# *Bumping algorithm for set-valued shifted tableaux*

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**Abstract.** We present an insertion algorithm of Robinson–Schensted type that applies to set-valued shifted Young tableaux. Our algorithm is a generalization of both set-valued non-shifted tableaux by Buch and non set-valued shifted tableaux by Worley and Sagan. As an application, we obtain a Pieri rule for a $K$-theoretic analogue of the Schur $Q$-functions.

**Résumé** Nous présentons un algorithme d'insertion de Robinson–Schensted qui s'applique aux tableaux décalés à valeurs sur des ensembles. Notre algorithme est une généralisation de l'algorithme de Buch pour les tableaux à valeurs sur des ensembles et de l'algorithme de Worley et Sagan pour les tableaux décalés. Comme application, nous obtenons une formule de Pieri pour un analogue en $K$-théorie des $Q$-functions de Schur.

**Keywords:** set-valued shifted tableaux, insertion, Robinson–Schensted, Pieri rule, $K$-theory, Schur $Q$-functions

## 1 Introduction

This article is an extended abstract of the paper [INN] of the same title. Most details of the proofs are omitted.

In [IN], we introduced a non-homogeneous ($K$-theoretic) analogue of Schur $Q$-functions. These functions are labeled by strict partitions (or shifted Young diagrams), as are the original $Q$-functions. For a strict partition $\lambda$, the corresponding $K$-theoretic Schur $Q$-function $GQ_\lambda(x)$ can be expressed as a weighted generating function of *shifted set-valued semistandard tableaux* of shape $\lambda$, which are the central concern of this article.

The main result of the paper is a Robinson–Schensted type insertion algorithm for the shifted set-valued tableaux (Thm 3.4). Our algorithm is a generalization of both set-valued non-shifted tableaux by Buch [Bu] and non set-valued shifted tableaux by Worley [Wo] and Sagan [Sa]. As an immediate consequence of our algorithm, we have a Pieri rule for $GQ_\lambda(x)$ (Cor. 3.5).

The original purpose for introducing functions $GQ_\lambda(x)$ was to apply them to Schubert calculus. In [IN] we introduced function $GQ_\lambda(x|b)$ (resp. $GP_\lambda(x|b)$) with the *equivariant* parameter $b = (b_1, b_2, \ldots)$,

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which represents the structure sheaf of the Schubert variety indexed by $\lambda$ in the $K$-ring of $T$-equivariant coherent sheaves on Langangian (resp. orthogonal) Grassmannian, where $T$ is the maximal torus acting on the Grassmannians. Thus our Pieri rule gives an explicit description of $K$-theoretic Schubert structure constant for an arbitrary Schubert class times a special (one row type) Schubert class in the $K$-ring of Lagrangian Grassmannian.

Recently, a $K$-theoretic Littlewood-Richardson rule in terms of the *jeu de taquin* for odd orthogonal Grassmannians of maximal isotropic subspaces has been obtained by Clifford, Thomas and Yong [CTY]. Their method starts from a Pieri rule for the $K$-theory by Buch and Ravikumar [BR], which applies to cominuscule Grassmannians. Our approach differs from them substantially. We proceeded independently a different approach of tableaux insertion to result in the same formula as [BR], i.e. the counting of KLG-tableaux. But our method is only applicable to the case of Lagrangian Grassmannians, although there is a set valued tableaux description for $GP_\lambda(x)$.

Organization of the paper is as follows. In Section 2, we give the definition of shifted set-valued tableaux, and $K$-theoretic Schur $Q$-functions $GQ_\lambda(x)$. In Section 3, we present our main result, an existence of a Robinson–Schensted type bijection for set-valued shifted tableaux. As a corollary, we have a Pieri rule for $GQ_\lambda(x)$. Precise description of the bijection is given by a bumping algorithm which is given in Section 4. In Section 5, we discuss a variant of the bijection, which is analogous to the results by Sagan and Worley. In Section 6, we give an outline of the proof of the main theorem.

