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The short toric polynomial

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Abstract. We introduce the short toric polynomial associated to a graded Eulerian poset. This polynomial contains the same information as Stanley's pair of toric polynomials, but allows different algebraic manipulations. Stanley's intertwined recurrence may be replaced by a single recurrence, in which the degree of the discarded terms is independent of the rank. A short toric variant of the formula by Bayer and Ehrenborg, expressing the toric h-vector in terms of the cd-index, may be stated in a rank-independent form, and it may be shown using weighted lattice path enumeration and the reflection principle. We use our techniques to derive a formula expressing the toric h-vector of a dual simplicial Eulerian poset in terms of its f-vector. This formula implies Gessel's formula for the toric h-vector of a cube, and may be used to prove that the nonnegativity of the toric h-vector of a simple polytope is a consequence of the Generalized Lower Bound Theorem holding for simplicial polytopes.

Résumé. Nous introduisons le polynôme torique court associé à un ensemble ordonné Eulérien. Ce polynôme contient la même information que le couple de polynômes toriques de Stanley, mais il permet des manipulations algébriques différentes. La récurrence entrecroisée de Stanley peut être remplacée par une seule récurrence dans laquelle le degré des termes écartés est indépendant du rang. La variante torique courte de la formule de Bayer et Ehrenborg, qui exprime le vecteur torique d'un ensemble ordonné Eulérien en termes de son cd-index, est énoncée sous une forme qui ne dépend pas du rang et qui peut être démontrée en utilisant une énumération des chemins pondérés et le principe de réflexion. Nous utilisons nos techniques pour dériver une formule exprimant le vecteur h-torique d'un ensemble ordonné Eulérien dont le dual est simplicial, en termes de son f-vecteur. Cette formule implique la formule de Gessel pour le vecteur h-torique d'un cube, et elle peut être utilisée pour démontrer que la positivité du vecteur h-torique d'un polytope simple est une conséquence du Théorème de la Borne Inférieure Généralisé appliqué aux polytopes simpliciaux.

Keywords: Eulerian poset, toric h-vector, Narayana numbers, reflection principle, Morgan-Voyce polynomial

Introduction

We often look for a "magic" simplification that makes known results easier to state, and provides the language to state new results. For Eulerian posets such a wonderful simplification was the introduction of the cd-index by Fine (see [6]) allowing to restate the Bayer-Billera formulas [2] in a simpler form and to formulate Stanley's famous nonnegativity conjecture [18], shown many years later by Karu [14].

A similar "magic" moment has yet to arrive regarding the toric polynomials $f(P, x)$ and $g(P, x)$ associated to an Eulerian poset $\widehat{P} = P \uplus \{\widehat{1}\}$ by Stanley [19]. The new invariant proposed here may not be the desired "magic simplification" yet, but it represents an improvement in some cases. The idea on which it is based is very simple and widely useful. There is a bijective way to associate each *multiplicatively symmetric* polynomial $p(x)$ (satisfying $p(x) = x^{\deg(p(x))}p(1/x)$) to an *additively symmetric polynomial* $q(x)$ (satisfying $q(-x) = (-1)^{\deg(q(x))}q(x)$) of the same degree, having the same set of coefficients (see Section 2). There is no change when we want to extract the coefficients of the individual polynomials only, but when we consider a sequence $\{p_n(x)\}_{n\geq 0}$ of multiplicatively symmetric polynomials, given by some rule, switching to the additively symmetric variant $\{q_n(x)\}_{n\geq 0}$ greatly changes the appearance of the rules, making them sometimes easier to manipulate.

The short toric polynomial $\mathrm{t}(P, x)$, associated to a graded Eulerian poset $\widehat{P}$ is defined in Section 3 as the multiplicatively symmetric variant of $f(P, x)$. The intertwined recurrence defining $f(P, x)$ and $g(P, x)$ is equivalent to a single recurrence for $\mathrm{t}(P, x)$. It is a tempting thought to use this recurrence to generalize the short toric polynomial to all ranked posets having a unique minimum element, even if in the cases of some lower Eulerian posets, "severe loss of information" may occur. We state the short toric variant of Fine's formula (see [1] and [3, Theorem 7.14]) expressing the toric h-vector in terms of the flag f-vector. By inspecting this formula, it is easy to observe that the generalization of $\mathrm{t}(P, x)$ makes most sense when the reduced Euler characteristic of the order complex of $P \setminus \{\widehat{0}\}$ is not zero.

