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Dual combinatorics of zonal polynomials

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Abstract. In this paper we establish a new combinatorial formula for zonal polynomials in terms of power-sums. The proof relies on the sign-reversing involution principle. We deduce from it formulas for zonal characters, which are defined as suitably normalized coefficients in the expansion of zonal polynomials in terms of power-sum symmetric functions. These formulas are analogs of recent developments on irreducible character values of symmetric groups. The existence of such formulas could have been predicted from the work of M. Lassalle who formulated two positivity conjectures for Jack characters, which we prove in the special case of zonal polynomials.

Résumé. Dans cet article, nous établissons une nouvelle formule combinatoire pour les polynômes zonaux en fonction des fonctions puissance. La preuve utilise le principe de l'involution changeant les signes. Nous en déduisons des formules pour les caractères zonaux, qui sont définis comme les coefficients des polynômes zonaux écrits sur la base des fonctions puissance, normalisés de manière appropriée. Ces formules sont des analogues de développements récents sur les caractères du groupe symétrique. L'existence de telles formules aurait pu être prédite à partir des travaux de M. Lassalle, qui a proposé deux conjectures de positivité sur les caractères de Jack, que nous prouvons dans le cas particulier des polynômes zonaux.

Keywords: zonal polynomials, zonal characters, Jack polynomials, Kerov polynomials, Stanley polynomials

The results of this extended abstract correspond to the ones of the full paper [FŚ11b], which will be published elsewhere. However, the method of proof used here is different and new.

1 Introduction

Zonal polynomials were introduced by James [Jam60, Jam61] (who credits also Hua [Hua63]) in order to solve some problems from statistics and multivariate analysis. They quickly became a fundamental tool in this theory as well as in the random matrix theory (an overview can be found in the book of Muirhead [Mui82]). They also play an important role in representation theory: they appear as zonal spherical functions of the pairs $(\mathfrak{S}_{2n}, H_n)$ (where H_n is the hyperoctahedral group) and $(\mathrm{GL}_d(\mathbb{R}), O_d)$, which means that they describe a canonical basis of the algebra of left and right H_n-invariant (resp. O_d-invariant) functions on $\mathfrak{S}_{2n}$ (resp. $\mathrm{GL}_d(\mathbb{R})$). This last property shows that zonal polynomials can be viewed as an analogue of Schur symmetric functions since the latter are zonal spherical functions for the Gelfand pairs $(\mathfrak{S}_n \times \mathfrak{S}_n, \mathfrak{S}_n)$ and $(\mathrm{GL}_d(\mathbb{C}), U_d)$. Besides, many of the properties of Schur functions can be extended to zonal polynomials and this article goes in this direction.

The main result of this article is a new combinatorial formula for zonal polynomials (Theorem 3). Note that, as the latter are a particular case of Jack symmetric functions, there already exists a combinatorial interpretation for them in terms of ribbon tableaux (due to Stanley [Sta89]). But our formula is of different kind: using it, one can describe combinatorially the coefficients of the zonal polynomial Z_λ expanded in the power-sum basis as a function of λ. In more concrete words, the combinatorial objects describing the coefficient of p_μ in Z_λ depend on μ, whereas the statistics on them depend on λ (in Stanley's result it is roughly the opposite). Note that the ring of shifted symmetric functions [OO97] is involved in this *dual* approach [Las08, Proposition 2]. The situation is analogous to recent developments concerning characters of the symmetric groups.

Zonal polynomials are known to be the special case of Jack symmetric functions. We conjecture that there exists some formula for Jack polynomials extending the one we present here for zonal polynomials, but unfortunately we have not been able to find it: even if our proof uses only general properties of Jack polynomials, we really need the special value of the parameter corresponding to zonal polynomials.

The paper is organized as follows. In Section 2 we recall the necessary definitions. In Section 3 we state our main theorem and in Section 4 we present its proof. Then, in Section 5 we sketch a few consequences of this theorem on zonal characters.

2 Notations

In this section, we give some definitions and notations on symmetric functions in general, on Jack polynomials and, in particular, zonal polynomials. We also give a few definitions of pair-partitions, which are the combinatorial objects involved in our formulas.

2.1 Symmetric functions

As much as possible, we use the notations of I.G. Macdonald's book [Mac95].

By definition, a *partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ of the integer n is a non-increasing sequence of non-negative integers of sum n. Its *length* $\ell(\lambda)$ is the number of non-zero terms in the sequence. For $i \geq 1$ we denote by $m_i(\lambda)$ the number of occurrences of i in λ. Finally, let us define $\mathrm{aut}(\lambda) := \prod_{i\geq 1} m_i(\lambda)!$ and $z_\lambda = \mathrm{aut}(\lambda) \cdot \prod_j \lambda_j$.

