A polynomial expression for the Hilbert series of the quotient ring of diagonal coinvariants (condensed version)

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Abstract. A special case of Haiman's identity [Invent. Math. **149** (2002), pp. 371–407] for the character of the quotient ring of diagonal coinvariants under the diagonal action of the symmetric group yields a formula for the bigraded Hilbert series as a sum of rational functions in q, t. In this paper we show how a summation identity of Garsia and Zabrocki for Macdonald polynomial Pieri coefficients can be used to transform Haiman's formula for the Hilbert series into an explicit polynomial in q, t with integer coefficients. We also provide an equivalent formula for the Hilbert series as the constant term in a multivariate Laurent series.

Résumé. Un cas spécial de l'identité de Haiman [Invent. Math. **149** (2002), pp. 371–407] pour le caractère de l'anneau quotient des coinvariants diagonaux sous l'action du groupe symétrique fournit une formule pour la série de Hilbert bigraduée comme somme de fonctions rationnelles en q, t. Dans cet article nous montrons comment une identité de sommation de Garsia et Zabrocki pour les coefficients de Pieri des polynômes de Macdonald peut être utilisée pour transformer la formule de Haiman pour la série de Hilbert en un polynôme explicite en q, t à coefficients entiers. Nous présentons également une formule équivalente pour la série de Hilbert comme terme constant d'une série de Laurent multivariée.

Keywords: Hilbert series, diagonal coinvariants

1 Introduction

Let $X_n = \{x_1, \ldots, x_n\}, Y_n = \{y_1, \ldots, y_n\}$ be two sets of variables and let

$$\mathrm{DR}_n = \mathbb{C}[X_n, Y_n]/\big\langle\{\sum_i x_i^h y_i^k, \forall h, k \geq 0, h+k>0\}\big\rangle \tag{1}$$

be the quotient ring of diagonal coinvariants. Let ∇ be the linear operator defined on the modified Macdonald polynomial basis $\{\tilde{H}_\mu(X_n; q, t)\}$, where $\mu \vdash n$ (i.e. μ is a partition of n), by

$$\nabla \tilde{H}_\mu(X_n; q, t) = T_\mu \tilde{H}_\mu(X_n; q, t), \tag{2}$$

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where $T_\mu = t^{n(\mu)}q^{n(\mu')}$ and $n(\mu) = \sum_i (i-1)\mu_i$. The symmetric group acts "diagonally" on a polynomial $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$ by $\sigma f = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)})$ and this action extends to DR_n. Haiman [Hai02] proved an earlier conjecture of Garsia and Haiman [GH96] that the Frobenius series of this action is given by $\nabla e_n(X_n)$, where e_n is the nth elementary symmetric function in a set of variables. (The Frobenius series is obtained by starting with the character and mapping the irreducible S_n-character χ^λ to the Schur function s_λ.) Since the Frobenius series of DR_n is given by ∇e_n, the Hilbert series $\mathrm{Hilb}(\mathrm{DR}_n)$ is given by $\langle \nabla e_n, h_1^n \rangle$ (See [Hag08, p. 24] for an explanation of why. Here $\langle , \rangle$ is the Hall scalar product, with respect to which the Schur functions are orthonormal, and $h_1(X) = \sum_i x_i$.) This results in a formula for $\mathrm{Hilb}(\mathrm{DR}_n)$ as an explicit sum of rational functions in q, t, described in detail in the next section. A corollary of this formula is that $\dim(\mathrm{DR}_n) = (n+1)^{n-1}$. See also [Hai94] and [Ber09] for background on this problem. We mention that many articles in the literature refer to the space of diagonal harmonics DH_n, which is known to be isomorphic to DR_n, and so $\mathrm{Hilb}(\mathrm{DH}_n) = \mathrm{Hilb}(\mathrm{DR}_n)$.

