}C_{D_2}\cdots C_D = C_D^N$. The collection of these idempotents $\{I_D\}$ form an orthogonal set of primitive idempotents that sum to 1.*

This follows from the previous result and the construction of the diagram demipotents.

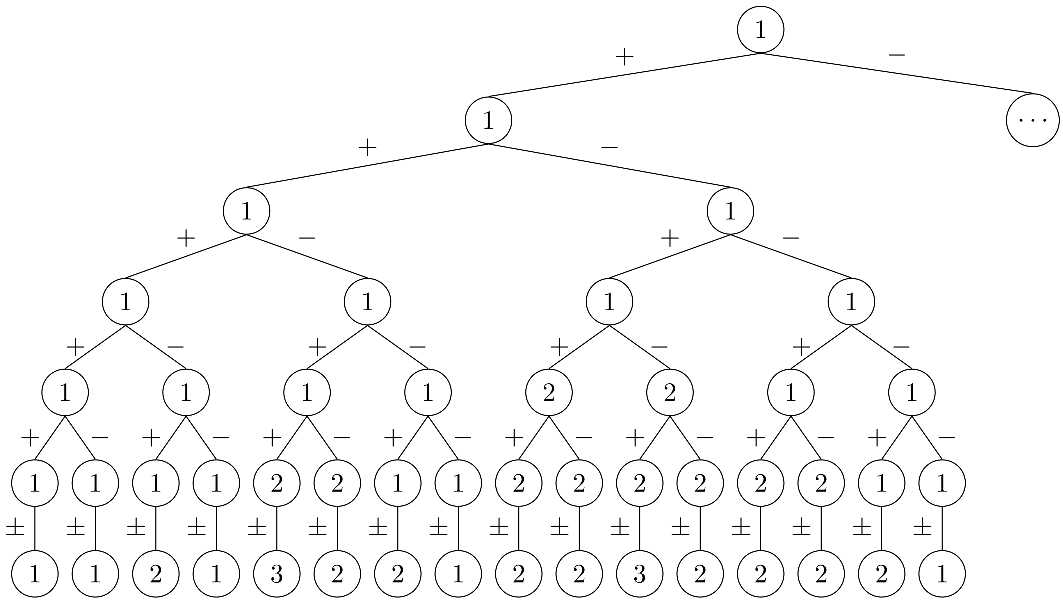

Fig. 2: Nilpotence degree of diagram demipotents. The root node denotes the diagram demipotent with empty diagram (the identity). Since sibling diagram demipotents have the same nilpotence degree, the lowest row has been abbreviated for readability.

5 Nilpotence Degree of Diagram Demipotents

Take any m in the 0-Hecke monoid whose descent set is exactly the set of positive nodes in the signed diagram D. Then $C_D m = m +$ (lower order terms), by a previous lemma, and $I_D m = (C_D)^k(m) = m +$ (lower order terms). The set $\{I_D m | Des(m) = \{\text{positive nodes in D}\}\}$ is thus linearly independent in $H_0(S_N)$, and gives a basis for the projective module corresponding to the idempotent I_D.

We have shown that for any diagram demipotent C_D, there exists a minimal integer k such that $(C_D)^k$ is idempotent. Call k the *nilpotence degree* of C_D. The nilpotence degree of all diagram demipotents for $N \leq 7$ is summarized in Figure 5.

The diagram demipotent $C_{+\dots+}$ with all nodes positive is given by the longest word in the 0-Hecke monoid, and is thus already idempotent. The same is true of the diagram demipotent $C_{-\dots-}$ with all nodes negative. As such, both of these elements have nilpotence degree 1.

The following Lemma is easily proved.

Lemma 5.1 *The nilpotence degree of sibling diagram demipotents C_{D+} and C_{D-} is at most one more than the nilpotence degree of the parent C_D. If the nilpotence degree of one sibling is less than or equal to the nilpotence degree of the parent, then the nilpotence degree of the other sibling is equal to the nilpotence degree of the parent.*

Lemma 5.2 *Let D be a signed diagram with a single sign change, or the sibling of such a diagram. Then C_D is idempotent (and thus has nilpotence degree* 1*).*

In particular, this lemma is enough to see why there is no nilpotence before $N = 5$; every signed Dynkin diagrams with three or fewer nodes has no sign change, one sign change, or is the sibling of a diagram with one sign change.

Theorem 5.3 *Let D be any signed diagram with n nodes, and let E be the largest prefix diagram such that E has a single sign change, or is the sibling of a diagram with a single sign change. Then if E has k nodes, the nilpotence degree of D is at most $n - k$.*

This result follows directly from the previous lemma and the fact that the nilpotence degree can increase by at most one with each branching.

This bound is not quite sharp for $H_0(S_N)$ with $N \leq 7$: The diagrams $+ - ++$, $+ - + + +$, and $+ - + + ++$ all have nilpotence degree 2. However, at $N = 7$, the highest expected nilpotence degree is 3 (since every diagram demipotent with three or fewer nodes is idempotent), and this degree is attained by 4 of the demipotents. These diagram demipotents are $+ + - + ++$, $+ - + - ++$, and their siblings.

An open problem is to find a formula for the nilpotence degree directly in terms of the diagram of a demipotent.

6 Remaining Questions

A number of questions still remain.

1. We conjecture that the diagram demipotents C_D have ± 1 coefficients when expanded over $\mathbb{C}$, as this holds for all of the diagram demipotents for $N \leq 8$.

2. Problem: Express the nilpotence degree of C_D as a function of the signed diagram D.

3. Problem: Extend the construction for the idempotents to a more general construction applicable to the 0-Hecke algebra of a general Coxeter group, or, even better, general $\mathcal{J}$-Trivial monoids. The key properties of the idempotents constructed in this paper are construction via a branching rule and invariance of the set of idempotents under the automorphism Ψ; one hopes that a more general construction would retain these properties. One of the impediments to extending to other Coxeter groups is that Lemma 4.1 does not hold for any families of finite Coxeter groups other than S_N, suggesting that other methods of branching must be found.

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