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Deformed diagonal harmonic polynomials for complex reflection groups

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Abstract. We introduce deformations of the space of (multi-diagonal) harmonic polynomials for any finite complex reflection group of the form $W = G(m, p, n)$, and give supporting evidence that this space seems to always be isomorphic, as a graded W-module, to the undeformed version.

Résumé. Nous introduisons une déformation de l'espace des polynômes harmoniques (multi-diagonaux) pour tout groupe de réflexions complexes de la forme $W = G(m, p, n)$, et soutenons l'hypothèse que cet espace est toujours isomorphe, en tant que W-module gradué, à l'espace d'origine.

Keywords: diagonal harmonic polynomials, complex reflection group, rational Steenrod algebra, deformations

1 Introduction

The aim of this work is to give support to an extension and a generalization of the main conjecture of [HT04], to the diagonal case as well as to the context of finite complex reflection groups. This is stated explicitly in the new Conjecture 1.2 below, after a few words concerning notations and a description of the overall context.

Let X denote a $\ell \times n$ matrix of variables

$$X := (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n),$$

with each of the columns $\mathbf{x}_j = (x_{ij})_{1 \le i \le \ell}$ containing ℓ variables. For any fixed i (a row of X), we say that the variables $x_{i1}, x_{i2}, \ldots, x_{in}$ form a *set of variables* (the i^{th} set), and thus X consists in ℓ sets of n variables. For $\mathbf{d} \in \mathbb{N}^\ell$, we set

$$|\mathbf{d}| := d_1 + d_2 + \cdots + d_\ell \qquad \text{and} \qquad \mathbf{d}! := d_1! d_2! \cdots d_\ell!\,,$$

and write $\mathbf{x}_j^{\mathbf{d}}$ for the column monomial of *degree* $\mathbf{x}_j^{\mathbf{d}}$:

$$\mathbf{x}_j^{\mathbf{d}} := \prod_{i=1}^{\ell} x_{ij}^{d_i} \ .$$

The ground field $\mathbb{K}$ is assumed to be of characteristic zero and, whenever needed, to contain roots of unity and/or a parameter q; typically, $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{C}(q)$, although algebraic or transcendental extensions of $\mathbb{Q}$ are better suited for some of the computer calculations. The parameter q is called formal if it is transcendental over $\mathbb{Q}$.

Let W be a complex reflection group of rank n. Elements of W may be thought of as $n \times n$ matrices with complex entries. The *diagonal action* of W on a polynomial $f(X)$ is defined, for $w \in W$, by:

$$w \cdot f(X) := f(X\,w), \tag{1.1}$$

where $X\,w$ stands for matrix multiplication. In other words, W acts in a similar "diagonal" manner on each set of variables in X. A polynomial is *diagonally W-invariant* if

$$w \cdot f(X) = f(X), \qquad \text{for all} \quad w \in W\,.$$

We denote by $\mathcal{I}_W^{(\ell)}$ the ideal generated by constant-term-free diagonally W-invariant polynomials.

For each of the variables $x_{ij} \in X$, there is an associated partial derivation denoted here by $\partial_{x_{ij}}$ or ∂_{ij} for short. For a polynomial $f(X)$, let $f(\partial_X)$ stand for the differential operator obtained by replacing variable in X by the corresponding derivation in ∂_X. The space $\mathcal{H}W^{(\ell)}$ of *diagonally W-harmonic polynomials* (or *harmonic polynomials* for short) is then defined as the set of the polynomials $g(X)$ that satisfy all of the linear partial differential equations

$$f(\partial_X)(g(X)) = 0, \qquad \text{for} \quad f(X) \in \mathcal{I}_W^{(\ell)}\,. \tag{1.2}$$

In the following, we first restrict ourselves to the complex reflection groups $W = G(m,n)$, for $m, n \in \mathbb{N}$, and then extend our discussion to the subgroups $G(m,p,n)$. Recall that the *generalized symmetric group* $G(m,n)$ may be constructed as the group of $n \times n$ matrices having exactly one non zero entry in each row and each column, whose value is a m-th root of unity. Since the cases $\ell = 1$ and $W = \mathfrak{S}_n$ have been extensively considered previously (see [Ber09]), we write for short $\mathcal{H}W = \mathcal{H}W^{(1)}$, $\mathcal{H}_n^{(\ell)} = \mathcal{H}\mathfrak{S}_n^{(\ell)}$, and $\mathcal{H}_n = \mathcal{H}\mathfrak{S}_n$.