# 2 Shifted Young diagrams, set-valued tableaux

## 2.1 Shifted Young diagrams

Let $\Delta$ denote the set $\{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le j\}$. Any element $\alpha = (i,j)$ is called a *box*. If $i = j$, then $(i,j)$ is called a *diagonal box*. A *shifted Young diagram* is any finite subset $\lambda$ of $\Delta$ such that for each $\alpha = (i,j) \in \lambda$, any box $\beta = (i',j') \in \Delta$ satisfying $i' \le i$ and $j' \le j$ belongs to $\lambda$.

We define $\mathbb{S}$ to be the set of shifted Young diagrams. For $\lambda \in \mathbb{S}$, we define $|\lambda|$ to be the number of boxes in $\lambda$. For $\lambda, \mu \in \mathbb{S}$ such that $\lambda \subset \mu$, we define the skew shifted Young diagram $\mu/\lambda$ to be the set-theoretic difference $\mu - \lambda$.

Let $\alpha = (i,j), \beta = (i',j') \in \Delta$. We say that $\alpha$ is *weakly below* (resp. *weakly right of*) $\beta$ if $i \ge i'$ (resp. $j \ge j'$). We say that $\alpha$ is *strictly below* (resp. *strictly right of*) $\beta$ if $i > i'$ (resp. $j > j'$). We say that $\alpha$ is *directly below* (resp. *directly right of*) $\beta$ if $i = i' + 1$ and $j = j'$ (resp. $i = i'$ and $j = j' + 1$).

We call a skew shifted diagram $\theta$ a *horizontal strip* (resp. *vertical strip*) if $\theta$ has no pair of boxes in the same column (resp. row). We call $\theta$ a *broken border strip* if $\theta$ contains no $2 \times 2$ square block.

## 2.2 Tableaux

Define a totally ordered set $\mathcal{B}$ to be disjoint union of sets $\mathcal{A} = \{1, 2, \ldots\}$ and $\mathcal{A}' = \{1', 2', \ldots\}$ with the following order:

$$1' < 1 < 2' < 2 < \cdots .$$

We define binary relations $\le_r$ and $\le_c$ on $\mathcal{B}$ by

$$x \le_r y \iff x = y \in \mathcal{A} \text{ or } x < y, \qquad x \le_c y \iff x = y \in \mathcal{A}' \text{or } x < y.$$

Note that $x \not\le_r y$ (resp. $x \not\le_c y$) is equivalent to $y \le_c x$ (resp. $y \le_r x$) for any $x, y \in \mathcal{B}$.

Let $\mathcal{X}$ denote the set of non-empty finite subsets of $\mathcal{B}$. We extend the relations $\leq_r, \leq_c$ on $\mathcal{X}$ by $A \leq_r B \iff \max A \leq_r \min B$ and $A \leq_c B \iff \max A \leq_c \min B$ for $A, B \in \mathcal{X}$.

**Definition 2.1 (Shifted set-valued semistandard tableaux)** *Let $\lambda$ be a shifted Young diagram. A* set-valued semistandard tableau *of shape $\lambda$ is a map $T$ from the set of boxes in $\lambda$ to $\mathcal{X}$ satisfying the following "semistandaredness" :*

1. *$T(\alpha) \leq_r T(\beta)$ if $\beta \in \lambda$ is directly right of $\alpha \in \lambda$.*
2. *$T(\alpha) \leq_c T(\beta)$ if $\beta \in \lambda$ is directly below $\alpha \in \lambda$.*

**Example 2.2** An example of a set-valued tableau is given by the following:

$$T = \begin{array}{|c|c|c|c|} \hline 1' & 12' & 23 & 34' \\ \hline \multicolumn{1}{c|}{} & 2' & 4' & 6 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 6 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}.$$

We denote by $\mathcal{T}(\lambda)$ the set of all set-valued tableaux of shape $\lambda$.

