Relative Node Polynomials for Plane Curves

Florian Block[†]

Department of Mathematics, University of Michigan, Ann Arbor, MI, USA

Abstract. We generalize the recent work of Fomin and Mikhalkin on polynomial formulas for Severi degrees.

The degree of the Severi variety of plane curves of degree d and δ nodes is given by a polynomial in d, provided δ is fixed and d is large enough. We extend this result to generalized Severi varieties parametrizing plane curves which, in addition, satisfy tangency conditions of given orders with respect to a given line. We show that the degrees of these varieties, appropriately rescaled, are given by a combinatorially defined "relative node polynomial" in the tangency orders, provided the latter are large enough. We describe a method to compute these polynomials for arbitrary δ, and use it to present explicit formulas for $\delta \leq 6$. We also give a threshold for polynomiality, and compute the first few leading terms for any δ.

Résumé. Nous généralisons les travaux récents de Fomin et Mikhalkin sur des formules polynomiales pour les degrés de Severi.

Le degré de la variété de Severi des courbes planes de degré d et à δ nœuds est donné par un polynôme en d, pour δ fixé et d assez grand. Nous étendons ce résultat aux variétés de Severi généralisées paramétrant les courbes planes et qui, en outre, satisfont à des conditions de tangence d'ordres donnés avec une droite fixée. Nous montrons que les degrés de ces variétés, rééchelonnés de manière appropriée, sont donnés par un "polynôme de noeud relatif", défini combinatoirement, en les ordres de tangence, dès que ceux-ci sont assez grands. Nous décrivons une méthode pour calculer ces polynômes pour $delta$ arbitraire, et l'utilisons pour présenter des formules explicites pour $\delta \leq 6$. Nous donnons aussi un seuil pour la polynomialité, et calculons les premiers termes dominants pour tout δ.

Keywords: enumerative geometry, floor diagram, Gromov-Witten theory, node polynomial, tangency conditions

1 Introduction and Main Results

The *Severi degree* $N^{d,\delta}$ is the degree of the Severi variety of (possibly reducible) nodal plane curves of degree d with δ nodes. Equivalently, $N^{d,\delta}$ is the number of such curves passing through $\frac{(d+3)d}{2} - \delta$ generic points in the complex projective plane $\mathbb{CP}^2$. Severi varieties have received considerable attention since they were introduced by F. Enriques [Enr12] and F. Severi [Sev21] around 1915. Much later, in 1986, J. Harris [Har86] achieved a celebrated breakthrough by showing their irreducibility.

In 1994, P. Di Francesco and C. Itzykson [DFI95] conjectured that the numbers $N^{d,\delta}$ are given by a polynomial in d, for a fixed number of nodes δ, provided d is large enough. S. Fomin and G. Mikhalkin [FM10, Theorem 5.1] established this polynomiality in 2009. More precisely, they showed that there exists, for every $\delta \geq 1$, a *node polynomial* $N_\delta(d)$ which satisfies $N^{d,\delta} = N_\delta(d)$ for all $d \geq 2\delta$.

[†]The author was partially supported by a Rackham One-Term Dissertation Fellowship & by the NSF grant DMS-055588.

The polynomiality of $N^{d,\delta}$ and the polynomials $N_\delta(d)$ were known in the 19th century for $\delta = 1, 2, 3$. For $\delta = 4, 5, 6$, this was only achieved by I. Vainsencher [Vai95] in 1995. In 2001, S. Kleiman and R. Piene [KP04] settled the cases $\delta = 7, 8$. In [Blo11], the author computed $N_\delta(d)$ for $\delta \leq 14$ and improved the threshold of S. Fomin and G. Mikhalkin by showing that $N^{d,\delta} = N_\delta(d)$ provided $d \geq \delta$.

Severi degrees can be generalized to incorporate tangency conditions to a fixed line $L \subset \mathbb{CP}^2$. More specifically, the *relative Severi degree* $N^\delta_{\alpha,\beta}$ is the number of (possibly reducible) nodal plane curves with δ nodes which have tangency of order i to L at α_i *fixed* points (chosen in advance) and tangency of order i to L at β_i *unconstrained* points, for all $i \geq 1$, and which pass through an appropriate number of generic points. Equivalently, $N^\delta_{\alpha,\beta}$ is the degree of the *generalized Severi variety* studied in [CH98, Vak00]. By Bézout's Theorem, the degree of a curve with tangencies of order (α, β) equals $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$. The number of point conditions (for a potentially finite count) is $\frac{(d+1)d}{2} - \delta + \beta_1 + \beta_2 + \cdots$. We recover non-relative Severi degrees by specializing to $\alpha = (0, 0, \dots)$ and $\beta = (d, 0, 0, \dots)$. The numbers $N^\delta_{\alpha,\beta}$ are determined by the rather complicated Caporaso-Harris recursion [CH98].

In this paper, we show that much of the story of (non-relative) node polynomials carries over to relative Severi degrees. Our main result is that, up to a simple combinatorial factor and for fixed $\delta \geq 0$, the relative Severi degrees $N^\delta_{\alpha,\beta}$ are given by a *multivariate polynomial* in $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$, provided that $\beta_1 + \beta_2 + \dots$ is sufficiently large. For a sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of non-negative integers with finitely many α_i non-zero we write

$$|\alpha| \stackrel{\text{def}}{=} \alpha_1 + \alpha_2 + \cdots, \qquad \alpha! \stackrel{\text{def}}{=} \alpha_1! \cdot \alpha_2! \cdot \cdots .$$

Throughout the paper we use the grading $\deg(\alpha_i) = \deg(\beta_i) = 1$ (so that d and $|\beta|$ are homogeneous of degree 1). The following is our main result.

Theorem 1.1 *For every $\delta \geq 0$, there is a combinatorially defined polynomial $N_\delta(\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots)$ of (total) degree 3δ such that, for all $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ with $|\beta| \geq \delta$, the relative Severi degree $N^\delta_{\alpha,\beta}$ is given by*

$$N^\delta_{\alpha,\beta} = 1^{\beta_1} 2^{\beta_2} \cdots \frac{(|\beta| - \delta)!}{\beta!} \cdot N_\delta(\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots). \tag{1.1}$$

We call $N_\delta(\alpha; \beta)$ the *relative node polynomial* and use the same notation as in the non-relative case if no confusion can occur. We do not need to specify the number of variables in light of the following stability condition.

