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Primitive orthogonal idempotents for R-trivial monoids

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Abstract. We construct a recursive formula for a complete system of primitive orthogonal idempotents for any R-trivial monoid. This uses the newly proved equivalence between the notions of R-trivial monoid and weakly ordered monoid.

Résumé. Nous construisons une formule récursive pour un système complet d'idempotents orthogonaux primitifs pour tout monoïde R-trivial. Nous employons une nouvelle équivalence entre les notions de monoïde R-trivial et de monoïde faiblement ordonné.

Keywords: monoids, primitive orthogonal idempotents, 0-Hecke algebras, left regular bands

1 Introduction

Recently, Denton ([6], [7]) gave a formula for a complete system of primitive orthogonal idempotents for the 0-*Hecke algebra* of type A, the first since the question was posed by Norton [9] in 1979. A complete system of primitive orthogonal idempotents for *left regular bands* was found by Brown [5] and Saliola [12]. Finding such collections is an important problem in representation theory because they decompose an algebra into projective indecomposable modules: if $\{e_J\}_{J\in\mathfrak{J}}$ is such a collection for a finite dimensional algebra A, then $A = \oplus_{J\in\mathfrak{J}} Ae_J$ for indecomposable modules Ae_J. They also allow for the explicit computation of the quiver, the Cartan invariants, and the Wedderburn decomposition of the algebra (see [4], [2]). For example, in [8], Denton, Hivert, Schilling, and Thiéry use a construction of a system of idempotents for any J-trivial monoid M to derive combinatorially the Cartan matrix and quiver of M.

Schocker [13] constructed a class of monoids, called *weakly ordered monoids*, to generalize 0-Hecke monoids and left regular bands, with the broader aim of finding a complete system of orthogonal idempotents for the corresponding monoid algebras. We realize this goal here.

A key step in being able to do so is recognizing that the notions of weakly ordered monoid and *R-trivial monoid* are one and the same. This was first pointed out to us by Nicolas M. Thiéry [17] after an intense

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discussion between the authors and Denton, Hivert, Schilling, and Thiéry. In Section 2, we fill out an outline of a proof provided by Steinberg [16], who independently made this same observation. In Section 3, we use this equivalence to construct a complete system of primitive orthogonal idempotents.

2 Weakly ordered monoids and R-trivial monoids

Given any monoid T, that is, a set with an associative multiplication and an identity element, we define a preorder $\leq$ as follows. Given $u, v \in T$, write $u \leq v$ if there exists $w \in T$ such that $uw = v$. We write $u < v$ if $u \leq v$ but $u \neq v$. Unless stated otherwise, the monoids throughout the paper are endowed with this "weak" preorder. (In the semigroup theory literature, the *dual* of this preorder is known as *Green's R-preorder*.)

Definition 2.1 *A finite monoid W is said to be a* **weakly ordered monoid** *if there is a finite upper semi-lattice $(\mathcal{L}, \preceq)$ together with two maps $C, D : W \to \mathcal{L}$ satisfying the following axioms.*

1. *C is a surjection of monoids.*
2. *If $u, v \in W$ are such that $uv \leq u$, then $C(v) \preceq D(u)$.*
3. *If $u, v \in W$ are such that $C(v) \preceq D(u)$, then $uv = u$.*

Remark 2.2 *This notion was introduced by Schocker [13] to generalize 0-Hecke monoids and left regular bands, with the broader aim of finding a complete system of orthogonal idempotents for the corresponding monoid algebras. In his paper, he actually calls these weakly ordered semigroups. However our understanding is that monoids include an identity element and semigroups do not. So throughout the paper we call these weakly ordered monoids.*

Definition 2.3 *A monoid S is* ***R*-trivial** *if, for all $x, y \in S$, $xS = yS$ implies $x = y$. It is easy to see that a monoid S is R-trivial if and only if the preorder $\leq$ defined above is a partial order.*

We restrict our discussion to *finite* R-trivial monoids.

Example 2.4 A monoid W is called a **left regular band** if $x^2 = x$ and $xyx = xy$ for all $x, y \in W$. Left-regular bands are R-trivial. Indeed, if $xW = yW$, then there exist $u, v \in W$ such that $xu = y$ and $x = yv$. But then, since $uv = uvu$,

$$x = yv = xuv = xuvu = yvu = xu = y.$$

Finitely generated left regular bands are also weakly ordered monoids, see Shocker [13], e.g. 2.4 and Brown [5], Appendix B.

Example 2.5 Let G be a Coxeter group with simple generators $\{s_i : i \in I\}$ and relations:

- $s_i^2 = 1$,
- $\underbrace{s_i s_j s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j s_i \cdots}_{m_{ij}}$ for positive integers m_{ij}.