### 2.3 $K$-theoretic $Q$-Schur functions

Let $x = (x_1, x_2, \ldots)$ be a sequence of variables. Let $\lambda \in \mathbb{S}$ and $T \in \mathcal{T}(\lambda)$. We define the corresponding monomial $x^T = \prod_{i=1}^{\infty} x_i^{e_i(T)}$ where $e_i(T)$ denotes the total number of $i$ and $i'$ appearing in $T$. The *weight* of $T \in \mathcal{T}(\lambda)$ is defined to be $\beta^{|T|-|\lambda|}x^T$, where $\beta$ is a formal parameter and $|T|$ is the total number of letters in $T$. The $K$-theoretic $Q$-Schur function $GQ_\lambda(x)$ is defined as the following formal sum of the weights of the elements in $\mathcal{T}(\lambda)$ :

$$GQ_\lambda(x) = \sum_{T \in \mathcal{T}(\lambda)} \beta^{|T|-|\lambda|} x^T.$$

When $\beta = 0$ this becomes the Schur Q-function $Q_\lambda(x)$, and when $\beta = -1$ this represents $K$-theory Schubert class corresponding to $\lambda$ for Lagrangian Grassmannians. See [IN] for other expressions of $GQ_\lambda(x)$ and geometric background.

## 3 Statements of main results

### 3.1 Admissible strips

Let $\theta = \lambda/\mu$ be a broken border strip. We consider a decomposition $\theta = C \sqcup C'$, with $C, C'$ skew diagrams, i.e. there is a diagram $\nu$ satisfying $\mu \subset \nu \subset \lambda$ and $C = \lambda/\nu$ and $C' = \nu/\mu$. Such a decomposition of $\theta$ is called *admissible* of if the following conditions are satisfied:

1. in each of the diagrams $C$ and $C'$, there is no pair of boxes in the same row or column.
2. there is no diagonal box in $C'$.

A non-empty broken border strip $\theta$ is called a 1-*admissible strip* if there exists an admissible decomposition of $\theta$. For a 1-admissible strip $\theta$, we denote by $\mathcal{C}(\theta)$ the set of all admissible decompositions of $\theta$. Later we define the notion of $m$-admissible decomposition of a broken border strip.

**Example 3.1** The following is an example of a 1-admissible strip and its 1-admissible decomposition,

$$
\begin{array}{cccccccc}
\square & \square & \square & \square & \square & \square & 1' & 1 \\
 & \square & \square & \square & 1' & 1 & & \\
 & & \square & \square & 1 & & & \\
 & & & 1 & & & &
\end{array}
,
$$

where the boxes with entry 1's form $C$ and $1'$'s form $C'$.

The next result shows the role of 1-admissible strip. The detailed construction of the map is given in Section 4. We define the weight of a 1-admissible strip $\theta$ to be $\beta^{|\theta|-1}$.

**Proposition 3.2** *There is a weight preserving bijection:*

$$\phi : \mathcal{T}(\lambda) \times \mathcal{X} \longrightarrow \bigsqcup_{\mu} \mathcal{T}(\mu) \times \mathcal{C}(\mu/\lambda)$$

*where $\mu \in \mathbb{S}$ runs for those $\mu$ such that $\mu/\lambda$ is a 1-admissible strip.*

## 3.2 Composable admissible strips

Let $\lambda, \mu, \nu \in \mathbb{S}$ be such that $\mu \subset \nu \subset \lambda$. Suppose $\theta_1 = \nu/\mu, \theta_2 = \lambda/\nu$ are 1-admissible strips. Let $(C'_i, C_i) \in \mathcal{C}(\theta_i)\,(i = 1, 2)$.We say that $(C'_1, C_1)$ *precedes* $(C'_2, C_2)$ and denote $(C'_1, C_1) \lhd (C'_2, C_2)$, if the following conditions are satisfied:

1. $C'_1 \cup C'_2$ is a vertical strip.
2. $C_1 \cup C_2$ is a horizontal strip.
3. Each box in $C'_2$ is strictly below any box in $C'_1$.
4. Each box in $C_2$ is strictly right of any box in $C_1$.
5. If $C_1 \neq \emptyset$, then $C'_2 = \emptyset$.