The ring Λ of symmetric functions has several classical linear bases, all indexed by partitions:

- *monomial symmetric functions*: let us use for monomials the short notation $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} x_2^{v_2} \dots$, then
$$m_\lambda := \sum_{\mathbf{v} \in S_\infty(\lambda)} \mathbf{x}^{\mathbf{v}},$$
where the sum runs over all vectors $\mathbf{v}$ which are permutations of λ (without multiplicities);
- *power-sum symmetric functions*: they are defined by
$$p_0 = 1, \qquad p_k = \sum_i x_i^k, \qquad p_\mu = \prod_j p_{\mu_j};$$
- *Schur functions* (s_λ): they have several equivalent definitions, one of them will be given in the next paragraph.

In addition to the additive and multiplicative structures, the ring of symmetric functions can be endowed with a scalar product (called *Hall scalar product*), for which:
$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}, \qquad \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

2.2 Jack symmetric functions and zonal polynomials

The set of partitions of a given integer n can be endowed with a partial order, called dominance order: by definition,

$$\lambda \preceq \mu \iff \forall_{i \geq 1}\ \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i.$$

With respect to this order, the matrix of Schur functions in the monomial basis is upper triangular: *i.e.*, if one writes a Schur functions s_λ on the monomial basis, the indices of the monomial symmetric functions with non-zero coefficients are dominated by λ. This triangularity property, together with the fact that they form an orthonormal basis with respect to Hall scalar product, determines entirely the Schur functions. Indeed, one can construct them by choosing any total order refining the dominance order and applying Gram-Schmidt orthogonalization process to the monomial symmetric function basis.

Hall scalar product can be deformed in the following way: $\langle \cdot, \cdot \rangle_\alpha$ is the bilinear form whose values on the power-sum basis is given by

$$\langle p_\lambda,\ p_\mu \rangle_\alpha = z_\lambda\ \alpha^{\ell(\lambda)}\ \delta_{\lambda,\mu}.$$

The following result corresponds to equations (VI,10.13) and (VI,10.14) of Macdonald's book [Mac95].

Proposition 1 *Fix $\alpha > 0$. There exists a (unique) family of functions $J_\lambda^{(\alpha)}$ such that the following three conditions are fulfilled:*

TR *The matrix of $J_\lambda^{(\alpha)}$ written in the monomial basis is upper triangular.*

OR *The $J_\lambda^{(\alpha)}$ form an orthogonal basis of the ring of symmetric functions endowed with the scalar product $\langle \cdot, \cdot \rangle_\alpha$.*

N *The coefficient of $p_1^{|\lambda|}$ in the power sum expansion of $J_\lambda^{(\alpha)}$ is equal to 1.*

These functions are called Jack symmetric functions. When $\alpha = 1$ they correspond, up to multiplication by a scalar, to Schur functions. When $\alpha = 2$ they are called zonal polynomials (because, as we already mentioned in Section 1, they appear in representation theory of some Gelfand pairs, see [Mac95, VII, 2]) and denoted Z_λ.

The uniqueness of such a family is easy to prove: it is essentially the same argument as the one given above for the characterization of Schur functions.

2.3 Pair partitions

Definition 2 *A pair-partition of $[2n] = \{1, \ldots, 2n\}$ is a set of pairwise disjoint two-element sets, such that their (disjoint) union is equal to $[2n]$.*

A pair-partition S can be also seen as an involution as the set $[2n]$. Therefore, we use the notation $S(i)$ for the partner of i, *i.e.* the other element in the same set of S.

For two pair-partitions S_1, S_2 of the same set $[2n]$ we consider the edge-bicolored graph G_{S_1,S_2} defined as follows: its vertex set is $[2n]$, its red (resp. blue) edges are the pairs in S_1 (resp. S_2). As every vertex has degree 2, this graph is a disjoint union of loops. We denote $\mathcal{L}(S_1, S_2)$ the corresponding set-partition of $[2n]$. Note that the length of each loop (or equivalently the size of each block of $\mathcal{L}(S_1, S_2)$) is an even number because of the edge bicoloration.

It is common to associate to a set-partition the integer partition formed by the non-increasing sequence of the part sizes. Here, as we deal only with set partitions Π with parts of even length, we will rather consider the non-increasing sequence of the *half-sizes* of the parts. This integer partition is called the type of the set partition Π and is denoted $\mu(\Pi)$.