A Dyck path is a lattice path in the first quadrant of the xy-plane from $(0,0)$ to (n,n) consisting of unit north N and east E steps which never goes below the diagonal $x = y$. A parking function is a placement of the integers $1, 2, \ldots, n$ (called "cars") just to the right of the N steps of a Dyck path, so there is strict decrease down columns. An open conjecture of Loehr and the author [HL05] expresses $\mathrm{Hilb}(\mathrm{DR}_n)$ as a positive sum of monomials, one for each parking function. In a recent preprint, Armstrong [Arm10] introduces a hyperplane arrangement model for $\mathrm{Hilb}(\mathrm{DR}_n)$ involving a pair of hyperplane arrangements with a statistic associated to each one. See also [AR]. He gives a bijection with parking functions which sends his pair of hyperplane arrangement statistics to the pair of statistics on parking functions introduced by Haglund and Loehr.

In this article we use a plethystic summation formula of Garsia and Zabrocki for Macdonald Pieri coefficients to show how $\langle \nabla e_n, h_1^n \rangle$ can be expressed as an element of $\mathbb{Z}[q,t]$. The most elegant way of expressing our result is to say that $\mathrm{Hilb}(\mathrm{DR}_n)$ is the coefficient of $z_1 z_2 \cdots z_n$ in a certain multivariate Laurent series (see (41)). We are currently unable to see how our result implies a positive formula such as the conjecture of Haglund and Loehr, but are hopeful that further work will lead to such applications.

2 Background Material

For $\mu \vdash n$, and s a square of the Ferrers diagram of μ, let $l(s), a(s), l'(s), a'(s)$ denote the leg, arm, coleg, coarm, respectively, of s, i.e. the number of squares above s, to the right of s, below s, and to the left of s, as in Figure 1. Furthermore let

$$M = (1-q)(1-t), \quad B_\mu = \sum_{s \in \mu} t^{l'} q^{a'}, \quad \Pi_\mu = \prod_{\substack{s \in \mu \\ s \neq (0,0)}} (1 - t^{l'} q^{a'}), \quad w_\mu = \prod_{s \in \mu} (q^a - t^{l+1})(t^l - q^{a+1}). \tag{3}$$

The known expansion

$$e_n(X) = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu(X;q,t) M \Pi_\mu B_\mu}{w_\mu} \tag{4}$$

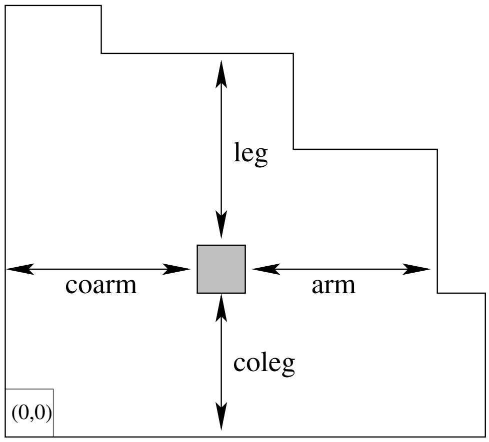

Fig. 1: The leg, coleg, arm, and coarm of a square

then implies

$$\nabla e_n(X) = \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu(X; q, t) M \Pi_\mu B_\mu}{w_\mu}. \tag{5}$$

Letting $F_\mu = \langle \tilde{H}_\mu, h_1^n \rangle$, by taking the scalar product of both sides of (5) with respect to h_1^n we get

$$\text{Hilb}(\text{DR}_n) = \sum_{\mu \vdash n} \frac{T_\mu F_\mu M \Pi_\mu B_\mu}{w_\mu}. \tag{6}$$

Let $^\perp$ be the operation on symmetric functions which is adjoint to multiplication with respect to the Hall scalar product, i.e. for any symmetric functions f, g, h,

$$\langle f^\perp g, h \rangle = \langle g, fh \rangle. \tag{7}$$

If $\mu \vdash n$ and $\nu \vdash n-1$, then $\nu \to \mu$ means ν is obtained from μ by removing some corner square of μ, and $\mu \leftarrow \nu$ means μ is obtained from ν by adding a single square to the Ferrers shape of ν. Define generalized skew Pieri coefficients $c^{f^\perp}_{\mu,\nu}(q,t)$ and Pieri coefficients $d^f_{\mu,\nu}(q,t)$ by the formulas

$$f^\perp \tilde{H}_\mu(X; q, t) = \sum_{\substack{\nu \\ \nu \to \mu}} c^{f^\perp}_{\mu,\nu}(q,t) \tilde{H}_\nu(X; q, t) \tag{8}$$

$$f \tilde{H}_\nu(X; q, t) = \sum_{\substack{\mu \\ \mu \leftarrow \nu}} d^f_{\mu,\nu}(q,t) \tilde{H}_\mu(X; q, t). \tag{9}$$