Arguably our nicest result is Theorem 4.6, expressing $\mathrm{t}(P, x)$ associated to a graded Eulerian poset $\widehat{P}$ by defining two linear operators $\mathcal{C}, \mathcal{D} : \mathbb{Q}[x] \to \mathbb{Q}[x]$ that need to be substituted into the reverse of the cd-index and applied to the constant polynomial 1. The fact that the toric h-vector may be computed by replacing the letters c and d in the reverse of the cd-index by some linear operators and applying the resulting linear operator to a specific vector is a direct consequence of the famous result by Bayer and Ehrenborg [3, Theorem 4.2], expressing the toric h-vector in terms of the cd-index. In applications, the use of this result may be facilitated by finding a linearly equivalent presentation that is easier to manipulate. Our Theorem 4.6 is analogous to Lee's result [15, Theorem 5] and it is the first result offering a rank-independent substitution rule. Theorem 4.2, the reason behind Theorem 4.6, also implies the short toric variant of [3, Theorem 4.2], and has a proof using weighted lattice path enumeration and the reflection principle. The use of weighted lattice paths is already present in the work Bayer and Ehrenborg [3, Section 7.4]. By finding the cd-index via calculating the ce-index first, and by using the short toric form, the applicability of the reflection principle becomes apparent.

Theorem 4.6 highlights the importance of the sequence of polynomials $\{\widetilde{Q}_n(x)\}_{n\geq 0}$, a variant of the sequence $\{Q_n(x)\}_{n\geq 0}$ in [3]. In Section 5 we take a closer look at this sequence, alongside the sequence of short toric polynomials $\{t_n(x)\}_{n\geq 0}$ associated to Boolean algebras. The polynomials $\{\widetilde{Q}_n(x)\}_{n\geq 0}$ turn out to be the dual basis to the *Morgan-Voyce polynomials*, whereas the polynomials $\{t_n(x)\}_{n\geq 0}$ may be used to provide a simple formula connecting $\mathrm{t}(P, x)$ to $g(P, x)$.

An application showing the usefulness of our invariant may be found in Section 6, where we express the toric h-vector of an Eulerian dual simplicial poset in terms of its f-vector. This question was raised by Kalai, see [19]. Besides using Theorem 4.6, the proof of the formula depends on a formula conjectured by Stanley [18, Conjecture 3.1] and shown in [10, Theorem 2], expressing the contribution of the h-vector entries of an Eulerian simplicial poset to its cd-index as weights of certain André permutations. An equivalent form of our formula implies that the nonnegativity of the toric h-vector of simple polytope is an elementary consequence of the Generalized Lower Bound Theorem (GLBT) holding for simplicial polytopes [20]. The word elementary has to be stressed since Karu [13] has shown that the GLBT holds for all polytopes.

1 Preliminaries

A poset P is *graded* if it has a unique minimal element $\widehat{0}$, a unique maximal element $\widehat{1}$ and a *rank function* $\rho : P \to \mathbb{N}$ satisfying $\rho(\widehat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ whenever y covers x. The rank of P is $\rho(\widehat{1})$. The *flag f-vector* of a graded poset P of rank $n+1$ is $(f_S : S \subseteq [1,n])$ where $f_S = f_S(P)$ is the number of maximal chains in the set $P_S := \{u \in P : \rho(u) \in S\}$. A graded poset is *Eulerian* if every interval $[u,v] \subseteq P$ with $u < v$ satisfies $\sum_{z\in[u,v]}(-1)^{\rho(z)} = 0$. All linear relations satisfied by the flag f-vector of an Eulerian poset were given by Bayer and Billera [2]. It was observed by Fine and proved by Bayer and Klapper [6] that the Bayer-Billera relations may be restated as the existence of the *cd-index*, as follows. Introducing the *flag h-vector* $(h_S : S \subseteq [1,n])$ of a graded poset of rank $(n+1)$ by setting $h_S := \sum_{T\subseteq S}(-1)^{|S|-|T|}f_T$, we define the *$ab$-index* as the polynomial $\Psi_P(a,b) = \sum_{S\subseteq[1,n]} h_S u_S$ in noncommuting variables a and b where the letter u_i in $u_S = u_1 \cdots u_n$ is a if $i \notin S$ and b if $i \in S$. The ab-index of an Eulerian poset is then a polynomial of $c = a + b$ and $d = ab + ba$. This polynomial $\Phi_P(c,d)$ is the *cd-index* of P. The existence of the cd index is equivalent to stating that the *ce-index*, obtained by rewriting the ab-index as a polynomial of $c = a + b$ and $e = a - b$, is a polynomial of c and e^2, see [18]. Let us denote by L_S the coefficient of the ce word $v_1 \cdots v_n$, where S is the set of indices i such that $v_i = e$. It was shown in [4] that the resulting *flag L-vector* $(L_S : S \subseteq [1,n])$ of a graded poset of rank $(n+1)$ is connected to the flag f-vector by the formulas

$$L_S = (-1)^{n-|S|} \sum_{T\supseteq[1,n]\setminus S} \left(-\frac{1}{2}\right)^{|T|} f_T \quad \text{and} \quad f_S = 2^{|S|} \sum_{T\subseteq[1,n]\setminus S} L_T. \tag{1}$$