In other words, the type of $\mathcal{L}(S_1, S_2)$ is the partition $\ell_1, \ell_2, \dots$ where $(2\ell_i)_{i\geq 1}$ is the non-increasing sequence of the lengths of the loops of G_{S_1,S_2}. We also define the sign of a couple of pair-partitions as follows:

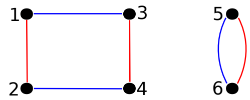
$$(-1)^{(S_1,S_2)} = (-1)^{(\ell_1-1)+(\ell_2-1)+\cdots} = (-1)^{n-|\mathcal{L}(S_1,S_2)|}.$$

Finally we use the short notations $p_\Pi = p_{\mu(\Pi)}$ and $m_\Pi = m_{\mu(\Pi)}$ for symmetric functions.

Example. We consider

$$S_1 = \big\{\{1,2\},\{3,4\},\{5,6\}\big\}; \qquad S_2 = \big\{\{1,3\},\{2,4\},\{5,6\}\big\}.$$

Then $G_{S_1,S_2} =$



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In this case, $\mathcal{L}(S_1, S_2) = \big\{\{1,2,3,4\},\{5,6\}\big\}$ and $\mu(\mathcal{L}(S_1, S_2)) = (2,1)$.

3 The main result

3.1 Zonal polynomials in terms of pair-partition

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n and T be a bijective filling of the Young diagram $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$ with the elements of $[2n]$. Any pair-partition of $[2n]$ can be viewed as a pairings of the boxes of T. We denote $S(T)$ the pair partition which matches the boxes $(i, 2j-1)$ and $(i, 2j)$ of T for all values of i and $1 \leq j \leq \lambda_i$. Besides, a couple of pair-partitions (S_1, S_2) is called T*-admissible* if each pair of boxes matched by S_2 is in the same row and if S_1 matches elements of the $2j-1$-th column of T with elements of its $2j$-th column.

Theorem 3 *Let λ be a partition and T be a bijective filling of 2λ. One has:*

$$Z_\lambda = \sum_{(S_1,S_2)\ T\text{-admissible}} (-1)^{(S(T),S_1)}\, p_{\mathcal{L}(S_1,S_2)}.$$

Example. Let $\lambda = (2,1)$ and $T = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}$. Then (S_1, S_2) is T-admissible if and only if:

$$S_1 \in \Big\{\{\{1,2\},\{3,4\},\{5,6\}\},\ \{\{1,6\},\{3,4\},\{2,5\}\}\Big\} \quad \text{and} \quad S_2 \in \Big\{\{\{1,2\},\{3,4\},\{5,6\}\},\ \{\{1,3\},\{2,4\},\{5,6\}\},\ \{\{1,4\},\{2,3\},\{5,6\}\}\Big\}.$$

The first possible value of S_1 gives $(-1)^{(S(T),S_1)} = 1$ and the types of $\mathcal{L}(S_1, S_2)$ for the three possible values of S_2 are, respectively, $(1,1,1)$, $(2,1)$ and $(2,1)$. For the second value of S_1 one has $(-1)^{(S(T),S_1)} = -1$ and the types of the corresponding set-partitions $\mathcal{L}(S_1, S_2)$ are, respectively, $(2,1)$, (3) and (3).

Finally, one obtains $Z_{(2,1)} = p_{(1,1,1)} + p_{(2,1)} - 2p_{(3)}$.

Remark. This theorem is an analogue of a known result on Schur symmetric functions:

$$\frac{n! \cdot s_\lambda}{\dim(\lambda)} = \sum (-1)^{|\sigma_1|}\, p_{\mathrm{type}(\sigma_1 \circ \sigma_2)}, \tag{1}$$

where the sum runs over pairs of permutations (σ_1, σ_2) of the boxes of the diagram λ such that σ_1 (resp. σ_2) preserves the columns (resp. the rows) of λ and $\mathrm{type}(\sigma_1 \circ \sigma_2)$ denotes the partition describing the lengths of the cycles of $\sigma_1 \circ \sigma_2$. This formula is a consequence of the explicit construction of the representation associated to λ via the Young symmetrizer. For a detailed proof, see [FŚ11a, Theorem 4]. In [Han88], the authors tries unsuccessfully to generalize it to Jack polynomials by introducing some statistics on couples of permutations. Our result shows, that, at least for $\alpha = 2$, a natural way to generalize it consists in using other combinatorial objects than permutations.

3.2 Application

Let us look at the coefficients $\theta_\nu^{(\alpha)}(\lambda)$ of the power sum expansion of Jack polynomials:

$$J_\lambda^{(\alpha)} = \sum \theta_\nu^{(\alpha)}(\lambda)\, p_\nu.$$

In the case $\alpha = 1$ we obtain, up to a multiplicative constant, the irreducible characters of the symmetric groups. By analogy, in the general case, we call these quantities *Jack characters* (or *zonal characters* in the case $\alpha = 2$). With the right choice of the normalizing constant, namely

$$\Sigma_\nu^{(\alpha)}(\lambda) = \binom{|\lambda| - |\nu| + m_1(\nu)}{m_1(\nu)} z_\nu\, \theta^{(\alpha)}_{\nu, 1^{|\lambda|-|\nu|}}(\lambda), \tag{2}$$

they are α-shifted symmetric functions of the partition λ.