Many of the identities in this paper are expressed using plethystic notation, defined as follows. If $p_k(X) = \sum_i x_i^k$ is the kth power sum, then for any expression E, the plethystic substitution of E into

p_k is obtained by replacing all indeterminates in E by their kth powers. We denote this by $p_k[E]$, so for example

$$p_k[X(1-t)] = p_k(X)(1-t^k). \tag{10}$$

For any symmetric function $f(X)$, we define $f[E]$ by first expressing f as a polynomial in the p_k, then replacing each p_k by $p_k[E]$.

The $c^{f^\perp}_{\mu,\nu}$ and the $d^f_{\mu,\nu}$ are related via [GH02, (3.5)]

$$c^{f^\perp}_{\mu,\nu} w_\nu = d^{\omega f[X/M]}_{\mu,\nu} w_\mu, \tag{11}$$

where ω is the linear operator on symmetric functions satisfying $\omega s_\lambda = s_{\lambda'}$. Note $d^{\omega h_1[X/M]}_{\mu,\nu} = d^{h_1[X]}_{\mu,\nu}/M$. We abbreviate $c^{h_1\perp}_{\mu,\nu}(q,t)$ by $c_{\mu,\nu}$ and $d^{h_1[X/M]}_{\mu,\nu}(q,t)$ by $d_{\mu,\nu}$.

A special case of Macdonald's Pieri formulas [Mac95, Section 6.6] gives an expression for $d_{\mu,\nu}$ as a quotient of factors of the form $(t^a q^b - t^c q^d)$, where a, b, c, d have simple combinatorial descriptions. Garsia found a simplification in this formula, which Garsia and Zabrocki used to obtain the $k = 1$ case of the following summation formula [GZ05]. The proof of the result for general k appears in [BGHT99] and [Gar10].

$$\sum_{\substack{\mu \\ \mu \leftarrow \nu}} d_{\mu,\nu} T^k = \begin{cases} 1/M & \text{if } k = 0, \\ (-1)^{k-1} e_{k-1}[MB_\nu - 1]/M & \text{if } k \geq 1, \end{cases} \tag{12}$$

where throughout this article T is an abbreviation for T_μ/T_ν. Eq. (12) is closely related to a corresponding summation formula involving the $c_{\mu,\nu}$ [GT96, Theorem 2.2].

Identity (12) can be recast in the following form.

Lemma 1

$$\sum_{\substack{\mu \\ \mu \leftarrow \nu}} d_{\mu,\nu}(1-T)T^k = \begin{cases} 0 & \textit{if } k = 0, \\ (-1)^{k-1} e_k[MB_\nu]/M & \textit{if } k \geq 1 \end{cases}. \tag{13}$$

The following simple fact will be useful later.

Lemma 2

$$(-1)^{k-1} e_k[M]/M = \frac{t^k - q^k}{t - q} \qquad k \geq 1. \tag{14}$$

3 A New Recursive Procedure to Generate the Hilbert Series

By definition we have

$$e_1^\perp \tilde{H}_\mu(X; q, t) = \sum_{\substack{\nu \\ \nu \to \mu}} c_{\mu,\nu} \tilde{H}_\nu(X; q, t). \tag{15}$$

Taking the scalar product of both sides with respect to h_1^{n-1} we get

$$\langle e_1^{\perp} \tilde{H}_\mu, h_1^{n-1} \rangle = \langle \tilde{H}_\mu, e_1 h_1^{n-1} \rangle = F_\mu = \sum_{\substack{\nu \\ \nu \to \mu}} c_{\mu,\nu} F_\nu. \tag{16}$$