The Bayer-Billera relations are thus also equivalent to stating that, for an Eulerian poset, $L_S = 0$ unless S is an *even set*, i.e., a disjoint union of intervals of even cardinality, see [5].

The *toric h-vector* associated to a graded Eulerian poset $[\widehat{0},\widehat{1}]$ was defined by Stanley [19] by introducing the polynomials $f([\widehat{0},\widehat{1}],x)$ and $g([\widehat{0},\widehat{1}],x)$ by the intertwined recurrences

$$f([\widehat{0},\widehat{1}],x) = \sum_{t\in[\widehat{0},\widehat{1})} g([0,p),x)(x-1)^{\rho(\widehat{1})-1-\rho(t)} \quad \text{and} \tag{2}$$

$$g([\widehat{0},\widehat{1}],x) = \sum_{i=0}^{\lfloor(\rho(\widehat{1})-1)/2\rfloor} ([x^i]f([\widehat{0},\widehat{1}],x) - [x^{i-1}]f([\widehat{0},\widehat{1}],x))x^i \tag{3}$$

and by the initial condition $f(\emptyset, x) = g(\emptyset, x) = 1$. Here the operator $[x^i]$ extracts the coefficient of x^i. The toric h-vector associated to $[\widehat{0}, \widehat{1})$ is then the vector of coefficients of the polynomial $x^{\rho(\widehat{1})-1} f([\widehat{0}, \widehat{1}), 1/x)$. The first formula expressing $f(P, x)$ in terms of the flag f-vector was found by Fine (see [1] and [3, Theorem 7.14]). Here we state it in an equivalent form that appears in the paper of Bayer and Ehrenborg [3, Section 7]:

$$f([\widehat{0}, \widehat{1}), x) = \sum_{S \subseteq [1,n]} f_S \sum_{\lambda \in \{-1,1\}^n : S(\lambda) \supseteq S} (-1)^{|S|+n-i_\lambda} x^{i_\lambda}, \tag{4}$$

where $S(\lambda) = \{s \in \{1, \ldots, n\} \ : \ \lambda_1 + \cdots + \lambda_s > 0\}$ and i_λ is the number of -1's in λ. Bayer and Ehrenborg [3, Theorem 4.2] also expressed the toric h-vector of an Eulerian poset in terms of its cd-index.

2 Additive and multiplicative symmetry of polynomials

Definition 2.1 *Let K be a field. We say that a polynomial $p(x) \in K[x]$ of degree n is* multiplicatively symmetric *if $x^n p(x^{-1}) = p(x)$ and it is* additively symmetric *if $p(x) = (-1)^n p(-x)$.*

Theorem 2.2 *A polynomial $p(x) \in K[x]$ of degree n is multiplicatively symmetric if and only if there is an additively symmetric polynomial $q(x) \in K[x]$ of degree n satisfying*

$$p(x) = x^{\frac{n}{2}} \left(q(\sqrt{x}) + q\left(\frac{1}{\sqrt{x}}\right) - q(0) \right). \tag{5}$$

Moreover, $q(x)$ is uniquely determined.

Definition 2.3 *Given a multiplicatively symmetric polynomial $p(x)$ we call the* additively symmetric variant of $p(x)$ *the additively symmetric polynomial $q(x)$ associated to $p(x)$ via* (5)*. Conversely, given an additively symmetric polynomial $q(x)$ we call the* multiplicatively symmetric variant of $q(x)$ *the polynomial $p(x)$ defined by* (5)*.*

To express the additively symmetric variant of a multiplicatively symmetric polynomial, we may use the following *truncation operators*.