Then the **0-Hecke monoid** $H^G(0)$ has generators $\{T_i : i \in I\}$ and relations:

- $T_i^2 = T_i$,
- $\underbrace{T_iT_jT_iT_j\cdots}_{m_{ij}} = \underbrace{T_jT_iT_jT_i\cdots}_{m_{ij}}$ for positive integers m_{ij}.

Of particular interest is the case when G is the symmetric group $\mathfrak{S}_n$. Norton [9] gave a decomposition of the monoid algebra $\mathbb{C}H^{\mathfrak{S}_n}(0)$ into left ideals and classified its irreducible representations. She raised the question of constructing a complete system of orthogonal idempotents for the algebra. Denton [6] gave the first construction of a set of orthogonal idempotents for $\mathbb{C}H^{\mathfrak{S}_n}(0)$.

The weakly ordered monoid $H^{\mathfrak{S}_n}(0)$ has maps C and D onto the lattice of subsets of $\{1,\ldots,n-1\}$. The map C is the *content set* of an element: $C(T_{i_1}T_{i_2}\cdots T_{i_k}) = \{i_1,i_2,\ldots,i_k\}$. The map D is the subset of right descents of an element: $D(x) = \{i \in \{1,\ldots,n-1\} : xT_i = x\}$. Note that the preorder for this monoid coincides with the weak order on the elements of the Coxeter group.

Example 2.6 Let S be the monoid with identity generated by the following matrices.

$$g_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad g_2 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $S = \{1,\ g_1,\ g_2,\ g_1g_2,\ g_2g_1\}$ and S is both an R-trivial monoid and a weakly ordered monoid. For example, we can take $\mathcal{L}$ be to be usual lattice of subsets of $\{1,2\}$, with $C : S \to \mathcal{L}$ given by

$$C(1) = \emptyset,\ C(g_1) = \{1\},\ C(g_2) = \{2\},\ C(g_1g_2) = C(g_2g_1) = \{1,2\},$$

and $D : S \to \mathcal{L}$ given by

$$D(1) = \emptyset,\ D(g_1) = \{1\},\ D(g_2) = D(g_1g_2) = \{2\},\ D(g_2g_1) = \{1,2\}.$$

The monoid S, however, is neither a left regular band, since g_1g_2 is not idempotent, nor isomorphic to the 0-Hecke monoid $H^{\mathfrak{S}_3}(0)$ on two generators, since the latter has six elements.

The fact that the above examples are both weakly ordered and R-trivial is no coincidence: the purpose of this section is to show that these two notions are equivalent.

Remark 2.7 *A weakly ordered monoid is an R-trivial monoid. Indeed, if W is a weakly ordered monoid, then Lemma* 2.1 *in [13] shows that the defining conditions of a weakly ordered monoid imply that the preorder on W is a partial order (see Definition 2.3).*

We will show that any finite R-trivial monoid S is a weakly ordered monoid using an argument outlined by Steinberg [16]. We must establish the existence of an upper semi-lattice $\mathcal{L}$ and two maps C and D from S to $\mathcal{L}$ that satisfy the conditions of Definition 2.1. We gather here the definitions of $\mathcal{L}$, C and D:

1. $\mathcal{L}$ is the set of left ideals Se generated by idempotents e in S, ordered by reverse inclusion;
2. $C : S \to \mathcal{L}$ is defined as $C(x) = Sx^\omega$, where x^ω is the idempotent power of x;
3. $D : S \to \mathcal{L}$ is defined as $D(u) = C(e)$, where e is a maximal element in the set $\{s \in S : us = u\}$ (with respect to the preorder $\leq$).

The remainder of this section is dedicated to showing that these objects are well-defined and that they satisfy the conditions of Definition 2.1. We begin by recalling some classical results from the semigroup literature. The following is [10, Proposition 6.1].

Lemma 2.8 *If S is a finite semigroup, then for each $x \in S$, there exists a positive integer $\omega = \omega(x)$ such that x^ω is idempotent, i.e. $(x^\omega)^2 = x^\omega$. Furthermore, if S is R-trivial, then we also have $x^\omega x = x^\omega$.*

Proof: Consider the elements $x, x^2, x^3, \ldots$. Since S is finite, there exists positive integers i and p such that $x^{i+p} = x^i$. Then $x^{k+p} = x^k$ for all $k \geq i$, so if we take $\omega = ip$, then $(x^\omega)^2 = x^{\omega+ip} = x^\omega$.