Plugging this recurrence for the F_μ into (6) yields

$$\begin{aligned} \mathrm{Hilb}(\mathrm{DR}_n) &= \sum_{\mu \vdash n} \frac{T_\mu M \Pi_\mu B_\mu}{w_\mu} \sum_{\substack{\nu \\ \nu \to \mu}} c_{\mu,\nu} F_\nu & (17) \\ &= \sum_{\nu \vdash n-1} F_\nu M \sum_{\substack{\mu \\ \mu \leftarrow \nu}} \frac{B_\mu \Pi_\mu c_{\mu,\nu} T_\mu}{w_\mu}. & (18) \end{aligned}$$

Now from (3) we see

$$B_\mu = B_\nu + T, \qquad \Pi_\mu = \Pi_\nu (1 - T). \tag{19}$$

Using this and the $f = e_1$ case of (11) in (18) we get

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\nu \vdash n-1} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} \sum_{\substack{\mu \\ \mu \leftarrow \nu}} d_{\mu,\nu} (B_\nu + T)(1 - T) T. \tag{20}$$

By (13) this implies

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\nu \vdash n-1} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} \left(\frac{e_1[M B_\nu]}{M} \frac{e_1[M B_\nu]}{M} - \frac{e_2[M B_\nu]}{M} \right). \tag{21}$$

(Although $e_1[M B_\nu]/M$ can be expressed more simply as $e_1[B_\nu]$, leaving (21) in the above form will prove more useful in the sequel.)

We now iterate the argument; first re-index the sum in (21) as a sum over $\mu \vdash n-1$, and replace F_μ by $\sum_{\nu \to \mu} c_{\mu,\nu} F_\nu$. Then write B_μ as $B_\nu + T$ as before, and reverse summation to get

$$\begin{aligned} \mathrm{Hilb}(\mathrm{DR}_n) = &\sum_{\nu \vdash n-2} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} & (22) \\ &\times \sum_{\substack{\mu \\ \mu \leftarrow \nu}} d_{\mu,\nu} (1 - T) T \left(\frac{e_1[M(B_\nu + T)]}{M} \frac{e_1[M(B_\nu + T)]}{M} - \frac{e_2[M(B_\nu + T)]}{M} \right). \end{aligned}$$

Now for any alphabets X, Y we have

$$e_k \left[X - Y \right] = \sum_{j=0}^{k} e_j \left[X \right] e_{k-j} \left[-Y \right]. \tag{23}$$

Hence for $k \geq 1$

$$(-1)^{k-1} \frac{e_k[M(B_\nu + T)]}{M} = b_k + T^k a_k + \sum_{j=1}^{k-1} -M b_j T^{k-j} a_{k-j}, \tag{24}$$

where we have abbreviated $(-1)^{j-1}e_j[M]/M$ by a_j and $(-1)^{j-1}e_j[MB_\nu]/M$ by $b_j = b_j(\nu)$. Here we have used the fact that $e_k[MT]/M = T^k e_k[M]/M$ (since for any expression $p_j[XT] = T^j p_j[X]$). Note also that $a_1 = 1$. The inner sum in (22) thus becomes

$$\sum_{\substack{\mu \\ \mu\leftarrow\nu}} d_{\mu,\nu}(1-T)T\left((b_1+Ta_1)^2 + b_2a_1 + T^2a_2 - Mb_1Ta_1\right) \tag{25}$$

$$= b_1^3 + 2b_1a_1b_2 + a_1^2b_3 + a_1b_2b_1 - Mb_1a_1^2b_2 + a_1a_2b_3 \tag{26}$$

by (13).