Definition 2.4 *Let K be a fixed field and $K[x, x^{-1}]$ the ring of Laurent polynomials. For any $z \in \mathbb{Z}$, the* truncation operator $U_{\geq z} : K[x, x^{-1}] \to K[x, x^{-1}]$ *is the linear operator defined by discarding all terms of degree less than z. Similarly $U_{\leq z} : K[x, x^{-1}] \to K[x, x^{-1}]$ is defined by discarding all terms of degree more than z.*

The notation $U_{\geq z}$ and $U_{\leq z}$ is consistent with the notation used in [3], where (3) is rewritten as

$$g([\widehat{0}, \widehat{1}), x) = U_{\leq \lfloor n/2 \rfloor}((1-x) f([\widehat{0}, \widehat{1}), x). \tag{6}$$

Lemma 2.5 *Let $p(x)$ be a multiplicatively symmetric polynomial of degree n. Then the additively symmetric variant $q(x)$ of $p(x)$ satisfies*

$$q(x) = U_{\geq 0}(x^{-n} p(x^2)) = U_{\geq 0}(x^n p(x^{-2})).$$

3 The short toric polynomial of an arbitrary graded poset

Stanley's generalization [19, Theorem 2.4] of the Dehn-Sommerville equations may be restated as follows.

Theorem 3.1 (Stanley) *For an Eulerian poset $[\widehat{0}, \widehat{1}]$ of rank $n+1$, the polynomial $f([\widehat{0}, \widehat{1}), x)$ is multiplicatively symmetric of degree n.*

Definition 3.2 *The short toric polynomial $\mathrm{t}([\widehat{0}, \widehat{1}), x)$ associated to an Eulerian poset $[\widehat{0}, \widehat{1}]$ is the additively symmetric variant of the toric polynomial $f([\widehat{0}, \widehat{1}), x)$.*

Note that the interval $[\widehat{0}, \widehat{1})$ is half open and $f(\emptyset, x) = 1$ implies $\mathrm{t}(\emptyset, x) = 1$.

Lemma 3.3 *If $[\widehat{0}, \widehat{1}]$ is an Eulerian poset of rank $n+1$ then we have*

$$U_{\geq 1}\left(\mathrm{t}([\widehat{0}, \widehat{1}), x) \cdot \left(x - \frac{1}{x}\right)\right) = x^{n+1} g\left([\widehat{0}, \widehat{1}), x^{-2}\right).$$

Using Lemmas 2.5 and 3.3 we may show the following *fundamental recurrence*.

Theorem 3.4 *The short toric polynomial satisfies the recurrence*

$$\mathrm{t}([\widehat{0}, \widehat{1}), x) = U_{\geq 0}\left((x^{-1} - x)^{\rho(\widehat{1})-1} + \sum_{\widehat{0} < p < \widehat{1}} U_{\geq 1}\left(\mathrm{t}([\widehat{0}, p), x)(x - x^{-1})\right)(x^{-1} - x)^{\rho(\widehat{1})-\rho(p)-1}\right).$$

We may use this fundamental recurrence to extend the definition of $\mathrm{t}([\widehat{0}, \widehat{1}), x)$ to all finite posets P having a unique minimal element $\widehat{0}$ and a rank function. This generalization is not equivalent to Stanley's generalization of $f(P, x)$ to *lower Eulerian posets* in [19, §4]. Recall that a finite poset is lower Eulerian if it has a unique minimal element $\widehat{0}$ and, for each $p \in P$, the interval $[\widehat{0}, p]$ is an Eulerian poset.

Proposition 3.5 *Let P be a lower Eulerian poset and let n be the length of the longest chain in P. Then $\mathrm{t}(P, x) = U_{\geq 0}(x^n f(P, x^{-2}))$.*

Remark 3.6 For a lower Eulerian poset P, the polynomial $\mathrm{t}(P,x)$ may not contain sufficient information to recover $f(P,x)$. If $P=[\widehat{0},\widehat{1}]$ is a graded Eulerian poset of rank $n+1$ then, using [19, (19)] and Proposition 3.5, we can show $\mathrm{t}([\widehat{0},\widehat{1}],x)=0$, yet $f([\widehat{0},\widehat{1}],x)$ is usually not zero.

Proposition 3.7 (Fine's formula)

$$\mathrm{t}(P,x)=\sum_{S\subseteq[1,n]} f_S(P)\cdot \sum_{\lambda\in\{-1,1\}^n\,:\,S(\lambda)\supseteq S, n-2i_\lambda\geq 0} (-1)^{n-i_\lambda+|S|}x^{n-2i_\lambda} \tag{7}$$

holds for all finite posets P having a unique minimal element $\widehat{0}$ and a rank function $\rho: P\rightarrow\mathbb{N}$, satisfying $\rho(\widehat{0})=0$ and $n=\max\{\rho(p)\,:\,p\in P\}$. Here, for any $S\subseteq[1,n]$, $f_S=f_S(P)$ is the number of maximal chains in $P_S=\{u\in P\,:\,\rho(u)\in S\}$.