Recently, M. Lassalle stated two positivity conjectures [Las08, Las09] suggesting the existence of some combinatorial description of this coefficient. Theorem 3 can be used to obtain such a combinatorial description in the particular case of zonal polynomials. Note that this particular case is especially interesting because the quantities $\theta_\nu^{(2)}(\lambda)$ have representation-theoretical interpretations (see [Mac95, VII,2]). We give precise statements in Section 5.

3.3 Combinatorial interpretation

The triplet of pair-partitions $(S(T), S_1, S_2)$ can be seen as a map (bipartite graph drawn on a — possibly non-orientable — two-dimensional, borderless surface), see [GJ96]. The statistics considered in Theorem 3, Theorem 9, Theorem 10 and Theorem 11 are natural in this context: $\mathcal{L}(S(T), S_1)$ and $\mathcal{L}(S(T), S_2)$ correspond respectively to white and black vertices, while $\mathcal{L}(S_1, S_2)$ corresponds to the faces.

Therefore, our results can be rephrased in a natural way using graphs on surfaces. For more details and precise statements see the complete version of this paper [FŚ11b].

3.4 Links between pair-partitions and zonal characters

It should be stressed that a previous link between pair-partitions and zonal characters can be found in the work of Goulden and Jackson [GJ96]. But their result goes in the opposite direction than ours: they count pair-partitions using zonal characters, while we express zonal characters using pair-partitions. The same picture exists for permutations and usual characters. It would be nice to understand the link between these two dual approaches.

4 Proof of Theorem 3

From now on we fix $\alpha = 2$. Let us define Y_λ as the right-hand side of Theorem 3 (it obviously does not depend on the choice of T). In this section we prove that Y_λ satisfies the three properties of Proposition 1. In this way we prove Theorem 3 and we also give an alternative proof of Proposition 1 in the special case $\alpha = 2$.

4.1 Combinatorial lemmas

The proof relies on three simple combinatorial lemmas.

Lemma 4 *Let λ and μ be two partitions of the same integer n. If λ does not dominate μ then for any bijective fillings T_λ and T_μ of the diagrams λ and μ with integers in $[n]$ there exist two elements i and j which are in the same row of T_μ and in the same column of T_λ.*

Proof: Let us replace in the tableau T_λ each number i by the index of the row of T_μ containing i. We have now a filling T'_λ of λ in which any number k appears μ_k times and we have to prove the existence of a column with some number appearing (at least) twice. Of course, reordering each column of T'_λ in the increasing order does not change anything. As λ does not dominate μ, there exists an integer i such that:

$$\mu_1 + \cdots + \mu_i > \lambda_1 + \cdots + \lambda_i.$$

But $\mu_1 + \cdots + \mu_i$ is exactly the number of integers smaller than $i+1$ in T'_λ. There is not enough room to fit all these numbers in the first i rows of λ, therefore there exists a box $\Box$ in the other rows of λ which contains a number which is smaller or equal than i. The column containing $\Box$ has therefore at least $i+1$ boxes with entries smaller or equal to i. The pigeon-hole principle shows that this column must contain a repeated entry. $\Box$

Consider the following action of S_{2n} on the pair-partitions of $[2n]$: if σ is a permutation in S_{2n} and S a pair-partition of $[2n]$, we denote by $\sigma(S)$ the pair partition such that $\{\sigma(i), \sigma(j)\}$ is a part of $\sigma(S)$ if and only if $\{i, j\}$ is a part of S. Of course, the graphs G_{S_1,S_2} and $G_{\sigma(S_1),\sigma(S_2)}$ are isomorphic thus the corresponding set of loops have the same type. Conversely:

Lemma 5 *The couples (S_1, S_2) such that $\mu(\mathcal{L}(S_1, S_2)) = \nu$ form exactly one orbit under the diagonal action of the symmetric group S_{2n}. Moreover, there are exactly $\frac{(2n)!}{z_\nu 2^{\ell(\nu)}}$ of them.*

Proof: Let us consider two couples (S_1, S_2) and (S'_1, S'_2), such that both graphs $G := G_{S_1,S_2}$ and $G' := G_{S'_1,S'_2}$ are collections of loops of lengths $2\nu_1, 2\nu_2 \dots$. These two graphs are isomorphic as edge-bicolored graphs. Let φ be any isomorphism of them. As it sends vertices of G on vertices of G', it can be seen as a permutation in S_{2n}. As it sends red (resp. blue) edges of G on red (resp. blue) edges of G', one has: $\varphi(S_1) = S'_1$ (resp. $\varphi(S_2) = S'_2$). Thus all couples of pair-partitions of type ν are in the same orbit.