The ring $\mathbb{K}[X]^W$ of diagonally W-invariant polynomials for $W = G(m,n)$ is generated by *polarized* powersums, this is to say the polynomials

$$P_{\mathbf{d}} = \sum_{j=1}^{n} \mathbf{x}_j^{\mathbf{d}}\,,$$

for $|\mathbf{d}| = m\,k$, with $1 \leq k \leq n$. Let us write $D_{\mathbf{d}}$ for the operator $P_{\mathbf{d}}(\partial_X)$:

$$D_{\mathbf{d}} = \sum_{j=1}^{n} \boldsymbol{\partial}_j^{\mathbf{d}}\,,$$

where

$$\boldsymbol{\partial}_j^{\mathbf{d}} := \partial_{1j}^{d_1}\partial_{2j}^{d_2}\cdots\partial_{\ell j}^{d_\ell}\,.$$

Then, the space $\mathcal{H}W^{(\ell)}$ is the intersection of the kernels of all the operators $D_{\mathbf{d}}$, for $|\mathbf{d}| = m\,k$, with $1 \leq k \leq n$. The space $\mathcal{H}W^{(\ell)}$ is graded by (multi-)degree, and thus decomposes as a direct sum

$$\mathcal{H}W^{(\ell)} = \bigoplus_{\mathbf{d}\in\mathbb{N}^\ell} \mathcal{H}W^{(\ell)}_{\mathbf{d}}\,,$$

of its homogeneous components of degree $\mathbf{d}$. Recall that $f(X)$ is homogeneous of degree $\mathbf{d}$, if and only if we have

$$f(\mathbf{t}\,X) = \mathbf{t}^{\mathbf{d}}\,f(X)\,,$$

where $\mathbf{t}\,X$ stands for the multiplication of the matrix X by the diagonal matrix

$$\begin{pmatrix} t_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & t_\ell \end{pmatrix}.$$

The *Hilbert series of the space* $\mathcal{H}W^{(\ell)}$ is defined as

$$\mathcal{H}W^{(\ell)}(\mathbf{t}) := \sum_{\mathbf{d}\in\mathbb{N}^\ell} \dim(\mathcal{H}W^{(\ell)}_{\mathbf{d}})\,\mathbf{t}^{\mathbf{d}}\,.$$

It is well known that, for $\ell = 1$, the graded space $\mathcal{H}W$ is isomorphic, as a W-module, to the graded regular representation of W. In particular, its Hilbert series is given by the formula

$$\mathcal{H}W(t) = \prod_{k=1}^{n} \frac{t^{k\,m}-1}{t-1}\,.$$

Our specific story starts with a q-deformation of the polarized powersums:

$$P_{q,\mathbf{d}} := \sum_{j=1}^{n} \mathbf{x}_j^{\mathbf{d}}\,(1 + q\,(x_{1j}\partial_{1j} + \ldots + x_{nj}\partial_{nj})). \tag{1.3}$$

and the corresponding q-deformation of the operators $D_{\mathbf{d}} = P_{\mathbf{d}}(\partial_X)$ above:

$$D_{q,\mathbf{d}} := \sum_{j=1}^{n} (1 + q\,(x_{1j}\partial_{1j} + \ldots + x_{nj}\partial_{nj}))\,\boldsymbol{\partial}_j^{\mathbf{d}}. \tag{1.4}$$

An homogeneous polynomial $f(X)$ is said to be *diagonally* W,q*-harmonic* (or q-harmonic for short) if

$$D_{q,\mathbf{d}}\,f(X) = 0, \tag{1.5}$$

for all $\mathbf{d} \in \mathbb{N}^\ell$ such that $|\mathbf{d}|$ is divisible by m. The W-module of all q-harmonic polynomials is henceforth denoted by $\mathcal{H}W_q^{(\ell)}$, and thus by $\mathcal{H}_{n,q}^{(\ell)}$ when $W = \mathfrak{S}_n$. Our aim here is to discuss for which q the following assertion holds.

Assertion 1.1 *$\mathcal{H}W_q^{(\ell)}$ is isomorphic (as a graded W-module) to $\mathcal{H}W^{(\ell)}$.*

Conjecture 1.2 *Let $W = G(m,n)$, $\ell \in \mathbb{N}$. Then, Assertion 1.1 holds for q a formal parameter. In particular $\mathcal{H}W_q^{(\ell)}(\mathbf{t}) = \mathcal{H}W^{(\ell)}(\mathbf{t})$.*

This conjecture is an extension of the main conjecture of [HT04] (case $W = \mathfrak{S}_n$, $\ell = 1$), which itself is a q-analogue of a conjecture of Wood [Woo98, Woo01] on the "hit polynomials" for the *rational Steenrod algebra* $\mathcal{S} := \mathbb{K}[P_{1,d} \mid d \geq 1]$. Beside the extensive computer exploration and results reported on in [HT04] for $W = \mathfrak{S}_n$ and $\ell = 1$, our supporting evidence for this conjecture, includes the following results:

(1) Applying a classical specialization argument at $q = 0$ (see e.g. [HT04]), gives $\dim \mathcal{H}W_q^{(\ell)} \leq \dim \mathcal{H}W^{(\ell)}$ (also homogeneous component by homogeneous component). Furthermore, equality holds if and only if Conjecture 1.2 does.