Theorem 1.2 *For $\delta \geq 0$ and vectors $\alpha = (\alpha_1, \dots, \alpha_m)$, $\beta = (\beta_1, \dots, \beta_{m'})$ with $|\beta| \geq \delta$, it holds that*

$$N_\delta(\alpha, 0; \beta) = N_\delta(\alpha; \beta) \quad \text{and} \quad N_\delta(\alpha; \beta, 0) = N_\delta(\alpha; \beta)$$

as polynomials. Therefore, there exists a formal power series $N^\infty_\delta(\alpha; \beta)$ in infinitely many variables $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ which specializes to all relative node polynomials under $\alpha_{m+1} = \alpha_{m+2} = \cdots = 0$ and $\beta_{m'+1} = \beta_{m'+2} = \cdots = 0$, for various $m, m' \geq 0$.

In fact, even more is true.

Proposition 1.3 *For $\delta \geq 0$ the relative node polynomial $N_\delta(\alpha, \beta)$ is a polynomial in d, $|\beta|$, $\alpha_1, \dots, \alpha_\delta$, and $\beta_1, \dots, \beta_\delta$, where $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$.*

Using the combinatorial description we provide a method for computing the relative node polynomials for arbitrary δ (see Sections 3 and 4). We utilize it to compute $N_\delta(\alpha; \beta)$ for $\delta \leq 6$. The polynomials N_0 and N_1 already appeared (implicitly) in [FM10, Section 4.2].

Theorem 1.4 *The relative node polynomials $N_\delta(\alpha;\beta)$, for $\delta \le 3$ (resp., $\delta \le 6$) are as listed in [Blo10, Appendix A] (resp., as provided in the ancillary files accompanying [Blo10]).*

The polynomial $N_\delta(\alpha;\beta)$ is of degree 3δ by Theorem 1.1. We compute its terms of degree $\ge 3\delta - 2$.

Theorem 1.5 *The terms of $N_\delta(\alpha;\beta)$ of (total) degree $\ge 3\delta - 2$ are given by*

$$\begin{aligned} N_\delta(\alpha;\beta) = \frac{3^\delta}{\delta!}\Big[& d^{2\delta}|\beta|^\delta + \frac{\delta}{3}\big[-\frac{3}{2}(\delta-1)d^2 - 8d|\beta| + |\beta|\alpha_1 + d\beta_1 + |\beta|\beta_1\big]d^{2\delta-2}|\beta|^{\delta-1} + \\ & + \frac{\delta}{9}\big[\frac{3}{8}(\delta-1)(\delta-2)(3\delta-1)d^4 + 12\delta(\delta-1)d^3|\beta| + (11\delta+1)d^2|\beta|^2 + \\ & - \frac{3}{2}\delta(\delta-1)(d^3\beta_1 + d^2|\beta|\alpha_1) - \frac{1}{2}(\delta+5)(3\delta-2)d^2|\beta|\beta_1 - 8(\delta-1)(d|\beta|^2\alpha_1 + d|\beta|^2\beta_1) + \\ & + \frac{1}{2}(\delta-1)(d^2\beta_1^2 + |\beta|^2\alpha_1^2 + |\beta|^2\beta_1^2) + (\delta-1)(d|\beta|\alpha_1\beta_1 + d|\beta|\beta_1^2 + |\beta|^2\alpha_1\beta_1)\big]d^{2\delta-4}|\beta|^{\delta-2} + \cdots\Big], \end{aligned}$$

where $d = \sum_{i\ge 1} i(\alpha_i + \beta_i)$.

Theorem 1.5 can be extended to terms of $N_\delta(\alpha,\beta)$ of degree $\ge 3\delta - 7$ (see Remark 5.2). We observe that all coefficients of $N_\delta(\alpha;\beta)$ in degree $\ge 3\delta - 2$ are of the form $\frac{3^\delta}{\delta!}$ times a polynomial in δ. Without computating the coefficients, we can extended this further. It is conceivable to expect this property to hold for arbitrary degrees.

Proposition 1.6 *Every coefficient of $N_\delta(\alpha;\beta)$ in degree $\ge 3\delta - 7$ is given, up to a factor of $\frac{3^\delta}{\delta!}$, by a polynomial in δ with rational coefficients.*

Our approach to planar enumerative geometry is combinatorial and inspired by *tropical geometry*, a piecewise-linear analogue of algebraic geometry (see, for example, [Gat06]). By the celebrated Correspondence Theorem of G. Mikhalkin [Mik05, Theorem 1] one can replace the algebraic curve count in $\mathbb{CP}^2$ by an enumeration of certain *tropical curves*. E. Brugallé and G. Mikhalkin [BM07, BM09] introduced a class of decorated graphs, called *(marked) floor diagrams* (see Section 2), which, if counted correctly, are equinumerous to such tropical curves. We use a version of these results which incorporates tangency conditions due to S. Fomin and G. Mikhalkin [FM10] (see Theorem 2.4). S. Fomin and G. Mikhalkin also introduced a *template decomposition* of floor diagrams which we extend to be suitable for the relative case. This decomposition is crucial in the proofs of all results in this paper, as is the reformulation of algebraic curve counts in terms of floor diagrams.

For some related work see [AB10], where F. Ardila and the author generalized the polynomiality of Severi degrees to a class of toric surfaces which contains $\mathbb{CP}^1 \times \mathbb{CP}^1$ and Hirzebruch surfaces but which are non-smooth in general. A main feature is that we show polynomiality not only in the multi-degree of the curves but also in the parameters of the surface. In [BGM11], A. Gathmann, H. Markwig and the author defined *Psi-floor diagrams* which enumerate plane curves which satisfy point and tangency conditions, and conditions given by *Psi-classes*. We proved a Caporaso-Harris type recursion for Psi-floor diagrams, and show that *relative descendant Gromov-Witten invariants* equal their tropical counterparts.