The statement may be shown in a very similar fashion to Fine's original formula. The role of equations (2) and (3) is taken over by the single recurrence given in Theorem 3.4.

Corollary 3.8 *The degree of* $\mathrm{t}(P,x)$ *equals* $\max\{\rho(p)\,:\,p\in P\}$ *if and only if* $\sum_{S\subseteq[1,n]}(-1)^{|S|}f_S(P)\neq 0.$

Note that $\sum_{S\subseteq[1,n]}(-1)^{|S|}f_S(P)$ is the reduced Euler characteristic of the *order complex* of $P\setminus\{\widehat{0}\}$.

4 The short toric polynomial and the cd-index of an Eulerian poset

Using (1) and the binomial theorem we may rewrite (7) as

$$\mathrm{t}([\widehat{0},\widehat{1}),x)=\sum_{T\subseteq[1,n]} L_S \sum_{\lambda\in\{-1,1\}^n\,:\,n-2i_\lambda\geq 0} x^{n-2i_\lambda}(-1)^{n-i_\lambda+|S(\lambda)\setminus S|}. \tag{8}$$

Just like in [3, Section 7.4], we represent each $\lambda\in\{-1,1\}^n$ by a lattice path starting at $(0,0)$ and containing $(1,\lambda_i)$ as step i for $i=1,\ldots,n$. The condition $n-2i_\lambda\geq 0$ restricts us to lattice paths whose right endpoint is on or above the horizontal axis. We introduce $R(\lambda):=\{i\in[1,n]\,:\,\lambda_1+\cdots+\lambda_i=0\}$ and say that a set S *evenly contains* the set R if $R\subseteq S$ and $S\setminus R$ is the disjoint union of intervals of even cardinality. We may use the "reflection principle" to match canceling terms, and obtain the following.

Theorem 4.1 *Let $[\widehat{0},\widehat{1}]$ be a graded Eulerian poset of rank $n+1$. Then we have*

$$\mathrm{t}([\widehat{0},\widehat{1}),x)=\sum_{S\subseteq[1,n]} L_S\cdot \mathrm{t}_{ce}(S,x).$$

Here $\mathrm{t}_{ce}(S,x)$ is the total weight of all $\lambda\in\{-1,1\}^n$ such that S evenly contains $R(\lambda)\cup(R(\lambda)-1)$ and $\lambda_1+\cdots+\lambda_i\geq 0$ holds for all $i\in\{1,\ldots,n\}$. The weight of each such λ is defined as follows: each $\lambda_i=-1$ contributes a factor of $-1/x$, each $\lambda_i=1$ contributes a factor of x and each element of $R(\lambda)$ contributes an additional factor of 2.

Theorem 4.1 gains an even simpler form when we rephrase it in terms of the cd-index.

Theorem 4.2 *Let $[\widehat{0},\widehat{1}]$ be a graded Eulerian poset of rank $n+1$. Then we have*

$$\mathrm{t}([\widehat{0},\widehat{1}],x)=\sum_{w}[w]\Phi_{[\widehat{0},\widehat{1}]}(c,d)\cdot \mathrm{t}(w,x).$$

Here the summation runs over all cd-words w of degree n. The polynomial $\mathrm{t}(w,x)$ is the total weight of all $\lambda\in\{-1,1\}^n$ such that the set of positions covered by letters d equals $R(\lambda)\cup(R(\lambda)-1)$ and $\lambda_1+\cdots+\lambda_i\geq 0$ holds for all $i\in\{1,\ldots,n\}$. The weight of each such λ is defined as follows: each $\lambda_i=-1$ contributes a factor of $-1/x$, each $\lambda_i=1$ contributes a factor of x, and each element of $R(\lambda)$ contributes an additional factor of -1.

Theorem 4.2 allows us to explicitly compute the contribution $\mathrm{t}(w,x)$. Thus we obtain the short toric equivalent of [3, Theorem 4.2], expressing $f([\widehat{0},\widehat{1}],x)$ in terms of the cd-index.

Proposition 4.3 *The polynomial $\mathrm{t}(c^{k_1}dc^{k_2}\cdots c^{k_r}dc^{k},x)$ is zero if at least one of k_1, k_2, …, k_r is odd. If k_1, k_2, …, k_r are all even then*

$$t(c^{k_1}dc^{k_2}\cdots c^{k_r}dc^{k},x)=(-1)^{\frac{k_1+\cdots+k_r}{2}}C_{\frac{k_1}{2}}\cdots C_{\frac{k_r}{2}}\widetilde{Q}_k(x).$$

Here $C_k=\frac{1}{k+1}\binom{2k}{k}$ is a Catalan number, and the polynomials $\widetilde{Q}_n(x)$ are given by $\widetilde{Q}_0(x)=1$ and

$$\widetilde{Q}_n(x):=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k\left(\binom{n-1}{k}-\binom{n-1}{k-1}\right)x^{n-2k}\quad \textit{for } n\geq 1.$$

Remark 4.4 The polynomials $\widetilde{Q}_n(x)$ are closely related to the polynomials $Q_n(x)$ introduced by Bayer and Ehrenborg [3]. They may be given by $\widetilde{Q}_n(x)=x^nQ_n(x^{-2})$.