We will prove now that the size of the centralizer of a fixed couple (S_1, S_2) is equal to $z_\nu 2^{\ell(\nu)} = \prod_i m_i(\nu)!(2i)^{m_i(\nu)}$. Indeed, an element in the centralizer is entirely determined by the image of one given point in each loop of $\mathcal{L}(S_1, S_2)$. Moreover, the image of an element in a loop of length 2ℓ must be in a loop of the same length 2ℓ and the images of two elements in two different loops must be in two different loops. $\Box$

Lemma 6 *Let λ be a partition of n, let T be a bijective filling of 2λ and let S_1 be a pair-partition matching elements of the $2j-1$-th column of T with elements in its $2j$-th column. If $\tau = (i\ j)$ with i and j in the same column of T, and $S = S(T)$ then*

$$(-1)^{(S,\tau(S_1))} = (-1)^{(S,S_1)+1}.$$

Proof: Let us first remark that $G_{S,\tau(S_1)}$ can be obtained from G_{S,S_1} by replacing the blue edges $\{S_1(i), i\}$ and $\{S_1(j), j\}$ by $\{S_1(i), j\}$ and $\{S_1(j), i\}$. We distinguish two cases:

- if i and j are in different loops in G_{S,S_1} then these loops are unified into one new loop in $G_{S,\tau(S_1)}$. Thus $\big|\mathcal{L}(S,\tau(S_1))\big| = \big|\mathcal{L}(S,S_1)\big| - 1$ and the lemma holds in this case;
- suppose i and j are in the same loop in G_{S,S_1}. As i and j are in the same column of T and both S and S_1 match elements of the $2j-1$-th column of T with elements of its $2j$-th column, the distance between i and j is even, where the distance is regarded as the number of steps within one loop. It is easy to check in this case that the loop containing i and j in G_{S,S_1} is split into two in $G_{S,\tau(S_1)}$. Therefore, $\big|\mathcal{L}(S,\tau(S_1))\big| = \big|\mathcal{L}(S,S_1)\big| + 1$ and the lemma also holds in this case. □

4.2 Triangularity

In this paragraph we prove that the functions Y_λ satisfy property **TR**.

Let us fix a bijective filling T of 2λ and set $S = S(T)$. The first step of the proof consist of writing

$$Y_\lambda = \sum_{(S_1,S_2)\ T\text{-admissible}} (-1)^{(S,S_1)}\ p_{\mathcal{L}(S_1,S_2)}$$

in the monomial basis. In order to do that we will write each $p_{\mathcal{L}(S_1,S_2)}$ on this basis.

Recall that, if Π and Π' are two set partitions, we say that Π is finer than Π' (and write $\Pi \le \Pi'$) if each part of Π' is the union of some parts of Π.

Lemma 7 *Let S_1, S_2 be two pair partitions of $\{1,\dots,2n\}$. Then*

$$p_{\mathcal{L}(S_1,S_2)} = \sum_{\Pi \ge \mathcal{L}(S_1,S_2)} \operatorname{aut}\big(\mu(\Pi)\big)\ m_\Pi.$$

Proof: It is a consequence of [Mac95, Chapter I, equation (6.9)]. □

Using this lemma, one can write Y_λ in the monomial basis:

$$\begin{aligned} Y_\lambda &= \sum_{(S_1,S_2)\ T\text{-admissible}}\ \sum_{\Pi\ge\mathcal{L}(S_1,S_2)} (-1)^{(S,S_1)}\ \operatorname{aut}\big(\mu(\Pi)\big)\ m_\Pi \\ &= \sum_{\substack{\Pi\text{set-}\\ \text{partition of }[2n]}} \left[\sum_{\substack{(S_1,S_2)\ T\text{-admissible}\\ \mathcal{L}(S_1,S_2)\le\Pi}} (-1)^{(S,S_1)} \right] \operatorname{aut}\big(\mu(\Pi)\big)\ m_\Pi. \end{aligned} \tag{3}$$

Remark. The inequality $\mathcal{L}(S_1,S_2) \le \Pi$ is equivalent to the following local condition: the partners of any integer i in the pair-partitions S_1 and S_2 are in the same part of Π as i.

Let us assume now that $\mu(\Pi)$ is not dominated by λ. By Lemma 4, there exist i and j in $[2n]$, which are in the same part of Π and in the same column of T. Let us denote $\tau = (i\ j) \in S_{2n}$. Then:

- $(\tau(S_1), S_2)$ is T-admissible $\iff$ (S_1, S_2) is T-admissible;
- $\mathcal{L}(\tau(S_1), S_2) \leq \Pi \iff \mathcal{L}(S_1, S_2) \leq \Pi$ (because of the remark above);
- $(-1)^{(S,\tau(S_1))} = (-1)^{(S,S_1)+1}$ (see Lemma 6).