## 3.3 Main results

Let $\theta = \mu/\lambda$ be a broken border strip, and $m$ be a positive integer. Suppose there is a nested sequence of shifted diagrams

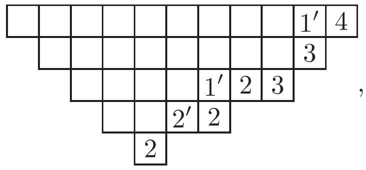
$$\lambda = \nu^{(0)} \subset \nu^{(1)} \subset \nu^{(2)} \subset \cdots \subset \nu^{(m)} = \mu \tag{1}$$

such that $\theta^{(i)} := \nu^{(i)}/\nu^{(i-1)}$ $(1 \le i \le m)$ are 1-admissible strips. If, moreover, there is a sequence of 1-admissible decompositions $(C'_i, C_i) \in \mathcal{C}(\theta^{(i)})$ $(1 \le i \le m)$ such that

$$(C'_i, C_i) \lhd (C'_{i+1}, C_{i+1}), \quad (1 \le i \le m-1). \tag{2}$$

then we say $\theta$ is an *$m$-admissible strip*. For an *$m$-admissible strip* $\theta$, let $\mathcal{C}_m(\theta)$ denote the set of pairs $\left(\{\nu^{(i)}\}_{i=1}^m,\ \{(C'_i, C_i)\}_{i=1}^m\right)$ satisfying the above conditions, which we call $m$-admissible decompositions of $\theta$. Note $\mathcal{C}_1(\theta) = \mathcal{C}(\theta)$ since condition (2) is vacant for $m = 1$.

**Example 3.3** The following is a 4-admissible strip



where the boxes with entry $i$ are $C_i$, and $i'$ are $C'_i$.

We denote by $(m)$ the shifted diagram consisting of one row with $m$ boxes. We simply denote $\mathcal{T}(m)$ for $\mathcal{T}((m))$. Recall that we define the weight of $T \in \mathcal{T}(\lambda)$ as $\beta^{|T|-|\lambda|}x^T$. Define the weight of $U \in \mathcal{C}_m(\theta)$ to be $\beta^{|\theta|-m}$.

**Theorem 3.4** *By algorithm 4.4, we have a weight preserving bijection:*

$$\phi_m : \mathcal{T}(\lambda) \times \mathcal{T}(m) \longrightarrow \bigsqcup_{\mu} \mathcal{T}(\mu) \times \mathcal{C}_m(\mu/\lambda), \tag{3}$$

*where $\mu$ runs for shifted diagrams $\mu$ such that $\mu \supset \lambda$ and $\mu/\lambda$ are $m$-admissible strips.*

As an immediate consequence, we have the following.

**Corollary 3.5 (Pieri rule)** *We have*

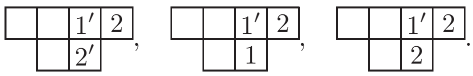
$$GQ_\lambda(x) \cdot GQ_m(x) = \sum_{\mu \supset \lambda} \beta^{|\mu|-|\lambda|-m} \# \mathcal{C}_m(\mu/\lambda) \times GQ_\mu(x),$$

*where $\mu$ runs for shifted diagrams $\mu$ such that $\mu \supset \lambda$ and $\mu/\lambda$ are $m$-admissible strips.*

For example we have

$$GQ_{2,1} \cdot GQ_2 = 2GQ_{4,1} + 2GQ_{3,2} + 3\beta GQ_{4,2} + \beta GQ_{5,1} + \beta GQ_{3,2,1} + \beta^2 GQ_{5,2} + \beta^2 GQ_{4,2,1}.$$

In order to give the coefficient of $GQ_{4,2}$, we count the elements in $\mathcal{C}_2(\mu/\lambda)$ with $\mu = (4,2), \lambda = (2,1)$ :



N.B. The elements in $\mathcal{C}_m(\mu/\lambda)$ are exactly the $KLG$-tableaux of shape $\mu/\lambda$ with content $\{1, 2, ..., m\}$ in [BR].