Theorem 4.2 also allows us to introduce two linear maps $\mathcal{C}:\mathbb{Q}[x]\rightarrow\mathbb{Q}[x]$ and $\mathcal{D}:\mathbb{Q}[x]\rightarrow\mathbb{Q}[x]$ in such a way that, for any graded Eulerian poset $[\widehat{0},\widehat{1}]$, the polynomial $\mathrm{t}([\widehat{0},\widehat{1}],x)$ may be computed by substituting $\mathcal{C}$ into c and $\mathcal{D}$ into d in the *reverse* of $\Phi_P(c,d)$ and applying the resulting linear operator to 1. Note that the definitions and the result below are *independent of the rank of P*.

Definition 4.5 *We define $\mathcal{C}:\mathbb{Q}[x]\rightarrow\mathbb{Q}[x]$ by setting $\mathcal{C}(1)=x$, $\mathcal{C}(x)=x^2$ and $\mathcal{C}(x^n)=x^{n+1}-x^{n-1}$ for $n\geq 2$. We define $\mathcal{D}:\mathbb{Q}[x]\rightarrow\mathbb{Q}[x]$ by setting $\mathcal{D}(1)=1$, $\mathcal{D}(x^2)=-1$ and $\mathcal{D}(x^n)=0$ for $n\notin\{0,2\}$.*

Theorem 4.6 *For any Eulerian poset $P=[\widehat{0},\widehat{1}]$ we have*

$$\mathrm{t}([\widehat{0},\widehat{1}],x)=\Phi_P^{\mathrm{rev}}(\mathcal{C},\mathcal{D})(1)$$

Here $\Phi_P^{\mathrm{rev}}(\mathcal{C},\mathcal{D})$ is obtained from $\Phi_P(c,d)$ by first taking the reverse of each cd-monomial and then replacing each c with $\mathcal{C}$ and each d with $\mathcal{D}$.

Proof: By Theorem 4.2, we only need to show that

$$\mathrm{t}(c^{k_1}dc^{k_2}\cdots c^{k_r}dc^k,x)=\mathcal{C}^k\,\mathcal{D}\,\mathcal{C}^{k_r}\,\mathcal{D}\,\mathcal{C}^{k_{r-1}}\,\mathcal{D}\cdots\mathcal{D}\,\mathcal{C}^{k_1}(1) \tag{9}$$

holds for any cd-word $w=c^{k_1}dc^{k_2}\cdots c^{k_r}dc^k$. This may be shown by induction on the degree of w, the basis being $\mathrm{t}(\varepsilon,t)=1$ where ε is the empty word. □

5 Two useful bases

Proposition 4.3 highlights the importance of the basis $\{\widetilde{Q}_n(x)\}_{n\geq 0}$ of the vector space $\mathbb{Q}[x]$. In this section we express the elements of the basis $\{x^n\}_{n\geq 0}$, as well as the operators $\mathcal{C}$ and $\mathcal{D}$, in this new basis. We also find the analogous results for the basis $\{t_n(x)\}_{n\geq 0}$ where $t_n(x)=\mathrm{t}(B_{n+1},x)$ for the Boolean algebra $\widehat{B_{n+1}}$ of rank $n+1$. This basis is useful in proving the main result of Section 6, as well as in finding a very simple formula connecting $\mathrm{t}(P,x)$ with $g(P,x)$.

Proposition 5.1 *For $n>0$ we have*

$$x^n=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-k}{k}\widetilde{Q}_{n-2k}(x).$$

We may rewrite Proposition 5.1 as $x^{2n}=\sum_{k=1}^{n}\binom{n-1+k}{n-k}\widetilde{Q}_{2k}(x)$ for even powers of x and as $x^{2n+1}=\sum_{k=0}^{n}\binom{n+k}{n-k}\widetilde{Q}_{2k+1}(x)$ for odd powers of x. The coefficients appearing in these equations are exactly the coefficients of the *Morgan-Voyce polynomials* $B_n(x)$ and $b_n(x)$ respectively, see [16, 23, 24]. Another connection between the toric g-polynomials of cubes and the Morgan-Voyce polynomials was noted in [11].