Thus, $(S_1, S_2) \mapsto (\tau(S_1), S_2)$ is a sign reversing involution proving that the expression in the bracket in the right-hand side of (3) is equal to zero as soon as $\mu(\Pi)$ is not dominated by λ.

This ends the proof of property **TR**.

4.3 Orthogonality

In this section we prove that the functions Y_λ satisfy property **OR**.

As the definition of Y_λ does not depend on the choice of the bijective filling $T \in F(\lambda)$ ($F(\lambda)$ is by definition the set of bijective fillings of 2λ), one can write:

$$\begin{aligned}(2n)!\, Y_\lambda &= \sum_{T \in F(\lambda)} \sum_{(S_1,S_2)\ T\text{-admissible}} (-1)^{(S(T),S_1)}\, p_{\mathcal{L}(S_1,S_2)} \\ &= \sum_{\substack{S_1,S_2 \\ \text{pair-partitions}}} \sum_{\substack{T \in F(\lambda) \text{ s.t.} \\ (S_1,S_2)\ T\text{-admissible}}} (-1)^{(S(T),S_1)}\, p_{\mathcal{L}(S_1,S_2)}. \end{aligned} \tag{4}$$

Let ν be a partition of n. A consequence of Lemma 5 is that each of the $\frac{(2n)!}{z_\nu 2^{\ell(\nu)}}$ couples of pair-partitions (S_1, S_2) such that $\mu(\mathcal{L}(S_1, S_2)) = \nu$ has the same contribution to the right-hand side of the previous equation. Therefore, if we fix such a couple (S_1, S_2), the coefficient of p_ν in Y_λ is given by:

$$(2n)!\, [p_\nu] Y_\lambda = \frac{(2n)!}{z_\nu 2^{\ell(\nu)}} \sum_{\substack{T \in F(\lambda) \text{ s.t.} \\ (S_1,S_2)\ T\text{-admissible}}} (-1)^{(S(T),S_1)}. \tag{5}$$

But, using equation (4) for Y_λ, one has the following expression for the scalar product:

$$(2n)!\, \langle Y_\lambda, Y_{\lambda'} \rangle_2 = \sum_{\substack{S_1,S_2 \\ \text{pair-partitions}}} \sum_{\substack{T \in F(\lambda) \text{ s.t.} \\ (S_1,S_2)\ T\text{-admissible}}} (-1)^{(S(T),S_1)}\, z_\nu\, 2^{\ell(\nu)}\, [p_\nu] Y_{\lambda'},$$

where $\nu = \mu\big(\mathcal{L}(S_1, S_2)\big)$ depends on the summation index. Now, $[p_\nu]Y_{\lambda'}$ can be evaluated *via* equation (5) and we obtain:

$$(2n)!\, \langle Y_\lambda, Y_{\lambda'} \rangle_2 = \sum_{\substack{S_1,S_2 \\ \text{pair-partitions}}} \sum_{\substack{T \in F(\lambda) \text{ s.t.} \\ (S_1,S_2)\ T\text{-admissible}}} \sum_{\substack{T' \in F(\lambda') \text{ s.t.} \\ (S_1,S_2)\ T'\text{-admissible}}} (-1)^{(S(T),S_1)}\, (-1)^{(S(T'),S_1)}. \tag{6}$$

Let us assume $\lambda \neq \lambda'$. One can assume without loss of generality that λ is not dominated by λ'; otherwise we swap λ and λ'. By Lemma 4 there exist two elements i and j which are in the same row of T' and in the same column of T. We choose the smallest ones in the lexicographic order. We denote by $\tau_{T,T'}$ the transposition $(i\ j)$ and we consider the function:

$$\varphi : (S_1, S_2, T, T') \mapsto (\tau_{T,T'}(S_1), S_2, T, T' \circ \tau_{T,T'}),$$

where $T' \circ \tau_{T,T'}$ is the tableau T' with the locations of i and j interchanged. It is easy to check that:

- φ is an involution (because $T' \circ \tau_{T,T'}$ and T' have the same rows and, therefore, the unordered pair $\{i, j\}$ is the same for the two pairs of fillings);
- (S_1, S_2) is T-admissible if and only if $(\tau_{T,T'}(S_1), S_2)$ is T-admissible;
- (S_1, S_2) is T'-admissible if and only if $(\tau_{T,T'}(S_1), S_2)$ is $T' \circ \tau_{T,T'}$-admissible;
- $(-1)^{(S(T),\tau_{T,T'}(S_1))} = (-1)^{(S(T),S_1)+1}$ (see Lemma 6);
- $(-1)^{\left(S(T'\circ\tau_{T,T'}),\tau_{T,T'}(S_1)\right)} = (-1)^{(S(T'),S_1)}$ (because $S(T' \circ \tau_{T,T'}) = \tau_{T,T'}(S(T'))$).