Let

$$A_1 = b_1 \tag{27}$$

$$A_2 = b_1^2 + b_2a_1 \tag{28}$$

$$A_3 = b_1^3 + 2b_1a_1b_2 + a_1^2b_3 + a_1b_2b_1 - Mb_1a_1^2b_2 + a_1a_2b_3. \tag{29}$$

The above discussion implies

Theorem 1 *For $p \in \mathbb{N}$, $1 \le p \le n$,*

$$\mathit{Hilb}(\mathit{DR}_n) = \sum_{\nu\vdash n-p+1} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} A_p, \tag{30}$$

where $A_p = A_p(\nu)$ is a certain polynomial in the a_i, b_i. Moreover, A_p can be calculated recursively from A_{p-1} by the following procedure. First replace each b_k in A_{p-1} by $b_k + T^k a_k - \sum_{j=1}^{k-1} Mb_jT^{k-j}a_{k-j}$. Then multiply the resulting expression out to form a polynomial in T, say

$$\sum_j c_jT^j. \tag{31}$$

Finally, replace T^j by b_{j+1}, i.e.

$$A_p = \sum_j c_jb_{j+1}. \tag{32}$$

(We replace T^j by b_{j+1} since, after multiplying the expression above out to get $\sum c_jT^j$, we still have another factor of T coming from the outer sum. Applying (13) replaces T^{j+1} by b_{j+1}.)

We now give a non-recursive expression for A_p. Let Q_n denote the set of all $n \times n$ upper-triangular matrices C of nonnegative integers which satisfy

$$-\sum_{i=1}^{j-1} c_{ij} + \sum_{i=j}^{n} c_{ji} = 1, \quad \text{for each } j,\ 1 \le j \le n. \tag{33}$$

For example,

$$Q_1 = \{[1]\} \tag{34}$$

$$Q_2 = \{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\} \tag{35}$$

$$Q_3 = \{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 2\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 2\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 3\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 3\end{bmatrix}\}. \tag{36}$$

Geometrically, the condition (33) says that for all j, if we add all the entries of C in the jth row together, and then subtract all the entries in the jth column above the diagonal, we get 1. Note that these conditions imply that each row of C must have at least one positive entry.

For $C \in Q_n$, let $\mathrm{Pos}(C)$ denote the multiset of positive entries in C, and $\mathrm{pos}(C)$ its cardinality. Matrices of this kind can be generated recursively, in a manner similar to the recursion generating the polynomials A_p, and using this one can prove the following.

Theorem 2 *For $1 \le p \le n$ and A_p, b_j, a_j as above,*

$$A_p = \sum_{C \in Q_p} (-M)^{pos(C)-n} \prod_{c_{ii} \in Pos(C)} b_{c_{ii}} \prod_{\substack{c_{ij} \in Pos(C)\\ i<j}} a_{c_{ij}}. \tag{37}$$

Corollary 1

$$Hilb(DR_n) = \sum_{C \in Q_n} (-M)^{pos(C)-n} \prod_{\substack{c_{ij} \in Pos(C)\\ 1 \le i \le j \le n}} [c_{ij}]_{q,t}, \tag{38}$$

where $[k]_{q,t} = (t^k - q^k)/(t - q)$ is the q, t-analog of the integer k.

Example 1 *The weights associated to the elements of Q_3, listed in the same left-to-right order as in (36) are*

$$1, \quad t+q, \quad t+q, \quad -M(t+q), \quad (t+q)(t^2+qt+q^2), \quad t+q, \quad t^2+qt+q^2. \tag{39}$$

Thus Hilb(DR_3) is the sum of these terms, namely

$$1 + 2q + 2t + 2q^2 + 3qt + 2t^2 + q^3 + q^2 t + qt^2 + t^3. \tag{40}$$

The sequence $1, 2, 7, 40, 357, 4820, \ldots$ consisting of the cardinalities of the sets $Q_1, Q_2, Q_3, \ldots$ form entry $A008608$ in Sloane's on-line encyclopedia of integer sequences. In fact, it was comparing the number of monomials in A_n for small n with sequences in Sloane's encyclopedia that led the author to the discovery of the non-recursive expression for the A_n in terms of the elements of Q_n. The sequence was introduced to Sloane's list by Glenn Tesler, who in a private conversation with the author said they arose in unpublished work of Tesler's from the late 1990's on plethystic expressions for Macdonald's $D_{n,r}$ operators. Although Tesler doesn't recall any further details about this work, we will refer to elements of Q_n as "Tesler matrices".