If S is R-trivial, then $x^\omega \leq x^\omega x \leq x^\omega x^\omega = x^\omega$, and so $x^\omega x = x^\omega$. □

Remark 2.9 *In what follows, if $x \in \mathbb{C}S$ and there exists an N such that $x^{N+1} = x^N$, we sometimes abuse notation by writing x^ω in place of x^N.*

We are now ready to construct a lattice corresponding to the R-trivial monoid S. Define

$$\mathcal{L} := \{Se \,:\, e \in S \text{ such that } e^2 = e\}.$$

That is, $\mathcal{L}$ is the set of left ideals generated by the idempotents of S. Define a partial order on $\mathcal{L}$ by

$$Se \preceq Sf \iff Se \supseteq Sf.$$

Proposition 2.10 *If e, f are idempotents in S, then $S(ef)^\omega$ is the least upper bound of Se and Sf in $\mathcal{L}$.*

Remark 2.11 *A fully detailed and elementary proof of this result for R-trivial monoids can be found in [3], although the motivated reader can deduce this from the above results and definitions. This is a special case of more general results in the semigroup theory literature. For example, it follows by restricting a result of Schützenberger to R-trivial monoids [14]. For a detailed discussion within the context of the representation theory of finite monoids, see [1] and [8].*

As a result, we may define the join of two elements Se and Sf in $\mathcal{L}$ by

$$Se \vee Sf = S(ef)^\omega.$$

That is, $\mathcal{L}$ is an upper semilattice with respect to this join operation. This observation proves the following.

Proposition 2.12 *The map $C : S \to \mathcal{L}$ defined by $C(x) = Sx^\omega$ is a surjective monoid morphism.*

Here is an alternate and useful characterization of $C(x)$.

Proposition 2.13 $C(x) = \{a \in S \,:\, ax = a\}$ *for all* $x \in S$.

Proof: Take an arbitrary element in $C(x) = Sx^\omega$, say tx^ω. Since $\big(tx^\omega\big)x = t\big(x^\omega x\big) = tx^\omega$ by Lemma 2.8, we see that $tx^\omega \in \{a \in S \,:\, ax = a\}$. On the other hand, take $b \in \{a \in S \,:\, ax = a\}$. Then

$$bx^\omega = (bx)x^{\omega-1} = bx^{\omega-1} = (bx)x^{\omega-2} = bx^{\omega-2} = \cdots = bx = b.$$

Therefore, $b \in Sx^\omega$. □

We now define the map $D : S \to \mathcal{L}$. Given $u \in S$, let $D(u) = C(e)$, where e is a maximal element in the set $\{s \in S \,:\, us = u\}$. To check that D is well-defined, let e and f be two distinct maximal elements in $\{s \in S \ : \ us = u\}$. Since $e \leq ef$ and $u(ef) = (ue)f = uf = u$, by the maximality of e, $e = ef$. Similarly, since $f \leq fe$ and $u(fe) = u$, the maximality of f implies $f = fe$. Then, by Proposition 2.12,

$$C(e) = C(ef) = C(e) \vee C(f) = C(f) \vee C(e) = C(fe) = C(f).$$

Note that the maximality of e and $ue^2 = u$ also implies that $e = e^2$, that is, e is idempotent.

The next proposition shows that the maps C and D interact in precisely the manner given in conditions 2 and 3 in Definition 2.1. The following lemma will help us prove this proposition.

Lemma 2.14 *Let $x, y \in S$. If $x \leq y$, then $C(x) \preceq C(y)$.*

Proof: If $s \in C(y)$, then $sy = s$. Since $x \leq y$, there exists $t \in S$ such that $y = xt$. So $sxt = s$, implying $sx \leq s$. That is, $s \in C(x)$. Hence $C(y) \subseteq C(x)$, or $C(x) \preceq C(y)$ since $s \leq sx$ and S is R-trivial. □

Proposition 2.15 *Let $u, v \in S$. (i) If $uv \leq u$, then $C(v) \preceq D(u)$. (ii) If $C(v) \preceq D(u)$, then $uv = u$.*

Proof: (i) Since $u \leq uv$, $u = uv$. Hence v lies in the set $\{s \in S \ : \ us = u\}$. Let e be a maximal element in this set such that $v \leq e$. Then, by Lemma 2.14, $C(v) \preceq C(e) = D(u)$.

(ii) By definition, $D(u) = C(e)$, where e is a maximal element of $\{s \in S \ : \ us = u\}$. So if $C(v) \preceq D(u)$, then $C(v) \preceq C(e)$. Hence $C(e) \subseteq C(v)$. Since $ue = u$, u lies in $C(e)$. So u is also a member of $C(v)$; that is, $uv = u$. □

Propositions 2.12 and 2.15 tell us that an R-trivial monoid is a weakly ordered monoid. Combining this with Corollary 2.7, we have the following result.

Theorem 2.16 *A monoid W is a weakly ordered monoid if and only if it is an R-trivial monoid.*

3 Constructing idempotents

We begin this section with a small technical lemma about R-trivial monoids. The proof is rather trivial, but we use it often enough in proofs to justify stating it at the onset.