The contributions of the quadruplets (S_1, S_2, T, T') and $\varphi(S_1, S_2, T, T')$ in equation (6) cancel, which proves that the functions Y_λ are orthogonal with respect to the deformed Hall scalar product $\langle \cdot, \cdot \rangle_2$.

4.4 Normalization

In this paragraph we prove that the functions Y_λ satisfy property **N**.

Let λ be a partition of n. We fix a bijective filling T of 2λ. We have to find the T-admissible couples (S_1, S_2) with $\mu(\mathcal{L}(S_1, S_2)) = 1^n$. The last condition is equivalent to $S_1 = S_2$ and a couple (S_1, S_1) is T-admissible if and only if $S_1 = S(T)$. In this case, $(-1)^{(S(T),S_1)} = 1$. Therefore, $[p_{1^n}]Y_\lambda = 1$ and Theorem 3 is proved. □

5 Combinatorial formulas for zonal characters

In this section we give new formulas for zonal characters (the latter are defined in Section 3.2), which are analogues of some recent formulas for characters of the symmetric groups and establish some particular cases of some conjectures of Lassalle. These formulas can be derived relatively easily from our main result, as done in the complete version of this paper [FŚ11b].

5.1 *Via* numbers of embeddings

Let S_0, S_1, S_2 be three pair-partitions of the set $[2k]$. We consider the following function on the set of Young diagrams:

Definition 8 $N^{(1)}_{S_0,S_1,S_2}(\lambda)$ *is the number of functions f from $[2k]$ to the boxes of the Young diagram λ such that for every $l \in [2k]$:*

(Q0) $f(l) = f(S_0(l))$, in other words f can be viewed as a function on the set of pairs constituting S_0;

(Q1) $f(l)$ and $f(S_1(l))$ are in the same column;

(Q2) $f(l)$ and $f(S_2(l))$ are in the same row.

Theorem 9 *Let ν be a partition of the integer k and S_1, S_2 be two fixed pair-partitions of the set $[2k]$ such that $\mu\big(\mathcal{L}(S_1, S_2)\big) = \nu$. Then one has the following equality between functions on the set of Young diagrams:*

$$\Sigma^{(2)}_\nu = \frac{1}{2^{\ell(\nu)}} \sum_{S_0} (-1)^{(S_0,S_1)}\, 2^{|\,\mathcal{L}(S_0,S_1)|}\, N^{(1)}_{S_0,S_1,S_2}, \tag{7}$$

where the sum runs over pair-partitions of $[2k]$.

5.2 In terms of Stanley's coordinates

The notion of Stanley's coordinates was introduced by Stanley [Sta04] who found a nice formula for normalized irreducible character values of the symmetric group corresponding to rectangular Young diagrams. In order to generalize this result, he defined, given two sequences $\mathbf{p}$ and $\mathbf{q}$ of positive integers of same size ($\mathbf{q}$ being non-increasing), the partition:

$$\mathbf{p} \times \mathbf{q} = (\underbrace{q_1, \dots, q_1}_{p_1 \text{ times}}, \dots, \underbrace{q_l, \dots, q_l}_{p_l \text{ times}}).$$

Then he suggested to consider the quantity $\Sigma_\nu^{(1)}(\mathbf{p} \times \mathbf{q})$ as a polynomial in $\mathbf{p}$ and $\mathbf{q}$. An explicit combinatorial interpretation of the coefficients was conjectured in [Sta06] and proved in [Fér10].

It is easy to deduce from the above expansion of $\Sigma_\nu^{(2)}$ in terms of the N functions a combinatorial description of the polynomial $\Sigma_\nu^{(2)}(\mathbf{p} \times \mathbf{q})$.

Theorem 10 *Let ν be a partition of the integer k and S_1, S_2 be two fixed pair-partitions of $[2k]$ such that $\mu\big(\mathcal{L}(S_1, S_2)\big) = \nu$. Then, one has:*

$$\Sigma_\nu^{(2)}(\mathbf{p} \times \mathbf{q}) = \frac{(-1)^k}{2^{\ell(\nu)}} \sum_{S_0} \left[\sum_{\phi: \mathcal{L}(S_1,S_0) \to \mathbb{N}^\star} \prod_{l \in \mathcal{L}(S_1,S_0)} (p_{\varphi(l)}) \cdot \prod_{l' \in \mathcal{L}(S_2,S_0)} (-2 q_{\psi(l')}) \right] \tag{8}$$

where $\psi(l') := \max_l \varphi(w)$ with l running over the loops of $\mathcal{L}(S_0, S_1)$ having at least one element in common with l'.