The explicit formula (38) for $\mathrm{Hilb}(\mathrm{DR}_n)$ can be formulated as a constant term identity.

Corollary 2 *For $n \geq 1$, $Hilb(DR_n)$ is the coefficient of $z_1 z_2 \cdots z_n$ in*

$$\frac{1}{(-M)^n} \prod_{i=1}^{n} \frac{(1-z_i)(1-qtz_i)}{(1-qz_i)(1-tz_i)} \prod_{1\leq i<j\leq n} \frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)}. \tag{41}$$

4 The m-parameter

The formula ∇e_n for the Frobenius series of DR_n is a special case of a more general result (also due to Haiman [Hai02]) which says that for any positive integer m, $\nabla^m e_n$ is the Frobenius series of a certain S_n-module $\mathrm{DR}_n^{(m)}$. Hence, from (5) we have

$$\mathrm{Hilb}(\mathrm{DR}_n^{(m)}) = \langle \nabla^m e_n, h_1^n \rangle \tag{42}$$

$$= \sum_{\substack{\mu \\ \mu\vdash n}} \frac{T_\mu^m F_\mu M \Pi_\mu B_\mu}{w_\mu}. \tag{43}$$

The methods of the previous section can be generalized to show that for any $1 \leq p \leq n$,

$$\mathrm{Hilb}(\mathrm{DR}_n^{(m)}) = \sum_{\mu\vdash n-p+1} \frac{T_\nu^m F_\nu M \Pi_\nu}{w_\nu} A_p^{(m)}, \tag{44}$$

where $A_p^{(m)} = A_p^{(m)}(\mu)$ is a polynomial in the b_j, a_j as before. We have $A_1^{(m)}(\mu) = b_1$, and for $p > 1$, we can construct $A_p^{(m)}$ recursively by the following procedure. First, replace each b_k in $A_{p-1}^{(m)}$ by $b_k + T^k a_k + \sum_{j=1}^{k-1} -M b_j T^{k-j} a_{k-j}$. Then, multiply the resulting expression out to form a polynomial in T say

$$\sum_j c_j T^j. \tag{45}$$

Finally, replace T^j by b_{j+m}, i.e.

$$A_p^{(m)} = \sum_j c_j b_{j+m}. \tag{46}$$

In terms of the Tesler matrices, we want the "hook sums" to be equal to $(1, m, m, \ldots, m)$ instead of $(1, 1, \ldots, 1)$. To be precise, define $Q_n^{(m)}$ to be the set of upper-triangular matrices C of nonnegative integers satisfying

$$-\sum_{i=1}^{j-1} c_{ij} + \sum_{i=j}^{n} c_{ji} = \begin{cases} 1 & \text{if } j = 1, \\ m & \text{if } 2 \leq j \leq n \end{cases}. \tag{47}$$

We get the following extensions of the earlier results.

Theorem 3 *For $1 \le p \le n$, $m \ge 1$, and $A_p^{(m)}$, b_j, a_j as above,*

$$A_p^{(m)} = \sum_{C \in Q_p^{(m)}} (-M)^{pos(C)-n} \prod_{c_{ii} \in Pos(C)} b_{c_{ii}} \prod_{\substack{c_{ij} \in Pos(C) \\ i<j}} a_{c_{ij}}. \tag{48}$$

Furthermore, the special case $p = n$ of (44) reduces to

$$Hilb(DR_n^{(m)}) = \sum_{C \in Q_n^{(m)}} (-M)^{pos(C)-n} \prod_{\substack{c_{ij} \in Pos(C) \\ 1 \le i \le j \le n}} [c_{ij}]_{q,t}. \tag{49}$$

Corollary 3 *For $n \ge 1$, $Hilb(DR_n^{(m)})$ is the coefficient of $z_1 z_2^m z_3^m \cdots z_n^m$ in*

$$\frac{1}{(-M)^n} \prod_{i=1}^n \frac{(1-z_i)(1-qtz_i)}{(1-qz_i)(1-tz_i)} \prod_{1 \le i < j \le n} \frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)}. \tag{50}$$

5 Conjectures and Open Questions

5.1 Tesler matrices with more general hook sums

In general the coefficient of $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ in (50) is not a positive polynomial in q, t, but Maple calculations suggest it is positive if the α_i are positive and nondecreasing.