This paper is organized as follows. In Section 2 we review the definition of floor diagrams and their markings. In Section 3 we introduce a new decomposition of floor diagrams which is compatible with tangency conditions. In Section 4 we discuss Theorems 1.1,1.2, 1.4 and Proposition 1.3. In Section 5 we discuss Theorem 1.5 and Proposition 1.6. For complete proofs of all statements see [Blo10].

Acknowledgements

I thank the two anonymous referees for their valuable comments and suggestions.

2 Floor diagrams and relative markings

Floor diagrams are a class of decorated graphs which, if counted with appropriate multiplicity, enumerate plane curves with prescribed properties. They were introduced by E. Brugallé and G. Mikhalkin [BM07, BM09] in the non-relative case and generalized to the relative setting by S. Fomin and G. Mikhalkin [FM10]. We begin with a review of the relative setup following notation of [FM10].

Definition 2.1 *A floor diagram $\mathcal{D}$ on a vertex set $\{1,\ldots,d\}$ is a directed graph (possibly with multiple edges) with edge weights $w(e) \in \mathbb{Z}_{>0}$ satisfying:*

1. *The edge directions preserve the vertex order, i.e., for each edge $i \to j$ of $\mathcal{D}$ we have $i < j$.*
2. *(Divergence Condition) For each vertex j of $\mathcal{D}$:*

$$\mathit{div}(j) \stackrel{\mathit{def}}{=} \sum_{\substack{\text{edges } e\\ j \xrightarrow{e} k}} w(e) - \sum_{\substack{\text{edges } e\\ i \xrightarrow{e} j}} w(e) \leq 1.$$

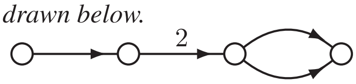
This means that at every vertex of $\mathcal{D}$ the total weight of the outgoing edges is larger by at most 1 than the total weight of the incoming edges.

The *degree* of a floor diagram $\mathcal{D}$ is the number of its vertices. It is *connected* if its underlying graph is. Note that in [FM10] floor diagrams are required to be connected. If $\mathcal{D}$ is connected its *genus* is the genus of the underlying graph (or the first Betti number of the underlying topological space). The *cogenus* of a connected floor diagram $\mathcal{D}$ of degree d and genus g is given by $\delta(\mathcal{D}) = \frac{(d-1)(d-2)}{2} - g$. If $\mathcal{D}$ is not connected let $d_1, d_2, \ldots$ and $\delta_1, \delta_2, \ldots$ be the degrees and cogenera, respectively, of its connected components. Then the *cogenus* of $\mathcal{D}$ is $\delta(\mathcal{D}) = \sum_j \delta_j + \sum_{j<j'} d_j d_{j'}$. Via the correspondence between algebraic curves and floor diagrams [BM09, Theorem 2.5] these notions correspond literally to the respective analogues for algebraic curves. Connectedness corresponds to irreducibility. Lastly, a marked floor diagram $\mathcal{D}$ has *multiplicity*[(i)]

$$\mu(\mathcal{D}) \stackrel{\text{def}}{=} \prod_{\text{edges } e} w(e)^2.$$

We draw floor diagrams using the convention that vertices in increasing order are arranged left to right. Edge weights of 1 are omitted.

Example 2.2 *An example of a floor diagram of degree $d = 4$, genus $g = 1$, cogenus $\delta = 2$, divergences $1, 1, 0, -2$, and multiplicity $\mu = 4$ is drawn below.*



To enumerate algebraic curves with floor diagrams we need the notion of markings of such diagrams. Our notation, which is more convenient for our purposes, differs slightly from [FM10] where S. Fomin and G. Mikhalkin define relative markings relative to the partitions $\lambda = \langle 1^{\alpha_1} 2^{\alpha_2} \cdots \rangle$ and $\rho = \langle 1^{\beta_1} 2^{\beta_2} \cdots \rangle$. In the sequel, all sequences are sequences of non-negative integers with finite support.

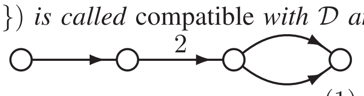
Definition 2.3 *For two sequences α, β we define an (α,β)-*marking *of a floor diagram $\mathcal{D}$ of degree $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$ by the following four step process which we illustrate in the case of Example 2.2 for $\alpha = (1,0,0,\ldots)$ and $\beta = (1,1,0,0,\ldots)$.*

Step 1: *Fix a pair of collections of sequences $(\{\alpha^i\},\{\beta^i\})$, where i runs over the vertices of $\mathcal{D}$, with:*

(i) If floor diagrams are viewed as floor contractions of tropical plane curves this corresponds to the notion of multiplicity of tropical plane curves.

1. *The sums over each collection satisfy* $\sum_{i=1}^{d} \alpha^i = \alpha$ *and* $\sum_{i=1}^{d} \beta^i = \beta$.
2. *For all vertices* i *of* $\mathcal{D}$ *we have* $\sum_{j\geq 1} j(\alpha_j^i + \beta_j^i) = 1 - div(i)$.

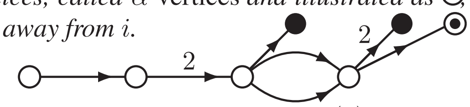
The second condition says that the "degree of the pair (α^i, β^i)*" is compatible with the divergence at vertex* i. *Each such pair* $(\{\alpha^i\}, \{\beta^i\})$ *is called* compatible *with* $\mathcal{D}$ *and* (α, β). *We omit writing down trailing zeros.*



$\alpha^i =$ (1)

$\beta^i =$ (1) $(0,1)$

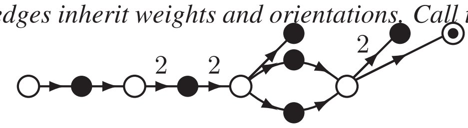
Step 2: *For each vertex* i *of* $\mathcal{D}$ *and every* $j \geq 1$ *create* β_j^i *new vertices, called* β-vertices *and illustrated as* ●*, and connected them to* i *with new edges of weight* j *directed away from* i. *For each vertex* i *of* $\mathcal{D}$ *and every* $j \geq 1$ *create* α_j^i *new vertices, called* α-vertices *and illustrated as* ⊙*, and connected them to* i *with new edges of weight* j *directed away from* i.