(2) Conjecture 1.2 holds for all groups $G(m,2)$ for $\ell = 1$ (see Section 5), as well as for all ℓ when $m \leq 5$. For example, with $W = G(3,2)$, we get

$$\begin{aligned}\mathcal{H}W_q^{(\ell)}(\mathbf{t}) \quad &= \quad 1 + 2\,h_1(\mathbf{t}) + 2\,h_2(\mathbf{t}) + h_1^2(\mathbf{t}) + h_3(\mathbf{t}) + 2\,h_2(\mathbf{t})h_1(\mathbf{t}) \\ &\qquad +2\,h_4(\mathbf{t}) + h_2^2(\mathbf{t}) + 3\,h_5(\mathbf{t}) + 2\,h_6(\mathbf{t}) + h_7(\mathbf{t}).\end{aligned}$$

(3) Conjecture 1.2 holds for all ℓ, in the case $W = \mathfrak{S}_n = G(1,n)$ for $n \leq 4$. For example, we get the Hilbert series

$$\mathcal{H}_{3,q}^{(\ell)}(\mathbf{t}) = 1 + 2\,h_1(\mathbf{t}) + h_{11}(\mathbf{t}) + h_2(\mathbf{t}) + h_3(\mathbf{t})\,.$$

(4) There seems to be an analogue of Conjecture 1.2 for the subgroups $G(m,p,n)$ of $G(m,n)$ (see Section 2); in particular, Conjecture 1.2 holds for $n = 2$ (including the dihedral groups $G(m,m,2)$) when $\ell = 1$ (see Section 5), and for any ℓ for small values of m,p,n.

Another interesting feature of the space $\mathcal{H}W_q^{(\ell)}$, is that it may be characterized as the intersection of the kernels of a much smaller family of operators than the set

$$\{D_{q,\mathbf{d}} \mid |\mathbf{d}| = m\,k, \quad 1 \leq k \leq n\}. \tag{1.6}$$

Indeed, a straightforward calculation shows that the usual Lie-bracket relation between the generators of the rational Steenrod algebra generalize naturally:

$$[D_{q,\mathbf{d}}, D_{q,\mathbf{d}'}] = q(|\mathbf{d}| - |\mathbf{d}'|)\,D_{q,\mathbf{d}+\mathbf{d}'}. \tag{1.7}$$

An efficient way to setup this calculation is to let both sides act on the generating function for all monomials, namely the formal series

$$\exp(Z.X) = \sum_{\mathbf{d}\in\mathbb{N}^{\ell\times n}} \mathbf{x}^{\mathbf{d}}\,\frac{\mathbf{z}^{\mathbf{d}}}{\mathbf{d}!}\,,$$

where Z stands for a matrix of variables just as X does, and $Z.X := \sum_{ij} z_{ij} x_{ij}$. It follows from (1.7) that a polynomial is in the kernel of $D_{q,\mathbf{d}+\mathbf{d}'}$, whenever it lies in the kernels of both $D_{q,\mathbf{d}}$ and $D_{q,\mathbf{d}'}$. From this, we can immediately deduce that

Proposition 1.3 *The space of q-harmonic polynomials for $W = G(m, n)$ can be obtained as*

$$\mathcal{H}W_q^{(\ell)} = \bigcap_{|\mathbf{d}|=m \text{ or } 2\,m} \mathrm{Ker}(D_{q,\mathbf{d}})\,.$$

For example, when $\ell = 1$, and as already noted in [HT04] in the case $W = \mathfrak{S}_n$, the space $\mathcal{H}W_q$ is defined by just two linear differential equations:

$$\mathcal{H}W_q = \mathrm{Ker}(D_{q,m}) \cap \mathrm{Ker}(D_{q,2\,m})\,.$$

Similarly, when $\ell = 2$, the space $\mathcal{H}W_q^{(2)}$ is the intersection of the kernels of only five operators:

$$D_{q,(1,0)}, \qquad D_{q,(2,0)}, \qquad D_{q,(1,1)}, \qquad D_{q,(0,1)}, \quad \text{and} \quad D_{q,(0,2)}\,.$$

This is striking because, assuming that Conjecture 1.2 holds, a direct calculation of this joint kernel and a specialization at $q = 0$ would yield back the famous space $\mathcal{H}_n^{(2)}$ of diagonally harmonic polynomials. Yet, even if the mysterious structure of $\mathcal{H}_n^{(2)}$ has been extensively studied in the past 20 years (see [Hai03]), no nice Gröbner basis for the ideal $\mathcal{I}_W^{(2)}$ is known, even for $W = \mathfrak{S}_n$.