5.3 In terms of free cumulants

The following observation is due to Lassalle [Las09]. Let $k \geq 1$ be a fixed integer and let α be fixed. Since $\Sigma_k^{(\alpha)}$ is an α-shifted symmetric function and the anisotropic free cumulants $(R_l^{(\alpha)})_{l \geq 2}$ form an algebraic basis of the ring of α-shifted symmetric functions (see [Las09] for their definition), there exists a polynomial $K_k^{(\alpha)}$ such that, for any Young diagram λ,

$$\Sigma_k^{(\alpha)}(\lambda) = K_k^{(\alpha)}\big(R_2^{(\alpha)}(\lambda), R_3^{(\alpha)}(\lambda), \dots\big).$$

This polynomial is called *Jack Kerov polynomial*.

Thus Jack Kerov polynomials express Jack characters on cycles in terms of free cumulants. For more complicated conjugacy classes it turns out to be more convenient to express not directly the characters $\Sigma_{(k_1,\dots,k_\ell)}^{(\alpha)}$ but rather *cumulant*

$$(-1)^{\ell-1} \kappa^{\mathrm{id}}(\Sigma_{k_1}^{(\alpha)}, \dots, \Sigma_{k_\ell}^{(\alpha)}). \tag{9}$$

This gives rise to *generalized Jack Kerov polynomials* $K_{(k_1,\dots,k_\ell)}^{(\alpha)}$. In the classical context $\alpha = 1$ these quantities have been introduced by one of us and Rattan [RŚ08]; in the Jack case they have been studied by Lassalle [Las09]. We skip the definitions and refer to the above papers for details since generalized Kerov polynomials are not of central interest for this paper.

Using the technology developed in [DFŚ10], one can deduce from Theorem 9 a combinatorial interpretation for the coefficients of generalized zonal Kerov polynomials (*i.e.* generalized Jack Kerov polynomials in the case $\alpha = 2$):

Theorem 11 *Let ν be a partition of an integer k and S_1, S_2 be two fixed pair-partitions of $[2k]$ such that $\mu\big(\mathcal{L}(S_1, S_2)\big) = \nu$. Consider also a sequence $s_2, s_3, \dots$ of non-negative integers with only finitely many non-zero elements. The rescaled coefficient*

$$(-2)^{\ell(\nu)}(-1)^{|\nu|+2s_2+3s_3+\cdots} \left[\left(R_2^{(2)}\right)^{s_2} \left(R_3^{(2)}\right)^{s_3} \cdots \right] K_\nu^{(2)}$$

of the (generalized) zonal Kerov polynomial is equal to the number of couples (S_0, q) with the following properties:

- *(a) the graph of vertex set $[2n]$ obtained by drawing an edge between i and j if they are in the same part of S_0, S_1 or S_2 is connected;*
- *(b) the number of loops in G_{S_0,S_1} is equal to $s_2 + s_3 + \cdots$;*
- *(c) the number of loops in G_{S_0,S_2} is equal to $s_2 + 2s_3 + 3s_4 + \cdots$;*
- *(d) q is a function from $\mathcal{L}(S_0, S_1)$ to the set $\{2, 3, \dots\}$; we require that each number $i \in \{2, 3, \dots\}$ is used exactly s_i times;*
- *(e) for every subset $A \subset \mathcal{L}(S_0, S_1)$ of loops which is nontrivial (i.e., $A \neq \emptyset$ and $A \neq \mathcal{L}(S_0, S_1)$), there are more than $\sum_{v\in A} \big(q(v) - 1\big)$ loops in $\mathcal{L}(S_0, S_2)$ which have a non-empty intersection with at least one loop from A.*

6 Conclusion

Our result on zonal polynomials and zonal characters are analogs of recent developments on Schur polynomials and characters of symmetric groups. Lassalle's work suggests that this kind of results hold for generic values of the parameter α, but we have not been able (yet) to generalize our argument.

Note that, unlike the proof in the full version of the paper [FŚ11b], which relies on the representation theoretical interpretation of zonal polynomials, the one presented here only uses the fact that zonal polynomials are special cases of Jack polynomials. So, if we manage to guess the good combinatorial objects for other values of α, a similar proof technique could be used.

Another approach to the general case is proposed in [DŚ11]: M. Dołęga and the second author give a combinatorial condition for an expression of the kind of Therorem 9 to be an α-shifted symmetric function. As $\Sigma_\nu^{(\alpha)}(\lambda)$ belongs to this class, this could help guessing a generalization of Therorem 9.

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