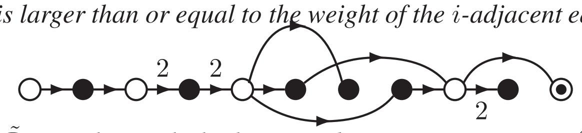
$\alpha^i =$ (1)

$\beta^i =$ (1) $(0,1)$

Step 3: *Subdivide each edge of the original floor diagram* $\mathcal{D}$ *into two directed edges by introducing a new vertex for each edge. The new edges inherit weights and orientations. Call the resulting graph* $\tilde{\mathcal{D}}$.



Step 4: *Linearly order the vertices of* $\tilde{\mathcal{D}}$ *extending the order of the vertices of the original floor diagram* $\mathcal{D}$ *such that, as in* $\mathcal{D}$*, each edge is directed from a smaller vertex to a larger vertex. Furthermore, we require that the* α*-vertices are largest among all vertices, and for every pair of* α*-vertices* $i' > i$*, the weight of the* i'*-adjacent edge is larger than or equal to the weight of the* i*-adjacent edge.*



We call the extended graph $\tilde{\mathcal{D}}$*, together with the linear order on its vertices, an* (α, β)-marked floor diagram*, or an* (α, β)-marking *of the floor diagram* $\mathcal{D}$.

We need to count (α, β)-marked floor diagrams up to equivalence. Two (α, β)-markings $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$ of a floor diagram $\mathcal{D}$ are *equivalent* if there exists a weight preserving automorphism of weighted graphs mapping $\tilde{\mathcal{D}}_1$ to $\tilde{\mathcal{D}}_2$ which fixes the vertices of $\mathcal{D}$. The *number of markings* $\nu_{\alpha,\beta}(\mathcal{D})$ is the number of (α, β)-marked floor diagrams $\tilde{\mathcal{D}}$ up to equivalence. Furthermore, we write $\mu_\beta(\mathcal{D})$ for the product $1^{\beta_1} 2^{\beta_2} \cdots \mu(\mathcal{D})$. The next theorem follows from [FM10, Theorem 3.18] by a straight-forward extension of the inclusion-exclusion procedure of [FM10, Section 1] which was used to conclude [FM10, Corollary 1.9] (the non-relative count of reducible curves via floor diagrams) from [FM10, Theorem 1.6] (the non-relative count of irreducible curves via floor diagrams).

Theorem 2.4 *For any* $\delta \geq 1$*, the relative Severi degree* $N_{\alpha,\beta}^{\delta}$ *is given by*

$$N_{\alpha,\beta}^{\delta} = \sum_{\mathcal{D}} \mu_\beta(\mathcal{D}) \nu_{\alpha,\beta}(\mathcal{D}),$$

where the sum is over (possibly disconnected) floor diagrams $\mathcal{D}$ *of degree* $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$ *and cogenus* δ.

3 Relative Decomposition of Floor Diagrams

In this section we introduce a new decomposition of floor diagrams compatible with tangency conditions. It is crucial in the proofs of all results stated in Section 1. The new decomposition is an extension of ideas of S. Fomin and G. Mikhalkin [FM10]. We start out by reviewing their key gadget.

Definition 3.1 *A* template Γ *is a directed graph (possibly with multiple edges) on vertices* $\{0, \ldots, l\}$, *where* $l \geq 1$, *and edge weights* $w(e) \in \mathbb{Z}_{>0}$, *satisfying:*

1. *If* $i \to j$ *is an edge then* $i < j$.
2. *Every edge* $i \xrightarrow{e} i+1$ *has weight* $w(e) \geq 2$. *(No "short edges.")*
3. *For each vertex* j, $1 \leq j \leq l-1$, *there is an edge "covering" it, i.e., there exists an edge* $i \to k$ *with* $i < j < k$.

Every template Γ comes with some numerical data associated with it. Its *length* $l(\Gamma)$ is the number of vertices minus 1. The product of squares of the edge weights is its *multiplicity* $\mu(\Gamma)$. Its *cogenus* $\delta(\Gamma)$ is

$$\delta(\Gamma) \stackrel{\text{def}}{=} \sum_{i \xrightarrow{e} j} \Big[(j-i)w(e) - 1 \Big].$$

For $1 \leq j \leq l(\Gamma)$ let $\varkappa_j = \varkappa_j(\Gamma)$ denote the sum of the weights of edges $i \xrightarrow{e} k$ with $i < j \leq k$ and define

$$k_{min}(\Gamma) \stackrel{\text{def}}{=} \max_{1 \leq j \leq l} (\varkappa_j - j + 1).$$

This makes $k_{min}(\Gamma)$ the smallest positive integer k such that Γ can appear in a floor diagram on $\{1, 2, \ldots\}$ with left-most vertex k. See [FM10, Figure 10] for a list of all templates Γ with $\delta(\Gamma) \leq 2$.

Our new decomposition of a floor diagram $\mathcal{D}$ depends on two (infinite) matrices A and B of non-negative integers. We require both to have only finitely many non-zero entries all of which lie above the respective dth row, where d is the degree of $\mathcal{D}$.