It is also noteworthy that systematic variations on the main conjecture in [HT04] have been extensively studied in [BGW10]. In an upcoming work, we plan to describe how these variations may be adapted to the context of the reflection groups considered here, including the diagonal point of view. In particular, since there is a close tie (*loc. sit.*) between the case $\ell = 1$, with $W = \mathfrak{S}_n$, and Wood's conjecture (stated in [Woo98] or [Woo01]), we also plan to analyze how to generalize it to our new expanded context.

2 Deformed harmonic polynomials for $G(m, p, n)$

This section presents work in progress toward generalizing the construction of q-harmonic polynomials, and Conjecture 1.2, to all finite complex reflection groups. For simplicity, we restrict ourselves to a single set of variables: namely $\ell = 1$. However, computer calculations suggest that the extension to the diagonal case is straightforward.

Recall that all but a small number of finite complex reflection groups are part of an infinite family of natural subgroups of the generalized symmetric groups which we consider now. For $m, n \in \mathbb{N}$, let p be a divisor of m. Then, the complex reflection group $G(m, p, n)$ is defined as:

$$G(m, p, n) := \{g \in G(m, n) \mid \det g^{m/p} = 1\}\,.$$

In particular, setting $p = 1$, we get back $G(m,n) = G(m,1,n)$. Recall, for example, that the classical dihedral groups correspond to the family $G(m,m,2)$.

The invariant ring for $W = G(m,p,n)$ is obtained by adjoining $e_n^{m/p}$ to the invariant ring of $G(m,n)$, with $e_n = e_n(\mathbf{x}) := x_1 \cdots x_n$ standing for the product of the variables. It is well known that the invariant ring $\mathbb{K}[\mathbf{x}]^W$ may then be described as the graded free commutative algebra:

$$\mathbb{K}[\mathbf{x}]^W = \mathbb{K}[p_m, \ldots, p_{(m-1)n}, e_n^{m/p}]\,,$$

with $\deg(p_k) = k$ and $\deg(e_n^{m/p}) = nm/p$. Notice the necessary suppression of the generator p_{mn} for this presentation to be free.

The choice of a canonical q-analogue of $e_n^{m/p}$ does not appear to be straightforward. Indeed, and as far as we know (see the discussion in [HT04, Section 7.1]), there is no natural analogue of the elementary symmetric polynomial inside the rational Steenrod algebra $\mathcal{S}_q$. Besides, experienced gained in [BGW10] suggests that (generically) any choice of q-analogue would give an isomorphic space. We therefore take the simplest option, which is to not deform $e_n^{m/p}$ at all. Hence, for $W = G(m,p,n)$, we define the *q-deformed rational W-Steenrod algebra* as

$$\mathcal{S}_q^W := \mathbb{K}[P_{q,m}, \ldots, P_{q,mn}, e_n^{m/p}].$$

Accordingly, we obtain the graded space $\mathcal{H}W_q$ of the q-harmonic polynomials for $W = G(m,p,n)$, just as previously. Specifically, writing ε for the operator $e_n(\partial_X)$, $\mathcal{H}W_q$ is the intersection of $\mathcal{H}G(m,n)_q$ and $\mathrm{Ker}(\varepsilon^{m/p})$. A natural question here is to ask whether Conjecture 1.2 holds for $G(m,p,n)$. We will show in Section 5 that it does for $n = 2$ and $\ell = 1$.

A first step to confirm the choice of $e_n^{m/p}$ would be to prove the following conjecture.

Conjecture 2.1 *The q-harmonic polynomials for $G(m,1,n)$, as defined above, coincide with those for $G(m,n)$.*

An equivalent but more concrete condition is that no q-harmonic polynomial for $G(m,n)$ shall contain a monomial divisible by e_n^m. This property holds for $n = 2$, and all q-harmonic polynomials for $\mathfrak{S}_n$ calculated in [HT04].