# 4 Bumping algorithm

The aim of this section is to describe the bijection of Prop 3.2.

The input of our algorithm is a pair $(T, w)$ with $T \in \mathcal{T}(\lambda)$ for some $\lambda \in \mathbb{S}$ and $w \in \mathcal{X}$. Basic output is a tableau $T'$ of some shape $\mu \in \mathbb{S}$ such that $\mu \supset \lambda$. The skew diagram $\theta = \mu/\lambda$, the set of "new boxes", turns out to be a 1-admissible strip. We also have some "recording data" on $\theta$ which gives an element of $\mathcal{C}(\theta)$.

### 4.1 Parts of "L" shape of a tableau

Let $\lambda \in \mathbb{S}$. Let $\ell(\lambda)$ be the number of rows of $\lambda$. For $1 \leq t \leq \lambda_1$ we define a subset of $\lambda$ by

$$L_t(\lambda) = \{(i,j) \in \lambda \mid i = t \text{ or } j = t\}.$$

For example, $L_1(\lambda)$ consists of the boxes in the first row. For $k \geq \ell(\lambda)$, $L_k(\lambda)$ is just the $k$-th column. In general, this is a subset of shape "L" including the diagonal box $(t,t)$. Let $T \in \mathcal{T}(\lambda)$. By restriction we have a map $L_t(T) : L_t(\lambda) \to \mathcal{X}$, which we call the $t$-th *part* of $T$.

Our algorithm starts from inserting $w = w^{(0)} \in \mathcal{X}$ into $L_1 = L_1(T)$, the first row of $T$, resulting a row $L_1'$ with possibly a new box at the right end, and a set $w^{(1)} \in \mathcal{X}$ "bumped out" from the procedure. Then we modify the original tableau $T = T^{(0)}$ by replacing $L_1$ with $L_1'$ to obtain $T^{(1)}$. Next we insert $w^{(1)}$ into the second part of the modified tableau $T^{(2)}$. We repeat this procedure until no boxes are bumped out.

### 4.2 Insertion into a part of "L" shape (a rough idea)

We define a procedure to insert some sets $w \in \mathcal{X}$ into an L part $X$ of a tableaux.

Here we present a rough idea of constructing the procedure. First, we look at the minimum letters of each boxes in order to decide the box into which a letter in $w$ to be inserted, in the same manner as the classical bumping procedure (some letters go into empty box at the end). If we might simply insert these letters into $X$, some letters in $w$ may violate the semistandardness, while some letters are not. So we eject some element in $X$ before inserting $w$. Let $\hat{w}$ be the set of letters in $w$ which do not conflict any original letters in $X$, and let $\check{w} := w - \hat{w}$ be the complement. If $\check{w} \neq \emptyset$, let $\check{u}$ be the set of elements in $X$ that conflict some element in $\check{w}$. To ensure the semistandardness, we first eject the elements in $\check{u}$ from the tableau. Furthermore, if a letter in $\hat{w}$ is inserted into a non-empty box, we eject all the remaining (original) entries of the box. Thus any letter inserted into a non-empty box "does some work" (bumps out at least one letter). This feature is important for constructing the inverse algorithm.

There is a flaw in this idea. For example, we consider a tableau $T = \boxed{1'}$ and $w = w^{(1)} = \{1'\}$. According to the naive algorithm above, the resulting tableau is $T^{(1)} = \boxed{1'}$, and the ejected set is $w^{(2)} = \{1'\}$. Since the second part is empty, the final result is $\boxed{1'}\boxed{1'}$, which is not semistandard. This is a reason why we need the "unmark" process introduced in the next section. In fact, we should care for the case of inserting elements into the diagonal boxes.

### 4.3 Insertion into a diagonal box

Let $X \in \mathcal{X}$, and $u$ be a subset of $X$. We insert $w \in \mathcal{X}$ into $X$, where we consider $X$ to be a diagonal box.