The triple $(\mathcal{D}, A, B)$ decomposes as follows. Let $l(A)$ and $l(B)$ be the largest row indices such that A and B have a non-zero entry in this row, respectively. After we remove all "short edges" from $\mathcal{D}$, i.e., all edges of weight 1 between consecutive vertices, the resulting graph is an ordered collection of templates $(\Gamma_1, \ldots, \Gamma_r)$, listed left to right. Let k_s be the smallest vertex in $\mathcal{D}$ of each template Γ_s. Record all pairs (Γ_s, k_s) which satisfy $k_s + l(\Gamma_s) \leq d - \max(l(A), l(B))$. Record the remaining templates together with all vertices i, for $i \geq \max(l(A), l(B))$ in *one* graph Λ on vertices $0, \ldots, l$ by shifting the vertex labels by $d - l$. See Example 3.4 for an example of this decomposition. Furthermore, by construction, if m is the number of recorded pairs (Γ_s, k_s), we have

$$\begin{cases} k_i & \geq & k_{\min}(\Gamma_i) & \text{for } 1 \leq i \leq m, \\ k_{i+1} & \geq & k_i + l(\Gamma_i) & \text{for } 1 \leq i \leq m-1, \\ k_m + l(\Gamma_m) & \leq & d - l(\Lambda). & \end{cases} \tag{3.1}$$

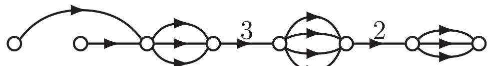
Fix a floor diagram $\mathcal{D}$. A partitioning of α and β into a compatible pair of collections $(\{\alpha^i, \beta^i\})$ (see Step 1 in Definition 2.3), where i runs over the vertices of $\mathcal{D}$, determines a pair of matrices A, B, if $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ are large enough as follows. Define the ith row a_i resp. b_i of the matrices A resp. B to be the sequence α^{d-i} resp. β^{d-i}, for $i \geq 1$, where d is the degree of $\mathcal{D}$. (If $d - i \leq 0$, set

$a_i = b_i = 0$.) In other words, the jth entry a_{ij} in row i of A is the number of α-edges of weight i adjacent to the $(j+1)$st vertex of Λ, counted from the right, and similarly for B (see Example 3.2). The sequences α^d and β^d (which are attached to the right-most vertex of $\mathcal{D}$) satisfy

$$\alpha^d = \alpha - \sum_{i \geq 1} a_i \quad \text{and} \quad \beta^d = \beta - \sum_{i \geq 1} b_i \tag{3.2}$$

if both expression are (component-wise) non-negative.

Example 3.2 *For $\alpha = (1,1)$, $\beta = (4,1)$ and the floor diagram $\mathcal{D}$ pictured below, the partitioning*

$\mathcal{D} =$ 

$\alpha^i =$ $(0,1)$ (1)

$\beta^i =$ (1) (1) $(2,1)$

determines the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

In light of (3.2) we consider, for given tangency conditions α and β, only triples $(\mathcal{D}, A, B)$ with

$$\sum_{i \geq 1} a_i \leq \alpha \quad \text{and} \quad \sum_{i \geq 1} b_i \leq \beta \qquad \text{(always component-wise)}, \tag{3.3}$$

For fixed d, the decomposition

$$\big(\mathcal{D}, A, B\big) \longrightarrow \big(\{(\Gamma_s, k_s)\}, \Lambda, A, B\big). \tag{3.4}$$

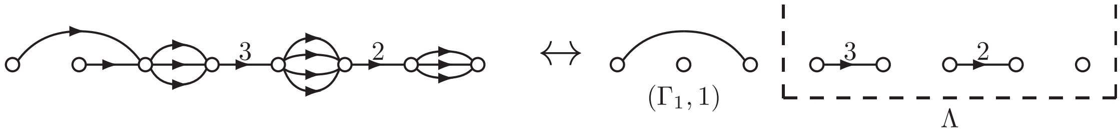
is reversible if the data on the right-hand side satisfies (3.1) and the tuple (Λ, A, B) is an "extended template."

Definition 3.3 *A tuple (Λ, A, B) is an* extended template *of* length $l = l(\Lambda) = l(\Lambda, A, B)$ *if Λ is a directed graph (possibly with multiple edges) on vertices $\{0, \ldots, l\}$, where $l \geq 0$, with edge weights $w(e) \in \mathbb{Z}_{>0}$, satisfying:*

1. *If $i \to j$ is an edge then $i < j$.*
2. *Every edge $i \xrightarrow{e} i+1$ has weight $w(e) \geq 2$. (No "short edges.")*

Moreover, we require A and B to be (infinite) matrices with non-negative integral entries and finite support, and we write $l(A)$ and $l(B)$ for the respective largest row indices of A and B of a non-zero entry. Additionally, we demand that $l(\Lambda) \geq \max(l(A), l(B))$ and that, for each $1 \leq j < l - \max(l(A), l(B))$, there is an edge $i \to k$ of Λ with $i < j < k$.

Example 3.4 *An example of a decomposition of a floor diagram $\mathcal{D}$ subject to the matrices A and B of Example 3.2 is pictured below. Once we fix the degree of the floor diagram the decomposition is reversible (here $d = 8$).*



The *cogenus* of an extended template (Λ, A, B) is the sum of the cogenera $\delta(\Lambda)$, $\delta(A)$ and $\delta(B)$, where

$$\delta(\Lambda) \stackrel{\text{def}}{=} \sum_{i \xrightarrow{e} j} \Big[(j-i)w(e) - 1\Big], \qquad \delta(A) \stackrel{\text{def}}{=} \sum_{i,j \geq 1} i \cdot j \cdot a_{i,j},$$

and similarly for B. It is not hard to see that the correspondence (3.4) is cogenus preserving in the sense that

$$\delta(\mathcal{D}) = \left(\sum_{i=1}^{m} \delta(\Gamma_i)\right) + \delta(\Lambda) + \delta(A) + \delta(B).$$

With an extended template (Λ, A, B) we associate the following numerical data. For $1 \leq j \leq l(\Lambda)$ let $\varkappa_j(\Lambda)$ denote the sum of the weights of edges $i \to k$ of Λ with $i < j \leq k$. Define $d_{min}(\Lambda, A, B)$ to be the smallest positive integer d such that (Λ, A, B) can appear (at the right end) in a floor diagram on $\{1, 2, \ldots, d\}$. We will see later that d_{min} is given by an explicit formula. For a matrix $A = (a_{ij})$ of non-negative integers with finite support define the "weighted lower sum sequence" $\mathrm{wls}(A)$ by

$$\mathrm{wls}(A)_i \stackrel{\text{def}}{=} \sum_{i' \geq i, j \geq 1} j \cdot a_{i'j}. \tag{3.5}$$

We now define the number of "markings" of templates and extended templates and relate them to the number of (α, β)-markings of the corresponding floor diagrams. To each template Γ we associate a polynomial as follows. For $k \geq k_{\min}(\Gamma)$ let $\Gamma_{(k)}$ denote the graph obtained from Γ by first adding $k + i - 1 - \varkappa_i$ short edges connecting $i - 1$ to i, for $1 \leq i \leq l(\Gamma)$, and then subdividing each edge of the resulting graph by introducing one new vertex for each edge. By [FM10, Lemma 5.6] the number of linear extensions (up to equivalence, see the paragraph after Definition 2.3) of the vertex poset of the graph $\Gamma_{(k)}$ extending the vertex order of Γ is given by a polynomial $P_\Gamma(k)$ in k, if $k \geq k_{min}(\Gamma)$(see [FM10, Figure 10]).