In fact, we expect $\mathcal{H}G(m,n)_q$ to decompose as a direct sum of m layers $L_0(q) \oplus \cdots \oplus L_{m-1}(q)$ such that all the elements of $L_k(q)$ are divisible by e_n^k but not by e_n^{k+1}, as in Figure 5. Furthermore, ε (depicted by the gray down arrows in this figure) would be an isomorphism from $L_k(q)$ to $L_{k-1}(q/(1+q))$, the change of q being due to the equation

$$e_n(\partial_X) D_{q,k} = (1+q) D_{\frac{q}{1+q},k} e_n(\partial_X)\,. \tag{2.1}$$

In that case, the q-harmonic polynomials for $G(m,p,n)$ would be given by

$$\mathcal{H}G(m,p,n)_q = L_0(q) \oplus \cdots \oplus L_{m/p-1}(q)\,. \tag{2.2}$$

For example, in Figure 5, the q-harmonic polynomials for the dihedral group $G(4,1,2)$ are given by $\mathcal{H}G(4,1,2)_q = L_0(q)$. Similarly, $\mathcal{H}G(4,2,2)_q = L_0(q) \oplus L_1(q)$. We expect that, in general, the space $\mathcal{H}G(m,n)_q$ consists of m copies of $\mathcal{H}G(m,n,1)_q$, that may be constructed by "lifting back" through ε.

This further suggests that the lack of operators commuting appropriately with the action of the rational Steenrod algebra, as reported in [HT04, Section 7] can be circumvented by allowing q to change during the commutation. For example, for $m = \ell = 1$, the harmonic polynomials are usually constructed from the Vandermonde determinant by iterative applications of operators ∂_i; it would be worth finding analogues of those operators which would construct new q-harmonic polynomials from q'-harmonic polynomials for some other q'.

3 Inflating q-harmonic polynomials from $\mathfrak{S}_n$ to $G(m,n)$ and $G(m,m,n)$

In this section, we do some preliminary steps in the following direction.

Problem 3.1 *Assume that a basis of the q-harmonic polynomials for $\mathfrak{S}_n$ is given. Is it possible to construct from it a basis of the q-harmonic polynomials for $G(m,n)$? for $G(m,m,n)$?*

Beware that the analogous problem for constructing diagonally q-harmonic polynomials from q-harmonic polynomials is already hard at $q = 0$.

To start with, any q-harmonic polynomial for $W = \mathfrak{S}_n$ remains q-harmonic for $G(m,n)$. It also remains q-harmonic for $G(m,p,n)$ as soon as Conjecture 2.1 holds for $\mathfrak{S}_n$. We now construct more q-harmonic polynomials by inflating those of $\mathfrak{S}_n$. To this end, we consider the inflation algebra morphism and its analogue (which is just a linear morphism) on the dual basis:

$$\phi_r : \begin{cases} \mathbb{K}[X] & \hookrightarrow\!\!\!\!\to \mathbb{K}[X^r] \\ \mathbf{x}^{\mathbf{d}} & \mapsto \mathbf{x}^{r\mathbf{d}} \end{cases} \qquad \overline{\phi}_r : \begin{cases} \mathbb{K}[X] & \hookrightarrow\!\!\!\!\to \mathbb{K}[X^r] \\ \mathbf{x}^{(\mathbf{d})} & \mapsto \mathbf{x}^{(r\mathbf{d})} \end{cases}, \tag{3.1}$$

where, by a slight notational abuse, X^r stands for the matrix of the r-th powers of the variables and $\mathbf{x}^{(\mathbf{d})} := \frac{1}{\mathbf{d}!}\mathbf{x}^{\mathbf{d}}$.

Proposition 3.2 *Let $W = G(m,n)$ and r be a divisor of m. Then, the morphism $\overline{\phi}_r$ restricts to a graded $\mathfrak{S}_n$-module embedding (resp. isomorphism if $r = m$) from $\mathcal{H}\mathfrak{S}_{nq}$ to $\mathcal{H}W_{q/m} \cap \mathbb{K}[X^r]$, up to an appropriate r-scaling of the grading. The statement extends to any $G(m,p,n)$ as soon as Conjecture 2.1 holds for $\mathfrak{S}_n$.*

The first step toward this proposition is to define an appropriate inflation on the q-rational Steenrod algebra. Note that the operators $P_{q,k}$ live inside the subalgebra $\mathbb{K}[X \cdot \partial_X, X]$ of the Weyl algebra, where $X \cdot \partial_X$ denotes the matrix of the Euler operators $x\partial_x$ for $x \in X$. The only non-trivial brackets in this algebra are $[x\partial_x, x] = x$, for $x \in X$, from which it follows that

$[x\partial_x, x^k] = kx^k$. Similarly, the operators $D_{q,k}$ live inside the subalgebra $\mathbb{K}[XPX, \partial_X]$, with analogous relations.