Lemma 3.1 *Suppose W is an R-trivial monoid. If $x, y, z \in W$ are such that $xyz = x$, then $xy = x$. Consequently, if $x, y_1, y_2, \ldots, y_m \in W$ are such that $xy_1 \cdots y_m = x$, then $xy_i = x$ for all $1 \leq i \leq m$.*

Proof: If $xyz = x$ then $xyW = xW$. Therefore $xy = x$ by the definition of W being R-trivial. The second statement immediately follows from the first. □

Definition 3.2 *Let A be a finite dimensional algebra with identity* 1. *We say that a set of nonzero elements $\Lambda = \{e_J : J \in \mathcal{I}\}$ of A is a* **complete system of primitive orthogonal idempotents for** *A if:*

1. *each e_J is* idempotent*: that is, $e_J^2 = e_J$ for all $J \in \mathcal{I}$;*
2. *the e_J are pairwise* orthogonal*: that is, $e_J e_K = 0$ for $J, K \in \mathcal{I}$ with $J \neq K$;*
3. *each e_J is* primitive *(meaning that it cannot be further decomposed into orthogonal idempotents): if $e_J = x + y$ with x and y orthogonal idempotents in A, then $x = 0$ or $y = 0$;*

4. $\{e_J : J \in \mathcal{I}\}$ *is* complete *(meaning that the elements sum to the identity):* $\sum_{J\in\mathcal{I}} e_J = 1$.

Remark 3.3 *If Λ is a* maximal *set of nonzero elements satisfying conditions 1 and 2, then Λ is a complete system of primitive orthogonal idempotents (that is, 3 and 4 also hold). Indeed, e_J is primitive, for if e_J could be written as $x+y$, then we could replace e_J in Λ with x and y, contradicting the maximality of Λ. To see 4, we just note that if $\sum_K e_K \neq 1$, then $1-\sum_K e_K$ is idempotent and orthogonal to all other e_K. Combining this element with Λ would again contradict the maximality of Λ.*

Let W denote a weakly ordered monoid with C and D being the associated "content" and "descent" maps from W to an upper semi-lattice $\mathcal{L}$. We let $\mathcal{G}$ denote a set of generators of W. The main goal of this paper is to build a method for finding a complete system of orthogonal idempotents for the monoid algebra $\mathbb{C}W$. In particular, this solves the problem posed by Norton about the 0-Hecke algebra for the symmetric group.

For each $J \in \mathcal{L}$, we define a **Norton element** $A_J T_J$. Let us begin by defining T_J:

$$T_J = \Big(\prod_{\substack{g\in\mathcal{G}\\ C(g)\preceq J}} g^{\omega} \Big)^{\omega} \in W.$$

Remark 3.4 *A different ordering of the set $\mathcal{G}$ of generators may produce different T_J's; so we fix an (arbitrarily chosen) order.*

We now define the A_J in the Norton element $A_J T_J$. First we let

$$B_J = \prod_{\substack{g\in\mathcal{G}\\ C(g)\not\preceq J}} (1-g^{\omega}) \in \mathbb{C}W.$$

In the spirit of Lemma 2.8, we would like to raise B_J to a sufficiently high power so that it is idempotent. However, B_J is not an element of the monoid W, so $(B_J)^{\omega}$ may not be well defined. The following lemma and corollary resolve this problem.

Definition 3.5 *Given $x = \sum_{w\in W} c_w w \in \mathbb{C}W$, the **coefficient** of w in x is c_w. We say w is a **term** of x if the coefficient of w in x is nonzero.*

Lemma 3.6 *Let $b \in W$ and suppose $bx^{\omega} = b$ for some $x \in \mathcal{G}$ with $C(x) \not\preceq J$. If c is a term of bB_J, then $c > b$.*

Proof: Let $\mathcal{D} = \{x^{\omega} : x \in \mathcal{G}, C(x) \not\preceq J, bx^{\omega} = b\}$. By assumption $\mathcal{D}$ is not empty. Let $g_1, g_2, \ldots, g_m$ be the generators which appear in the definition of B_J. Then

$$B_J = \sum_{i_1<i_2<\cdots<i_k} (-1)^k g_{i_1}^{\omega} g_{i_2}^{\omega} \cdots g_{i_k}^{\omega}.$$

It follows from Lemma 3.1 that the coefficient of b in bB_J is counting the terms in B_J where each of $g_{i_1}, \ldots, g_{i_k}$ come from $\mathcal{D}$, weighted with sign $(-1)^k$. If $|\mathcal{D}| = m \geq 1$ then this is $1 - m + \binom{m}{2} - \binom{m}{3} + \cdots + (-1)^m = 0$. Therefore $c \neq b$. The statement now follows from the definition of order, as every term c of bB_J must be of the form $c = bz$ for some term z appearing in B_J, and hence $c \geq b$. $\square$

Lemma 3.7 *For every $J \in \mathcal{L}$, there exists an integer N such that $y^\omega B_J^N = 0$ for all $y \in \mathcal{G}$ with $C(y) \not\preceq J$.*

Proof: Let $N = \ell + 1$, where ℓ is the length of the longest chain of elements in the poset $(W, \leq)$.