For each pair of sequences (α, β) and each extended template (Λ, A, B) satisfying (3.3) and $d \geq d_{min}$, where $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$, we define its "number of markings" as follows. Write $l = l(\Lambda)$ and let $\mathcal{P}(\Lambda, A, B)$ be the poset obtained from Λ by

1. first creating an additional vertex $l + 1$ $(> l)$,
2. then adding b_{ij} edges of weight j between $l - i$ and $l + 1$, for all $1 \leq i \leq l$ and $j \geq 1$,
3. then adding $\beta_j - \sum_{i \geq 1} b_{ij}$ edges of weight j between l and $l + 1$, for $j \geq 1$,
4. then adding
$$d - l(\Lambda) + i - 1 - \varkappa_i(\Lambda) - \mathrm{wls}(A)_{l+1-i} - \mathrm{wls}(B)_{l+1-i} \tag{3.6}$$
("short") edges of weight 1 connecting $i - 1$ and i, for $1 \leq i \leq l$, and finally
5. subdividing all edges of the resulting graph by introducing a midpoint vertex for each edge.

Denote by $Q_{(\Lambda, A, B)}(\alpha; \beta)$ the number of linear orderings on $\mathcal{P}(\Lambda, A, B)$ (up to equivalence) which extend the linear order on Λ. As $d \geq d_{\min}(\Lambda, A, B)$ if and only if (3.6) is non-negative, for $1 \leq i \leq l$, we have

$$d_{min}(\Lambda, A, B) = \max_{1 \leq i \leq l(\Lambda)} (l(\Lambda) - i + 1 + \varkappa_i(\Lambda) + \mathrm{wls}(A)_{l(\Lambda)+1-i} + \mathrm{wls}(B)_{l(\Lambda)+1-i}).$$

For sequences $s, t_1, t_2, \dots$ with $s \geq \sum_i t_i$ (component-wise) we denote by $\binom{s}{t_1, t_2, \dots} \stackrel{\text{def}}{=} \frac{s!}{t_1! t_2! \cdots (s - \sum_i t_i)!}$ the multinomial coefficient of sequences.

We obtain all (α, β)-markings of the floor diagram $\mathcal{D}$ that come from a compatible pair of sequences $(\{\alpha^i\}, \{\beta^i\})$ by independently ordering the α-vertices and the non-α-vertices. The number such markings is given (via the correspondence (3.4)) by

$$\Big(\prod_{s=1}^{m} P_{\Gamma_s}(k_s) \Big) \cdot \binom{\alpha}{a_1^T, a_2^T, \dots} \cdot Q_{(\Lambda, A, B)}(\alpha; \beta), \tag{3.7}$$

where $a_1^T, a_2^T, \dots$ are the *column* vectors of A. We conclude this section by recasting relative Severi degrees in terms of templates and extended templates.

Proposition 3.5 *For any $\delta \geq 1$, the relative Severi degree $N^{\delta}_{\alpha, \beta}$ is given by*

$$\sum_{\substack{(\Gamma_1, \dots, \Gamma_m),\\ (\Lambda, A, B)}} \Big(\prod_{s=1}^{m} \mu(\Gamma_s) \sum_{k_1, \dots k_m} \prod_{s=1}^{m} P_{\Gamma_s}(k_s) \Big) \cdot \Big(\mu(\Lambda) \prod_{i \geq 1} i^{\beta_i} \binom{\alpha}{a_1, a_2, \dots} Q_{(\Lambda, A, B)}(\alpha; \beta) \Big), \tag{3.8}$$

where the first sum is over all collections $(\Gamma_1, \dots, \Gamma_m)$ of templates and all extended templates (Λ, A, B) satisfying (3.3), $d \geq d_{min}(\Lambda, A, B)$ and

$$\sum_{i=1}^{m} \delta(\Gamma_i) + \delta(\Lambda) + \delta(A) + \delta(B) = \delta,$$

and the second sum is over all positive integers $k_1, \dots, k_m$ which satisfy (3.1).

4 Relative Severi Degrees and Polynomiality

We now turn to the discussion of the proofs of our main results by first mentioning a number of technical lemmata whose proofs can be found in [Blo10]. For a graph G, we denote by $\#E(G)$ the number of edges of G. We write $||A||_1 = \sum_{i,j \geq 1} a_{ij}$ for the 1-norm of a (possibly infinite) matrix $A = (a_{ij})$.

Lemma 4.1 *For every extended template (Λ, A, B) there is a polynomial $q_{(\Lambda, A, B)}$ in $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ of degree $\#E(\Lambda) + ||B||_1 + \delta(B)$ such that for all α and β satisfying (3.3) the number $Q_{(\Lambda, A, B)}(\alpha; \beta)$ of linear orderings (up to equivalence) of the poset $\mathcal{P}(\Lambda, A, B)$ is given by*

$$Q_{(\Lambda, A, B)}(\alpha; \beta) = \frac{(|\beta| - \delta(B))!}{\beta!} \cdot q_{(\Lambda, A, B)}(\alpha; \beta)$$

provided $\sum_{i \geq 1} i(\alpha_i + \beta_i) \geq d_{min}(\Lambda, A, B)$.