Remark 3.3 *Fix $r \in \mathbb{N}$. Then, the two mappings*

$$x\partial_x \mapsto 1/r(x\partial_x), \quad x \mapsto x^r, \text{ for } x \in X \qquad \text{and} \qquad x\partial_x \mapsto 1/r(x\partial_x), \quad \partial_x \mapsto \partial_x^r, \text{ for } x \in X$$

extend respectively to algebra isomorphisms

$$\Phi_r : \mathbb{K}[X \cdot \partial_X, X] \hookrightarrow\!\!\!\!\!\rightarrow \mathbb{K}[X \cdot \partial_X, X^m] \qquad \text{and} \qquad \overline{\Phi}_r : \mathbb{K}[X \cdot \partial_X, \partial_X] \hookrightarrow\!\!\!\!\!\rightarrow \mathbb{K}[X \cdot \partial_X, \partial_X^m]\,.$$

Furthermore, those isomorphisms are compatible with the action on inflated polynomials: for $f \in \mathbb{K}[X]$ and F in $\mathbb{K}[X \cdot \partial_X, X]$ (resp. in $\mathbb{K}[X \cdot \partial_X, \partial_X]$), we have

$$\Phi_r(F)(\phi_r(f)) = \phi_r(F(f)) \qquad \text{and} \qquad \overline{\Phi}_r(F)(\overline{\phi}_r(f)) = \overline{\phi}_r(F(f))\,. \tag{3.2}$$

Using this remark, a straightforward calculation shows that:

$$\Phi_r(P_{q,\mathbf{d}}) = P_{q/r,r\mathbf{d}} \qquad \text{and} \qquad \overline{\Phi}_r(D_{q,\mathbf{d}}) = D_{q/r,r\mathbf{d}}\,. \tag{3.3}$$

This readily implies that Φ_m restricts to an isomorphism from the q-rational Steenrod algebra for $\mathfrak{S}_n$ to that for $G(m,n)$. Computer exploration suggests that the Gröbner basis for the right ideal generated by the Steenrod algebra for $G(m,n)$ is simply obtained by inflating that for $\mathfrak{S}_n$. This possibly opens the door for controlling the leading monomials of "q-hit polynomials" for $G(m,n)$ from those for $\mathfrak{S}_n$.

Returning to our main goal, we now have all the ingredients to prove Proposition 3.2.

Proof of Proposition 3.2: Let $f \in \mathbb{K}[X]$. Then, using Equation (3.2),

$$D_{q/r,\,r\mathbf{d}}(\phi(f)) = \overline{\Phi}_r(D_{q,\mathbf{d}})(\phi(f)) = \phi(D_{q,\mathbf{d}}(f))\,.$$

Hence $D_{q/r,r\mathbf{d}}(\phi(f)) = 0$ if and only if $D_{q,\mathbf{d}}(f) = 0$. The statements follows. □

4 Singular values

As discussed in [HT04], Assertion 1.1 may fail for very specific values of q. In that case, q is said to be *singular*. Computer exploration (see Table A.3 of [HT04]) and the complete analysis of the case $n = 2$ suggested that the only such singular values for $W = \mathfrak{S}_n$ and $\ell = 1$ are negative rational numbers of the form $-a/b$ with $a \leq n$. In [DM10] D'Adderio and Moci refined this statement to $a \leq n \leq b$ (with $q = -a/b$ not necessarily reduced), and proved that all such values are indeed singular by constructing explicit exceptional q-harmonic polynomials.

Proposition 4.1 *Let $W = G(m,n)$, $\ell \in \mathbb{N}$, and $q = -a/b$ with $a \leq n \leq b$, for $a, b \in \mathbb{N}$. Then, q is singular.*

Proof (sketch of): Let $f(x_1, \ldots, x_n)$ be the q-harmonic polynomial for $\mathfrak{S}_n$ which was constructed in [DM10]. Then, as stated in Proposition 3.2, $f(x_1^m, \ldots, x_n^m)$ is a q/m-harmonic for W of high enough degree (as in [DM10]) to disprove the statement of Conjecture 1.2. Going from $\ell = 1$ to ℓ arbitrary is then straightforward, since the intersection of $\mathcal{H}W_q^{(\ell)}$ with the polynomial ring in the first set of variables is $\mathcal{H}W_q$. □

It is worth noting that, for $n = 2$, and $\ell = 1$ the singular values are exactly those listed in Proposition 4.1 (see Section 5). However, at this stage, we lack computer data to extend this to a conjecture for all n and ℓ.

5 Complete study for $n = 2$

In this section, we prove Conjecture 1.2 for any group $W = G(m, p, 2)$ in the case $\ell = 1$. We denote for short the two variables x and y instead of x_1 and x_2. Naturally ∂_x and ∂_y are the corresponding differential operators. We also introduce the following q-analogue of the Pockhammer symbol $(d)_k$:

$$\langle d \rangle_k := d\,(d-1)\,\cdots\,(d-k+1)\,(1 + q\,(d-k)).$$

Then, for any monomial $x^\alpha y^\beta$, one has:

$$D_{q,k}(x^\alpha y^b) = \langle \alpha \rangle_k\, x^{\alpha-k} y^\beta + \langle \beta \rangle_k\, x^\alpha y^{\beta-k}\,, \tag{5.1}$$

which is well defined for any nonnegative numbers α and β, since $\langle \alpha \rangle_k = 0$ whenever $\alpha < k$.