Suppose $y^\omega B_J^N \neq 0$. Let c_N be a term of B_J^N. Then c_N is a term of $c_{N-1}B_J$ for some term c_{N-1} in $y^\omega B_J^{N-1}$. Since $y^\omega y^\omega = y^\omega$, Lemma 3.6 implies that y^ω is not a term of $y^\omega B_J^k$ for any $k \geq 1$, so that $c_{N-1} = y^\omega g_1^\omega \cdots g_m^\omega$ for some $m \geq 1$ and $g_i \in \mathcal{G}$ with $C(g_i) \not\preceq J$. In particular, $c_{N-1}g_m^\omega = c_{N-1}$, and so, again by Lemma 3.6, $c_N > c_{N-1}$. Repeated application of this argument produces a decreasing chain

$$c_N > c_{N-1} > c_{N-2} > \cdots > c_1$$

of elements in W, contradicting the fact that the length of the longest chain of elements in $(W, \leq)$ is ℓ. □

Corollary 3.8 *For every $J \in \mathcal{L}$ there exists an N such that $B_J^{N+1} = B_J^N$.*

Proof: By Lemma 3.7, $(B_J - 1)B_J^N = 0$ for a sufficiently large N since every element of $B_J - 1$ is of the form αy^ω where $\alpha \in \mathbb{C}$, $y \in \mathcal{G}$ and $C(y) \not\preceq J$. □

This now allows us to define $A_J = B_J^\omega$.

Lemma 3.9 *Let $J \in \mathcal{L}$. Then:*

1. *$T_J x = T_J$ for all x such that $C(x) \preceq J$;*
2. *$y^\omega A_J = 0$ for all y such that $C(y) \not\preceq J$ and $y \in \mathcal{G}$.*

Proof: Since $J = C(T_J)$, $C(x) \preceq J$ implies $C(x) \supseteq C(T_J)$. We also know that $T_J \in C(T_J)$ because T_J is idempotent. So $T_J \in C(x)$, that is, $T_J x = T_J$.

The second part follows from Lemma 3.7 since $A = B^N$. □

Remark 3.10 *Although T_J and A_J are idempotents individually, their product, the Norton element z_J, need not be. For example, take the 0-Hecke algebra $H_6(0)$ corresponding to the symmetric group $\mathfrak{S}_6$. Let J be the subset $\{1,4,5\}$ of $\{1,2,3,4,5\}$. Then $T_J = T_1T_4T_5T_4$, $A_J = (1-T_2)(1-T_3)(1-T_2)$ and z_J is their product. No power of z_J is idempotent.*

Lemma 3.11 *The coefficient of T_J in $z_J = A_J T_J$ is 1. All other terms y in z_J have $C(y) \succ J$.*

Proof: The coefficient of the identity element 1 in A_J is 1. Each term of $A_J T_J$ is of the form aT_J for a term a of A_J. If $a \neq 1$, then $C(a) \not\preceq J$ so $C(aT_J) = C(a) \vee C(T_J) \succ C(T_J) = J$. Hence the coefficient of T_J in $A_J T_J$ is 1 and all other terms have content greater than J. □

Lemma 3.12 *If $J \not\preceq K$ then $z_J z_K = 0$.*

Proof: Since $J \not\preceq K$, there exists a $g \in \mathcal{G}$ with $C(g) \preceq J$ but $C(g) \not\preceq K$. Then, using Lemma 3.9 (1) and Lemma 3.9 (2), $z_J z_K = A_J T_J A_K T_K = A_J(T_J g^\omega)A_K T_K = A_J T_J(g^\omega A_K)T_K = 0$. □

Lemma 3.13 *For all $J \in \mathcal{L}$, there exists an N such that $(1 - z_J)^N z_J^2 = 0$.*

Proof (Outline): The proof is somewhat involved, so we only include an outline of the main argument here. A complete and detailed proof can be found in [3]. To simplify the notation, we temporarily set $T = T_J$, $A = A_J$ and $z = AT$. Fisrt note that $(1-z)^k z^2 = A(T(1-A)T)^k AT$. The idea is to argue that $(T(1-A)T)^N A = 0$ for N larger than the length of the largest chain in $(W, \leq)$.