Recall that, for an extended template (Λ, A, B), we defined $d_{\min} = d_{\min}(\Lambda, A, B)$ to be the smallest $d \geq 1$ such that $d - l(\Lambda) + i - 1 \geq \varkappa_i(\Lambda) + \text{wls}(A)_{l(\Lambda)+1-i} + \text{wls}(B)_{l(\Lambda)+1-i}$ for all $1 \leq i \leq l(\Lambda)$. Let i_0 be the smallest i for which equality is attained (it is easy to see that equality is attained for some i). Define the quantity $s(\Lambda, A, B)$ to be the number of edges of Λ from $i_0 - 1$ to i_0 (of any weight). See [Blo10, Figure 2] for examples.

Lemma 4.2 *For any extended template* (Λ, A, B) *and any* $\alpha, \beta \geq 0$ *(component-wise) with*

$$d_{\min}(\Lambda, A, B) - s(\Lambda, A, B) \leq \sum_{i \geq 1} i(\alpha_i + \beta_i) \leq d_{\min}(\Lambda, A, B) - 1$$

we have $q_{(\Lambda, A.B)}(\alpha; \beta) = 0$*, where* $q_{(\Lambda, A, B)}$ *is the polynomial of Lemma 4.1.*

The next lemma specifies which extended templates are compatible with a given degree.

Lemma 4.3 *For every extended template* (Λ, A, B) *we have*

$$d_{min}(\Lambda, A, B) - s(\Lambda, A, B) \leq \delta(\Lambda) + \delta(A) + \delta(B) + 1.$$

Proof of Theorems 1.1 and 1.2: The basic idea of the proof is to show that all summands of (3.8) are polynomial in $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ (up to the combinatorial factor), thus contribute polynomially to the relative node polynomial $N_\delta(\alpha; \beta)$. The first factor of each summand is an iterated "discrete integral" of polynomials and thus polynomial in d. For the second factor we use Lemma 4.1. Then we use Lemmata 4.2 and 4.3 to reduce the polynomiality threshold. For a detailed proof see [Blo10]. $\square$

Remark 4.4 *Expression (3.8) gives, in principle, an algorithm to compute the relative node polynomial* $N_\delta(\alpha; \beta)$*, for any* $\delta \geq 1$*. In [Blo11, Section 3] we explain how to generate all templates of a given cogenus, and how to compute the first factor in (3.8). The generation of all extended templates of a given cogenus from the templates is straightforward, as is the computation of the second factor in (3.8).*

Proof of Theorem 1.4: Proposition 3.5 gives a combinatorial description of relative Severi degrees. The proof of Lemma 4.1 (see [Blo10]) provides a method to calculate the polynomial $Q_{(\Lambda, A, B)}(\alpha; \beta)$. All terms of expression (3.8) are explicit or can be evaluated using the techniques of [Blo11, Section 3]. This reduces the calculation to a (non-trivial) computer calculation. $\square$

5 Coefficients of Relative Node Polynomials

We now turn towards the computation of the coefficients of the relative node polynomial $N_\delta(\alpha; \beta)$ of large degree for any δ. We propose a method to compute all terms of $N_\delta(\alpha; \beta)$ of degree $\geq 3\delta - t$, for any given $t \geq 0$. This method was used (with $t = 2$) to compute the terms in Theorem 1.5.

The main idea of the algorithm is that, even for general δ, only a small number of summands of (3.8) contribute to the terms of $N_\delta(\alpha; \beta)$ of large degree. A summand of (3.8) is indexed by a collection of templates $\tilde{\Gamma} = \{\Gamma_s\}$ and an extended template (Λ, A, B). To determine whether this summand might contribute to $N_\delta(\alpha; \beta)$ we define the *(degree) defects*

- of the collection of templates $\tilde{\Gamma}$ by $\mathrm{def}(\tilde{\Gamma}) \stackrel{\text{def}}{=} \sum_{s=1}^{m} \left[\delta(\Gamma_i)\right] - m$, and
- of the extended template (Λ, A, B) by $\mathrm{def}(\Lambda, A, B) \stackrel{\text{def}}{=} \delta(\Lambda) + 2\delta(A) + 2\delta(B) - ||A||_1 - ||B||_1$.

The following lemma restricts the indexing set of (3.8) to the relevant terms, if only the leading terms of $N_\delta(\alpha; \beta)$ are of interest. For a proof see [Blo10].

Lemma 5.1 *The summand of (3.8) indexed by* $\tilde{\Gamma}$ *and* (Λ, A, B) *is of the form*

$$1^{\beta_1}2^{\beta_2}\cdots\frac{(|\beta|-\delta)!}{\beta!}\cdot P(\alpha;\beta),$$

where $P(\alpha;\beta)$ *is a polynomial in* $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ *of degree* $\leq 3\delta - \mathrm{def}(\tilde{\Gamma}) - \mathrm{def}(\Lambda, A, B)$.

Therefore, to compute the coefficients of degree $\geq 3\delta - t$ of $N_\delta(\alpha;\beta)$ for some $t \geq 0$, it suffices to consider only summands of (3.8) with $\mathrm{def}(\tilde{\Gamma}) \leq t$ and $\mathrm{def}(\Lambda, A, B) \leq t$.

One can proceed as follows. First, we can compute, for some formal variable $\tilde{\delta}$, the terms of degree $\geq 2\tilde{\delta} - t$ of the first factor of (3.8) to $N_{\tilde{\delta}}(\alpha;\beta)$, that is the terms of degree $\geq 2\tilde{\delta} - t$ of

$$R_{\tilde{\delta}}(d) \stackrel{\mathrm{def}}{=} \sum\prod_{i=1}^{m}\mu(\Gamma_i)\sum_{k_m=k_{min}(\Gamma_m)}^{d-l(\Gamma_m)}P_{\Gamma_m}(k_m)\cdots\sum_{k_1=k_min(\Gamma_1)}^{k_2-l(\Gamma_1)}P_{\Gamma_1}(k_1), \tag{5.1}$$

where the first sum is over all collections of templates $\tilde{\Gamma} = (\Gamma_1, \ldots, \Gamma_m)$ with $\delta(\tilde{\Gamma}) = \tilde{\delta}$. The leading terms of $R_{\tilde{\delta}}(d)$ can be computed with a slight modification of [Blo11, Algorithm 2] (by replacing, in the notation of [Blo11], C^{end} by C and M^{end} by M). The algorithm relies on the polynomiality of solutions of certain polynomial difference equations, which has been verified for $t \leq 7$, see [Blo11, Section 5] for more details.