Conjecture 1 *For $n \ge 1$ and α the reverse of a partition (so $1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$)*

$$\frac{1}{(-M)^n} \prod_{i=1}^n \frac{(1-z_i)(1-qtz_i)}{(1-qz_i)(1-tz_i)} \prod_{1 \le i < j \le n} \frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)} \Big|_{z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}} \in \mathbb{N}[q,t]. \tag{51}$$

Equivalently, the weighted sum over Tesler matrices with hook sums $\alpha_1, \ldots, \alpha_n$ is in $\mathbb{N}[q,t]$.

Remark 1 *The argument proving Theorem 3 shows that if $\alpha_1 = 1$, the coefficient of $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ in (51) can be obtained by starting with $\nabla^{\alpha_2} e_n$, applying $e_1^\perp$, then applying $\nabla^{\alpha_3-\alpha_2}$, applying $e_1^\perp$ again, then applying $\nabla^{\alpha_4-\alpha_3}$, etc. The author doesn't know if these polynomials have a representation-theoretic interpretation for $\alpha \ne (1, m, m, \ldots, m)$.*

5.2 A refinement of the q, t-positivity

Note that $[k]_{q,t}$ can be expressed as $s_{k-1}(\{q,t\})$, i.e. the $(k-1)$st complete homogeneous symmetric function evaluated in the set of variables $\{q,t\}$. Also, $-M = t + q - 1 - qt$ equals $s_1 - 1 - s_{1,1}$, also in the set of variables $\{q,t\}$. In (38) we can substitute in these Schur function formulations for $[k]_{q,t}$ and $-M$, multiply everything out using the Pieri rule for Schur function multiplication, and thereby obtain a formula for $\mathrm{Hilb}(\mathrm{DR}_n)$ in terms of Schur functions in the set of variables $\{q,t\}$. If we then cancel terms of the form s_λ where λ has more than two parts (which becomes zero since our set of variables has only two elements) it appears that the resulting expression is Schur-positive. For example, for $n = 3$ the terms from (39) become

$$1, \quad s_1, \quad s_1, \quad (s_1 - 1 - s_{1,1})s_1, \quad s_1 s_2, \quad s_1, \quad s_2, \tag{52}$$

and the sum of these equals $1 + 2s_1 + 2s_2 + s_{1,1} + s_2 - s_{1,1,1} + s_3$. Since $s_{1,1,1}(\{q,t\}) = 0$, we can remove this leaving

$$\mathrm{Hilb}(\mathrm{DR}_3) = 1 + 2s_1 + 2s_2 + s_{1,1} + s_2 + s_3. \tag{53}$$

F. Bergeron [Ber09, p.196] has previously conjectured a stronger statement, namely that

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\sigma \in S_n} h_{\lambda(\sigma)}(\{q,t\}), \tag{54}$$

i.e. that for each permutation on n elements, there is some way of defining a partition $\lambda(\sigma)$ such that the sum of the $h_{\lambda(\sigma)}$ gives Hilb(DR_n). Here $h_\lambda = \prod_i s_{\lambda_i}$ as before. When $n = 3$, the expansion is

$$\mathrm{Hilb}(\mathrm{DR}_3) = 1 + 2h_1 + h_2 + h_{1,1} + h_3, \tag{55}$$

in agreement with (53). Bergeron further conjectures that these sums have the remarkable property that if we evaluate them in the set of variables $\{q_1, q_2, \ldots, q_k\}$ we get the Hilbert series of diagonal coinvariants in k sets of variables, for any $k \geq 1$. We hope that further study of how the cancellation in identity (38) results in positivity will lead to progress on the $k = 2$ case of Bergeron's conjecture.

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