Corollary 5.2 *The linear transformation $\mathbb{Q}[x]\to\mathbb{Q}[x]$ given by $x^n\mapsto\widetilde{Q}_n(x)$ takes $B_n(x^2)$ into x^{2n} and $xb_n(x^2)$ into x^{2n+1}.*

Comparing Proposition 4.3 with (9) yields the following consequence.

Corollary 5.3 *The operators $\mathcal{C}$ and $\mathcal{D}$ are equivalently given by*

$$\mathcal{C}(\widetilde{Q}_n(x)):=\widetilde{Q}_{n+1}(x)\quad\textit{and}\quad\mathcal{D}(\widetilde{Q}_n(x))=\begin{cases}0 & \textit{for odd } n,\\ (-1)^{n/2}C_{n/2} & \textit{for even } n.\end{cases}$$

We now turn to the polynomials $t_n(x):=\mathrm{t}(B_{n+1},x)$. Stanley's result [19, Proposition 2.1] may be rewritten as

$$t_n(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}x^{n-2k}\quad\text{for } n\geq 0. \tag{10}$$

Inverting the summation given in (10) yields

$$x^n = \begin{cases} t_n(x) - t_{n-2}(x) & \text{if } n \geq 2, \\ t_n(x) & \text{if } n \in \{0, 1\}. \end{cases} \tag{11}$$

As an immediate consequence of Definition 4.5 and (11) we obtain

$$\mathcal{C}(t_n(x)) = t_{n+1}(x) - t_{n-1}(x) \quad \text{and} \quad \mathcal{D}(t_n(x)) = \delta_{n,0} \quad \text{for } n \geq 0. \tag{12}$$

Here we set $t_{-1}(x) := 0$ and $\delta_{n,0}$ is the Kronecker delta function. Finally, the most remarkable property of the basis $\{t_n(x)\}_{n\geq 0}$ is its role in the following result connecting the polynomials $g(P, x)$ and $\mathrm{t}(P, x)$.

Proposition 5.4 *Let $[\widehat{0}, \widehat{1}]$ be any Eulerian poset of rank $n + 1$. Then $\mathrm{t}([\widehat{0}, \widehat{1}), x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_k t_{n-2k}(x)$ holds for some integers $c_0, c_1, \dots, c_{\lfloor n/2 \rfloor}$ if and only if the same integers satisfy $g([\widehat{0}, \widehat{1}), x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_k x^k$.*

6 The toric h-vector associated to an Eulerian dual simplicial poset

Given any graded poset P of rank $n+1$, let f_i denote number of elements of rank $i+1$ in P. The resulting vector $(f_{-1}, f_0, \dots, f_n)$ is the *f-vector* of P. A graded poset P is *simplicial* if for all $t \in P \setminus \{\widehat{1}\}$, the interval $[\widehat{0}, t]$ is a Boolean algebra. A graded poset P is *dual simplicial* if its dual P^* is a simplicial poset. It was first observed by Kalai that the toric h-polynomials coefficients of a dual simplicial graded Eulerian poset P depend only on the entries f_i in the f-vector P. This linear combination is not unique, and a simple explicit formula was not known before. We will express the toric h-polynomial coefficients of P in terms of its *h-vector* $(h_0, \dots, h_n)$, given by

$$h_k = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} f_i,$$

Since $f_i(P) = f_{n-1-i}(P^*)$, by [19, Corollary 2.2] this h-vector coincides with the toric h-vector of the simplicial poset P^*. Our main result is the following:

Theorem 6.1 *The short toric polynomial $\mathrm{t}([\widehat{0}, \widehat{1}), x)$ associated to a graded dual simplicial Eulerian poset $P = [\widehat{0}, \widehat{1}]$ of rank $n + 1$ may be written as*

$$\begin{aligned} \mathrm{t}([\widehat{0}, \widehat{1}), x) = {} & h_0(t_n(x) - (n-1)t_{n-2}(x)) \\ & + \sum_{i=1}^{n-1} h_i \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n-i}{k}\binom{i-1}{k-1} - \binom{n-i-1}{k}\binom{i}{k-1} \right) t_{n-2k}(x) \end{aligned}$$

The proof uses Stanley's description [18, Theorem 3.1] of the cd-index of an Eulerian simplicial poset in terms of its h-vector as a combination $\Phi_P(c, d) = \sum_{i=0}^{n-1} h_i \cdot \check{\Phi}_i^n$ and the combinatorial description of the polynomials $\check{\Phi}_i^n$ stated in [10, Theorem 2], originally conjectured by Stanley [18, Conjecture 3.1]. These results are combined with (9) to obtain recurrence formulas that allow proving the result by induction. See [12] for details. An important equivalent form of Theorem 6.1 is the following statement.