Let $\mathcal{A}$ be the set of terms in $1 - A$. Every term of $(T(1-A)T)^N$ is of the form $Ta_1Ta_2T\cdots a_NT$ with $a_i \in \mathcal{A}$. If we write $x_i = Ta_1Ta_2T\cdots a_iT$, then in the R-order we have $x_1 \leq x_2 \leq \cdots \leq x_N$. For some i we must have $x_i = x_{i+1}$, so by Lemma 3.1, $x_i = x_i a_{i+1}$. This implies that $x_i(1-A)T = x_i a_{i+1}(1-A)T = x_i T = x_i$, from which it follows that $x_i A = 0$. □

Definition 3.14 *Let $J \in \mathcal{L}$. Let*

$$P_J := \sum_{n,m\geq 0} (1-z_J)^{n+m}\, z_J^2 \;=\; \sum_{k\geq 0} (k+1)\,(1-z_J)^k\, z_J^2.$$

(In Remark 3.20 we establish a summation-free formula for P_J.)

Remark 3.15 *Lemma 3.13 shows there are only finitely many terms in the summation of P_J. Therefore P_J is a well-defined element of $\mathbb{C}W$ for each $J \in \mathcal{L}$.*

Remark 3.16 *A monoid S is called J-trivial if $SxS = SyS$ implies $x = y$ for all $x, y \in S$. When S is J-trivial it suffices to define*

$$P_K = \sum_{n\geq 0} (1-z_K)^n z_K.$$

Lemma 3.17 *The coefficient of T_J in P_J is 1 and all other terms y of P_J have $C(y) \succ J$.*

Proof: If $n + m > 0$ then, using that T_J is idempotent,

$$A_J T_J A_J T_J (1 - A_J T_J)^{n+m} = A_J T_J A_J (T_J - T_J A_J T_J)^{n+m}.$$

Each term x in $(T_J - T_J A_J T_J)^{n+m}$ has $C(x) \succ J$, so no T_J appears in $z_J^2(1-z_J)^{n+m}$. The coefficient of T_J in z_J is 1, by Lemma 3.11. Hence T_J appears in $z_J^2(1-z_J)^0$ with coefficient 1. By Lemma 3.11, since all of the terms $y \neq T_J$ of z_J have $C(y) \succ J$ and P_J is a polynomial in z_J, all other terms w of P_J must have $C(w) \succ J$. □

Remark 3.18 *As polynomials in x we have for any nonnegative integer N:*

$$x \sum_{n=0}^{N} (1-x)^n = 1 - (1-x)^{N+1}.$$

Proposition 3.19 *For each $J \in \mathcal{L}$, the element P_J is idempotent.*

Proof: Let $J \in \mathcal{L}$ be fixed and let N be such that $(1 - z_J)^N z_J^2 = 0$. Let us temporarily denote z_J by z. We can use Lemma 3.18 to rewrite P_J as

$$\begin{aligned} P_J &= \sum_{n,m \geq 0} z^2(1-z)^{n+m} = \sum_{n=0}^{N} \sum_{m=0}^{N-n} z^2(1-z)^{n+m} \\ &= \sum_{n=0}^{N}(1-z)^n \left(z^2 \sum_{m=0}^{N-n}(1-z)^m \right) = \sum_{n=0}^{N}(1-z)^n \left(z - z(1-z)^{N-n+1} \right) \\ &= z\left(\sum_{n=0}^{N}(1-z)^n \right) - (N+1)z(1-z)^{N+1} = 1-(1-z)^{N+1} - (N+1)z(1-z)^{N+1}. \end{aligned}$$

This implies that $z^2 P_J = z^2$ since $z^2(1-z)^{N+1} = 0$, and so

$$P_J^2 = \left(\sum_{n=0}^{N} \sum_{m=0}^{N-n} (1-z)^{n+m} z^2 \right) P_J = \sum_{n=0}^{N} \sum_{m=0}^{N-n} (1-z)^{n+m} z^2 = P_J.$$

□

Remark 3.20 *As shown in the calculation above, one could define P_J as*

$$P_J = 1 - (1 + (N+1)z_J)(1 - z_J)^{N+1},$$

where N is the length of the longest chain in the monoid. For a J-trivial monoid, it suffices to take $P_J = 1 - (1 - z_J)^{N+1}$.

Lemma 3.21 *For all $J, K \in \mathcal{L}$, with $J \not\preceq K$, $P_J P_K = 0$.*

Proof: Follows from Lemma 3.12 and the fact that P_J is a polynomial in z_J with no constant term. □

Definition 3.22 *For each $J \in \mathcal{L}$, let*

$$e_J := P_J \left(1 - \sum_{K \succ J} e_K \right).$$

Lemma 3.23 *T_J occurs in e_J with coefficient 1. All other terms y of e_J have $C(y) \succ J$. In particular, $e_J \neq 0$.*

Proof: We proceed by induction. If J is maximal, then $e_J = P_J$, so the statement is implied by Lemma 3.17.

Now suppose the statement is true for all $M \succ J$. Then $e_J = P_J(1 - \sum_{M \succ J} e_M)$. By induction, all terms x of e_M have $C(x) \succeq M \succ J$. So terms y from $P_J e_M$ have $C(y) \succeq M \succ J$. The only other terms are those from P_J, for which the statement was proved in Lemma 3.17. □

Lemma 3.24 *$e_K P_J = 0$ for $K \not\preceq J$.*

Proof: The proof is by a downward induction on the semilattice. If K is maximal, then $e_K = P_K$, so by Lemma 3.21, $e_K P_J = P_K P_J = 0$.