Remark 5.1 *Let $W = G(m, m, 2)$ be the dihedral group. Then, the space $\mathcal{H}W_q$ is isomorphic to $\mathcal{H}W$, and in fact coincides with it, if and only if q is not of the form $-1/b$ with $1 \le b \le m$. In that case, it is of dimension $2m$ and a basis is given by*

$$\{1, x, x^2, x^3, \ldots, x^{m-1}, x^m - y^m, y^{m-1}, y^{m-2}, \ldots, y^2, y\}\,. \tag{5.2}$$

Otherwise, the basis of $\mathcal{H}W_q$ contains additionally the monomials x^{b+m} and y^{b+m}, or just the binomial $x^{2m} - y^{2m}$ if $b = m$.

Proof: Let $f = f(x, y)$ be an homogeneous q-harmonic polynomial in $\mathbb{K}[x, y]$. It satisfies:

$$D_{q,m}(f) = 0, \qquad D_{q,2m}(f) = 0, \qquad \textit{and} \qquad \varepsilon(f) = 0,$$

where $\varepsilon = \partial_x \partial_y$. By the last equation, f is of the form $\lambda x^d + \mu y^d$, and the two other equations rewrite as $(d)_k(\lambda x^{d-km} + \mu y^{d-km})$ for $k = 1, 2$. The statement follows. □

Proposition 5.2 *Let $W = G(m,2)$ and $\ell = 1$. Then, the space $\mathcal{H}W_q$ is isomorphic as a graded W-module to $\mathcal{H}W$ if and only if q is not of the form $-a/b$ with $1 \leqslant a \leqslant 2 \leqslant b$, and $a, b \in \mathbb{N}$. In that case, it is of dimension $2m^2$ and a basis is given by*

$$\{x^\alpha y^\beta\}_{\substack{0 \leqslant a < m \\ 0 \leqslant b < m}} \cup \{\langle\beta + m\rangle_m x^{\alpha+m} y^\beta - \langle\alpha + m\rangle_m x^\alpha y^{\beta+m}\}_{\substack{0 \leqslant \alpha < m \\ 0 \leqslant \beta < m}} . \tag{5.3}$$

Proof: As suggested by Equation (5.1), the implicit combinatorial ingredient is the length of the longest string $\dots, x^{\alpha-m}y^{\beta+m}, x^\alpha y^\beta, x^{\alpha+m}y^{\beta-m}, \dots$ containing any given monomial.

Obviously, whenever $\alpha < m$ and $\beta < m$, the monomial $x^\alpha y^\beta$ is killed by both operators $D_{q,m}$ and $D_{q,2\,m}$, and is therefore q-harmonic. This gives the first m^2 monomials in (5.3). Using Equation 5.1, a direct calculation shows that the remaining m^2 binomials

$$\langle\beta + m\rangle_m x^{\alpha+m} y^\beta - \langle\alpha + m\rangle_m x^\alpha y^{\beta+m}$$

for $\alpha < m$ and $\beta < m$ are also q-harmonic.

We now consider a monomial $x^{\alpha'} y^{\beta'}$ that does not appear in any of the q-harmonic polynomials of (5.3), and prove that it cannot appear in any other q-harmonic polynomial. It is straightforward that we can choose α and β such that

$$x^{\alpha'} y^{\beta'} \in \{x^{\alpha+m}y^{\beta-m}, x^\alpha y^\beta, x^{\alpha-m}y^{\beta+m}\} .$$

Let $h = c_1 x^{\alpha+m}y^{\beta-m} + c_2 x^\alpha y^\beta + c_3 x^{\alpha-m}y^{\beta+m} + \cdots$ be a q-harmonic polynomial. Then,

$$0 = D_{q,m}(h)_{|x^\alpha y^{\beta-m}} = c_1\langle\alpha + m\rangle_m + c_2\langle\beta\rangle_m .$$

Looking similarly at $D_{q,m}(h)_{|x^{\alpha-m}y^\beta}$ and $D_{q,2\,m}(h)_{|x^{\alpha-m}y^{\beta-m}}$, shows that the coefficients c_1, c_2, and c_3 must satisfy the following system of equations:

$$\begin{array}{rcccccl}
\langle\alpha + m\rangle_m \;\; c_1 & + & \langle\beta\rangle_m \, c_2 & & & = & 0 \\
 & & \langle\alpha\rangle_m \, c_2 & + & \langle\beta + m\rangle_m \;\; c_3 & = & 0 \\
\langle\alpha + m\rangle_{2m} \, c_1 & & & + & \langle\beta + m\rangle_{2m} \, c_3 & = & 0
\end{array}$$

whose determinant is:

$$\frac{(\alpha + m)!(\beta + m)!}{(\alpha - m)!(\beta - m)!}(1 + q\,(\alpha - m))(1 + q\,(\beta - m))(2 + q\,(\alpha + \beta)) .$$