Proposition 6.2 *Let $[\widehat{0},\widehat{1}]$ be a graded dual simplicial Eulerian poset of rank $n+1$. Then we have*

$$\mathrm{t}([\widehat{0},\widehat{1}),x) = h_0 t_n(x) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (h_i - h_{i-1}) \sum_{k=1}^{\min\{i,n-i\}} \frac{n+1-2i}{k} \binom{n-i}{k-1} \binom{i-1}{k-1} t_{n-2k}(x).$$

Corollary 6.3 *Let $[\widehat{0},\widehat{1}]$ be a graded dual simplicial Eulerian poset of rank $n+1$. Then*

$$\begin{aligned} g([\widehat{0},\widehat{1}),x) &= h_0(1-(n-1)x) \\ &\quad + \sum_{i=1}^{n-1} h_i \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n-i}{k}\binom{i-1}{k-1} - \binom{n-i-1}{k}\binom{i}{k-1} \right) x^k. \end{aligned}$$

Corollary 6.4 *Let $[\widehat{0},\widehat{1}]$ be a graded dual simplicial Eulerian poset of rank $n+1$. Then*

$$g([\widehat{0},\widehat{1}),x) = 1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (h_i - h_{i-1}) \sum_{k=1}^{\min\{i,n-i\}} \frac{n+1-2i}{k} \binom{n-i}{k-1} \binom{i-1}{k-1} x^k.$$

The most important consequence of Corollary 6.4 is the following.

Corollary 6.5 *Let $[\widehat{0},\widehat{1}]$ be a graded dual simplicial Eulerian poset of rank $n+1$. If the h-vector $(h_0,\ldots,h_n)$ satisfies $h_0 \le h_1 \le \cdots \le h_{\lfloor n/2 \rfloor}$, then $f([\widehat{0},\widehat{1}],x)$ has nonnegative coefficients.*

Indeed, by Corollary 6.4 above, $g([\widehat{0},\widehat{1}),x)$ has nonnegative coefficients and the statement follows from [19, (19)].

Example 6.6 Let $[\widehat{0},\widehat{1}]$ be the face lattice of an n-dimensional simple polytope $\mathcal{P}$. By Corollary 6.5, the fact that the toric h-vector of $\mathcal{P}$ has nonnegative entries is a consequence of the Generalized Lower Bound Theorem [20] for simplicial polytopes.

Remark 6.7 In the case when $n=2i$, the coefficients $N(i,k) = \binom{i-1}{k-1}\binom{i}{k-1}/k$, contributed by $h_{\lfloor n/2 \rfloor} - h_{\lfloor n/2 \rfloor -1}$ in Corollary 6.4 are the *Narayana numbers*, see sequence A001263 in [17]. The same numbers appear also as the coefficients of the contributions of $h_{\lfloor n/2 \rfloor}$ and $h_{\lceil n/2 \rceil}$ for any n in Corollary 6.3.

Motivated by Example 6.9 below, we rewrite Corollary 6.3 in the basis $\{(x-1)^k\}_{k \ge 0}$.

Proposition 6.8 *Let $[\widehat{0},\widehat{1}]$ be a graded dual simplicial Eulerian poset of rank $n+1$. Then we have*

$$g([\widehat{0},\widehat{1}],x) = h_0(n-(n-1)(x-1)) + \sum_{i=1}^{n-1} h_i \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n-i}{k}\binom{n-k-1}{i-k} - \binom{n-i-1}{k}\binom{n-k-1}{i+1-k} \right)(x-1)^k.$$

Example 6.9 Let $[\widehat{0},\widehat{1}]$ be the face lattice $[\widehat{0},\widehat{1}]$ of an n-dimensional cube. Starting with Proposition 6.8, after repeated use of Pascal's identity and the Chu-Vandermonde identity, one can show

$$g([\widehat{0},\widehat{1}],x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} C_{n-k}(x-1)^k. \tag{13}$$

It was noted in [11, Lemma 3.3] that (13) is equivalent to Gessel's result [19, Proposition 2.6], stating

$$g([\widehat{0},\widehat{1}],x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{n-k+1}\binom{n}{k}\binom{2n-2k}{n}(x-1)^k. \tag{14}$$

The first combinatorial interpretation of the right hand side of (14) is due to Shapiro [21, Ex. 3.71g] the proof of which was published by Chan [9, Proposition 2].

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