Now suppose that for every $L \succ K$, $e_L P_J = 0$ for $L \not\preceq J$, and we will show that $e_K P_J = 0$ for $K \not\preceq J$. We expand $e_K P_J$:

$$e_K P_J = P_K \left(1 - \sum_{L \succ K} e_L\right) P_J = P_K P_J - \sum_{L \succ K} P_K e_L P_J.$$

Since $K \not\preceq J$, we have $P_K P_J = 0$ by Lemma 3.21, and $e_L P_J = 0$ by induction, since $L \succ K$ and $K \not\preceq J$ implies $L \not\preceq J$. □

Corollary 3.25 *e_J is idempotent.*

Proof: We expand $e_J e_J$:

$$e_J e_J = P_J \left(1 - \sum_{M \succ J} e_M\right) P_J \left(1 - \sum_{M \succ J} e_M\right) = P_J \left(P_J - \sum_{M \succ J} e_M P_J\right) \left(1 - \sum_{M \succ J} e_M\right)$$
$$\overset{(1)}{=} P_J^2 \left(1 - \sum_{M \succ J} e_M\right) \overset{(2)}{=} P_J \left(1 - \sum_{M \succ J} e_M\right) = e_J,$$

where (1) follows from Lemma 3.24, and (2) follows from Lemma 3.19. □

Lemma 3.26 $e_J e_K = 0$ *for* $J \neq K$.

Proof: The proof is by downward induction on the lattice $\mathcal{L}$. For a maximal element $M \in \mathcal{L}$, $e_M = P_M$, so $e_M e_K = P_M P_K (1 - \sum e_L) = 0$ by Lemma 3.21. Now suppose that for all $M \succ J$, $e_M e_K = 0$ for $M \neq K$ and we will show that $e_J e_K = 0$ for $J \neq K$. We expand $e_J e_K$:

$$e_J e_K = P_J (1 - \sum_{L \succ J} e_L) e_K = P_J (e_K - \sum_{L \succ J} e_L e_K) \tag{1}$$

If $K \not\succ J$, then $\sum_{L \succ J} e_L e_K = 0$ by our induction hypothesis, so $P_J(e_K - \sum_{L \succ J} e_L e_K) = P_J e_K = P_J P_K (1 - \sum_{M \succ K} e_M) = 0$ by Lemma 3.21.

If $K \succ J$, then $\sum_{L \succ J} e_L e_K = e_K$ since e_K is idempotent and $e_L e_K = 0$ for $L \neq K$ by the inductive hypothesis. Therefore $e_K - \sum_{L \succ J} e_L e_K = 0$ and hence the right hand side of (1) is zero. □

Theorem 3.27 *The set* $\{e_J \,:\, J \in \mathcal{L}\}$ *is a complete collection of orthogonal idempotents for* $\mathbb{C}W$.

Proof: From [13], we know that the maximal number of such idempotents is the cardinality of $\mathcal{L}$. The rest of the claim is just Lemma 3.23, Corollary 3.25 and Lemma 3.26. □

Appendix: An example

We show by example how to use the above construction to create orthogonal idempotents for the free left regular band on two generators.

Idempotents for the free left regular band on two generators

Let S be the left regular band freely generated by two elements a, b. Then $S = \{1, a, b, ab, ba\}$. All elements of S are idempotent. Also $aba = ab$ and $bab = ba$. The lattice $\mathcal{L}$ has four elements: $\emptyset := S, \mathfrak{a} := Sa, \mathfrak{b} := Sb$ and $\mathfrak{ab} := Sab = Sba$, where $\emptyset \prec \mathfrak{a} \prec \mathfrak{ab}$ and $\emptyset \prec \mathfrak{b} \prec \mathfrak{ab}$, but $\mathfrak{a}$ and $\mathfrak{b}$ have no relation. We begin by computing the elements P_J.