Therefore $c_1 = c_2 = c_3 = 0$ whenever q is not one of the announced singular values. □

Corollary 5.3 *For $W = G(m,p,2)$ and $q \neq -a/b, 1 \leqslant a \leqslant 2 \leqslant b$, the space $\mathcal{H}W_q$ is isomorphic as a graded W-module to $\mathcal{H}W$. Its basis is obtained by considering the layers $L_0(q), \dots, L_{m/p-1}(q)$ of the q-harmonic polynomials for $G(m,2)$.*

Proof: Select out of Equation (5.3) the elements which satisfy the extra equation $\varepsilon^{m/p}(f) = 0$. See also, in Figure 5, the vertical arrows which depict the action of ε. □

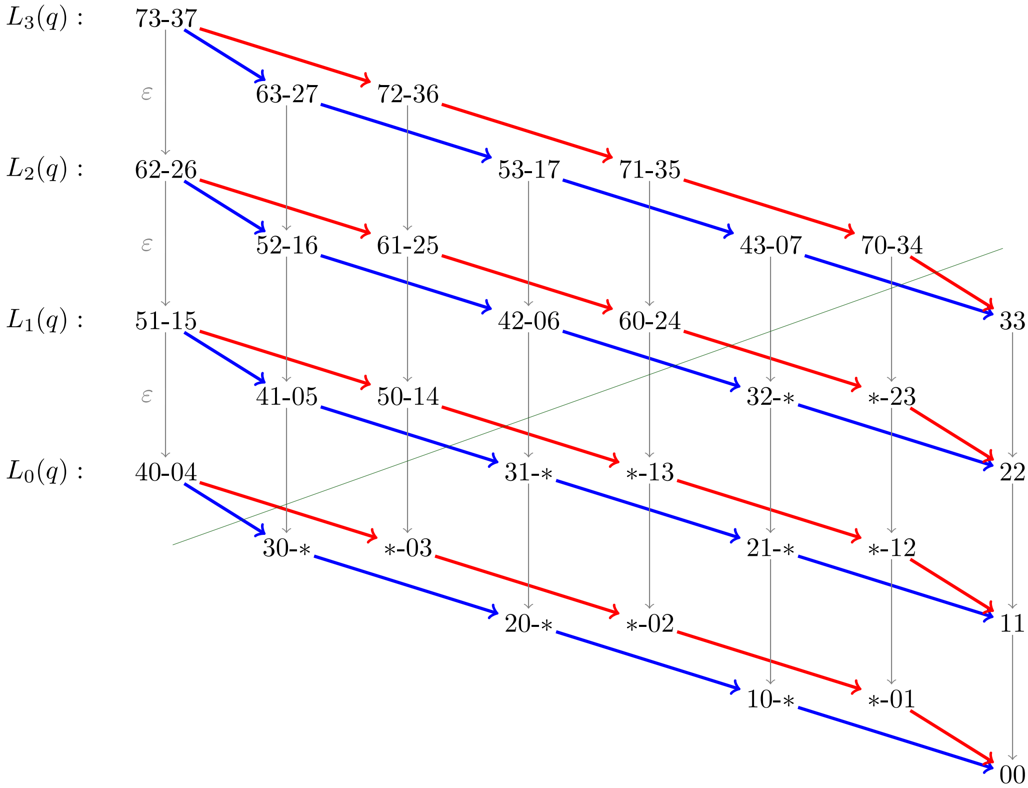

Fig. 1: Structure of q-Harmonic polynomials of $G(4,2)$. For short, the q-harmonic binomial $\langle d\rangle_4 x^\alpha y^\beta - \langle a\rangle_4 x^{\alpha'} y^{\beta'}$ is denoted "$\alpha\beta$-$\alpha'\beta'$". Similarly, the q-harmonic monomial $x^\alpha y^\beta$ is denoted "$\alpha\beta$", "$\alpha\beta$-$*$", or "$*$-$\alpha\beta$". The blue (resp. red) arrows denote the action of the would be q-analogues of the operators ∂_x and ∂_y within each layer L_i. The gray arrows denote the action of the operator $\varepsilon = e_2(\partial_X) = \partial_x\partial_y$ (recall that it changes the value of q). The green line separates the q-harmonic monomials and binomials.

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