Finally, to compute the coefficients of degree $\geq 3\delta - t$, it remains to compute all extended templates (Λ, A, B) with $\mathrm{def}(\Lambda, A, B) \leq t$ and collect the terms of degree $\geq 3\delta - t$ of the polynomial

$$R_{\tilde{\delta}}(d - l(\Lambda))\cdot\mu(\Lambda)\binom{\alpha}{a_1^T, a_2^T, \ldots}\prod_{i=\delta(B)}^{\delta-1}(|\beta| - i)\cdot q_{(\Lambda,A,B)}(\alpha;\beta), \tag{5.2}$$

where, as before, $a_1^T, a_2^T, \ldots$ denote the column vectors of the matrix A, $q_{(\Lambda,A,B)}(\alpha;\beta)$ is the polynomial of Lemma 4.1, and $\tilde{\delta} = \delta - \delta(\Lambda, A, B)$. Notice that, for an indeterminant x and integers $c \geq 0$ and $\delta \geq 1$, we have the expansion

$$\prod_{i=c}^{\delta-1}(x-i) = \sum_{t=0}^{\delta-c}s(\delta-c, \delta-c-t)(x-c)^{\delta-c-t},$$

where $s(n,m)$ is the *Stirling number of the first kind* [Sta97, Section 1.3] for integers $n, m \geq 0$. Furthermore, with $\delta' = \delta - c$ the coefficients $s(\delta', \delta' - t)$ of the sum equal $\delta'(\delta'-1)\cdots(\delta'-t)\cdot S_t(\delta')$, where S_t is the t-th *Stirling polynomial* [GKP94, (6.45)], for $t \geq 0$, and thus are polynomial in δ'. Therefore, we can compute the leading terms of the product in (5.2) by collecting the leading terms in the sum expansion above. Theorem 1.5 is proved by an implementation of this method.

Proof of Proposition 1.6: Using [Blo11, Algorithm 2] we can compute the terms of the polynomial $R_{\tilde{\Gamma}}(d)$ of degree $\geq 2\tilde{\delta} - 7$ (see [Blo11, Section 5]) and observe that all coefficients are polynomial in $\tilde{\delta}$. By the previous paragraph the coefficients of the expansion of the sum of (5.2) are polynomial in δ. This completes the proof. □

Remark 5.2 *It is straight-forward to compute the coefficients of* $N_\delta(\alpha;\beta)$ *of degree* $\geq 3\delta - 7$ *(and thereby to extend Theorem 1.5). Algorithm 3 of [Blo11] computes the coefficients of the polynomials* $R_{\tilde{\delta}}(d)$ *of degree* $\geq 2\tilde{\delta} - 7$*, and thus the desired terms can be collected from (5.2). We expect this method to compute the leading terms of* $N_\delta(\alpha,\beta)$ *of degree* $\geq 3\delta - t$ *for arbitrary* $t \geq 0$ *(see [Blo11, Section 5], especially Conjecture 5.5).*

References

[AB10] F. Ardila and F. Block. Universal polynomials for severi degrees of toric surfaces. Preprint, arXiv:1012.5305, 2010.

[BGM11] F. Block, A. Gathmann, and H. Markwig. Psi-floor diagrams and a Caporaso-Harris type recursion. *Israel J. Math. (to appear)*, 2011.

[Blo10] F. Block. Relative node polynomials for plane curves. Preprint, arXiv:1009.5063, 2010.

[Blo11] F. Block. Computing node polynomials for plane curves. *Math. Res. Lett. (to appear)*, 2011.

[BM07] E. Brugallé and G. Mikhalkin. Enumeration of curves via floor diagrams. *C. R. Math. Acad. Sci. Paris*, 345(6):329–334, 2007.

[BM09] E. Brugallé and G. Mikhalkin. Floor decompositions of tropical curves: the planar case. In *Proceedings of Gökova Geometry-Topology Conference 2008*, pages 64–90. Gökova Geometry/Topology Conference (GGT), Gökova, 2009.

[CH98] L. Caporaso and J. Harris. Counting plane curves of any genus. *Invent. Math.*, 131(2):345–392, 1998.

[DFI95] P. Di Francesco and C. Itzykson. Quantum intersection rings. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 81–148. Birkhäuser Boston, Boston, MA, 1995.

[Enr12] F. Enriques. Sui moduli d'una classe di superficie e sul teorema d'esistenza per funzioni algebriche di due variabilis. *Atti Accad. Sci. Torino*, 47, 1912.

[FM10] S. Fomin and G. Mikhalkin. Labeled floor diagrams for plane curves. *J. Eur. Math. Soc. (JEMS)*, 12(6):1453–1496, 2010.

[Gat06] A. Gathmann. Tropical algebraic geometry. *Jahresber. Deutsch. Math.-Verein.*, 108(1):3–32, 2006.

[GKP94] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete mathematics*. Addison-Wesley Publishing Company, Reading, MA, second edition, 1994. A foundation for computer science.

[Har86] J. Harris. On the Severi problem. *Invent. Math.*, 84(3):445–461, 1986.

[KP04] S. Kleiman and R. Piene. Node polynomials for families: methods and applications. *Math. Nachr.*, 271:69–90, 2004.

[Mik05] G. Mikhalkin. Enumerative tropical geometry in $\mathbb{R}^2$. *J. Amer. Math. Soc.*, 18:313–377, 2005.

[Sev21] F. Severi. *Vorlesungen über Algebraische Geometrie*. Teubner, Leipzig, 1921.

[Sta97] R. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[Vai95] I. Vainsencher. Enumeration of n-fold tangent hyperplanes to a surface. *J. Algebraic Geom.*, 4(3):503–526, 1995.

[Vak00] R. Vakil. Counting curves on rational surfaces. *Manuscripta Math.*, 102(1):53–84, 2000.