$J = \emptyset$: Neither of the generators satisfies $C(g) \preceq J$, so $T_\emptyset = 1 \in S$. $B_\emptyset = (1-a)(1-b)$. Also

$$\begin{aligned} B_\emptyset^2 &= (1-a)(1-b)(1-a)(1-b) = (1-a-b+ab)(1-a)(1-b) \\ &= (1-a-b+ab)(1-b) = (1-a-b+ab) = B_\emptyset. \end{aligned}$$

Therefore $A_\emptyset = B_\emptyset = 1 - a - b + ab$, so $z_\emptyset = 1 - a - b + ab$ is idempotent and

$$P_\emptyset = 1 - a - b + ab.$$

$J = \mathfrak{a}$: Then $C(a) \preceq \mathfrak{a}$ and $C(b) \not\preceq \mathfrak{a}$, so $T_\mathfrak{a} = a$ and $B_\mathfrak{a} = 1 - b = A_\mathfrak{a}$ since $1 - b$ is idempotent. Therefore $z_\mathfrak{a} = (1-b)a = a - ba$. $z_\mathfrak{a}^2 = a - ab$ and one can check that $z_\mathfrak{a}^3 = z_\mathfrak{a}^2$, so

$$P_\mathfrak{a} = z_\mathfrak{a}^2(1 + (1 - z_\mathfrak{a}) + (1 - z_\mathfrak{a})^2 + \dots) = z_\mathfrak{a}^2 = a - ab.$$

One can check that $P_\mathfrak{a}$ is idempotent.

$J = \mathfrak{b}$: Similarly,

$$P_\mathfrak{b} = b - ba.$$

$J = \mathfrak{ab}$: $C(a), C(b) \preceq \mathfrak{ab}$, so $T_\mathfrak{ab} = ab$ and $A_\mathfrak{ab} = 1$. $z_\mathfrak{ab} = ab$ is idempotent, so

$$P_\mathfrak{ab} = ab.$$

We can now compute the idempotents e_J. Since $\mathfrak{ab}$ is maximal,

$$e_\mathfrak{ab} = ab.$$

Since $P_\mathfrak{a} e_\mathfrak{ab} = (a - ab)ab = ab - ab = 0$,

$$e_\mathfrak{a} = P_\mathfrak{a}(1 - e_\mathfrak{ab}) = P_\mathfrak{a} = a - ab$$

and similarly,

$$e_\mathfrak{b} = b - ba.$$

Finally, note that $P_\emptyset e_\mathfrak{a} = (1 - a - b + ab)(a - ab) = 0$ and similarly $P_\emptyset e_\mathfrak{b} = 0$, so that

$$e_\emptyset = P_\emptyset(1 - e_\mathfrak{a} - e_\mathfrak{b} - e_\mathfrak{ab}) = P_\emptyset - P_\emptyset e_\mathfrak{ab} = 1 - a - b + ab - ab + ba = 1 - a - b + ba.$$

One can check that $\{e_\emptyset, e_\mathfrak{a}, e_\mathfrak{b}, e_\mathfrak{ab}\}$ is a collection of mutually orthogonal idempotents.

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References

[1] J. Almedia, S. Margolis, B. Steinberg, and M. Volkov. Representation theory of finite semigroups, semigroup radicals and formal language theory. *Trans. AMS*, 361(3):1429–1461, 2009.

[2] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1998. Basic representation theory of finite groups and associative algebras.

[3] C. Berg, N. Bergeron, S. Bhargava, and F. Saliola. Primitive orthogonal idempotents for R-trivial monoids. *ArXiv e-prints*, 2010. arXiv:1009.4943v1 [math.RT].

[4] M. Bremner. How to compute the wedderburn decomposition of a finite-dimensional associative algebra. *ArXiv e-prints*, 2010. preprint arXiv:1008.2006v1 [math.RA].

[5] K. Brown. Semigroups, rings, and Markov chains. *J. Theoret. Probab.*, 13(3):871–938, 2000.

[6] T. Denton. A combinatorial formula for orthogonal idempotents in the 0-Hecke algebra of S_N. *DMTCS proc.*, AN(01):701–712, 2010.

[7] T. Denton. A combinatorial formula for orthogonal idempotents in the 0-Hecke algebra of the symmetric group. *ArXiv e-prints*, 2010. preprint arXiv:1008.2401v1 [math.RT].

[8] T. Denton, F. Hivert, A. Schilling, and N. M. Thiéry. On the representation theory of finite *J*-trivial monoids. preprint arXiv:1010.3455, 2010.

[9] P. Norton. 0-Hecke algebras. *J. Austral. Math. Soc. Ser. A*, 27:337–57, 1979.

[10] J.-É. Pin. *Mathematical Foundations of Automata Theory*. 2010.

[11] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008. `http://combinat.sagemath.org`.

[12] F. Saliola. The Quiver of the Semigroup Algebra of a Left Regular Band. *International Journal of Algebra and Computation*, 17(8):1593–1610, 2007.

[13] M. Schocker. Radical of weakly ordered semigroup algebras. *J. Algebr. Comb.*, 28:231–234, 2008.

[14] M. P. Schützenberger. Sur le produit de concaténation non ambigu. *Semigroup Forum*, 13(1):47–75, 1976/77.

[15] W. A. Stein et al. *Sage Mathematics Software (Version 4.6)*. The Sage Development Team, 2010. `http://www.sagemath.org`.

[16] B. Steinberg. Email communications, February 2010.

[17] N. M. Thiéry. Email communications, January 2010.