**Algorithm 4.1 (Bumping for a diagonal box)**

**input** $X, w, u \in \mathcal{X}$ satisfying $u \subset X$ and $\max w \leq_c \min X$.

**output** $Y, v$.

**procedure**

1. If $X \neq u$, then let $Y = (X - u) \cup w$ and $v = u$; and return $Y, v$.
2. If $i' = \max(w) \in \mathcal{A}'$ and $i \in X,\ i' \notin X$, then let $Y = \{\, i \,\} \cup (w - \{\, i' \,\})$ and $v = X$; and return $Y, v$.

3. If $i' = \max(w) \in \mathcal{A}'$ and $i' \in X$, $i \notin X$, then let $Y = w$ and $v = \{ i \} \cup (X - \{ i' \})$; and return $Y$, $v$.
4. If $i' = \max(w) \in \mathcal{A}'$ and $i, i' \in X$, then let $Y = \{ i \} \cup w$ and $v = X - \{ i' \}$; and return $Y$, $v$.
5. Otherwise, let $Y = w$ and $v = X$; and return $Y$, $v$.

For example, if $u = X = \boxed{34}$ and $w = 13'$, then we apply (2) to obtain $Y = \boxed{13}$ rather than $\boxed{13'}$, and $u = 34$. Thus letter $3'$ is unprimed to be 3 in $u$. If $u = X = \boxed{3'4}$ and $w = 13'$, then we apply (3) to obtain $Y = \boxed{13'}$ and $u = 34$, rather than $u = 3'4$. In this case, two $3'$ are involved, and one may think of this process as umpriming "bigger" $3'$. Case (4) is a bit strange. If $u = X = \boxed{3'3}$ and $w = 3'$, then we have $Y = \boxed{3'3}$ and $u = 3$. This case we are unpriming "bigger" $3'$ also, and let it remain in the box.

## 4.4 Insertion into a part of "L" shape (definition)

Let $T$ be a tableau of shape $\lambda$, and $t$ be a positive integer such that $t \le \lambda_1$. Let $X = L_t(T)$ be the $t$-th part of $T$. If $t = 1$, then $X$ is a row: $X = (X_{(1,1)} \le_r X_{(1,2)} \le_r \cdots \le_r X_{(1,\lambda_1)})$. If $t > \ell(\lambda)$ then $X$ is a column: $X = (X_{(1,t)} \le_c \cdots \le_c X_{(k,t)})$ for some $k < t$. We say that $X$ is a *pure column* in this case (note that $X$ does not contain diagonal box). If $1 < t \le \ell(\lambda)$ then $X = L_t(T)$ is a sequence of elements in $\mathcal{X}$ :

$$X = (X_{(1,t)} \le_c \cdots \le_c X_{(t-1,t)} \le_c X_{(t,t)} \le_r X_{(t,t+1)} \le_r \cdots \le_r X_{(t,t+\lambda_t-1)}).$$

The following algorithm takes as an input a sequence of elements in $\mathcal{X}$ satisfying

$$X = (X_{-k} \le_c \cdots \le_c X_{-1} \le_c X_0 \le_r X_1 \le_r \cdots \le_r X_l),$$

for some $k, l \ge 0$, and $w \in \mathcal{X}$. If $k = 0$, we consider $X$ as a row. Output is a triple $(Y, Y_+, v)$, where $Y$ is a sequence $Y = (Y_i)_{i=-k}^{l}$ satisfying the same condition as $X$, and $Y_+, v \in \mathcal{X} \cup \emptyset$. If $Y_+ \ne \emptyset$ we will make a new box with entry $Y_+$ at the right end of $Y$.

**Algorithm 4.2 (Bumping rule for an L part)**

**input** $X = (X_i)_{i=-k}^{l}$ : tableau of $L$ shape, i.e.

$$X = (X_{-k} \le_c \cdots \le_c X_{-1} \le_c X_0 \le_r X_1 \le_r \cdots \le_r X_l),$$

and $w \in \mathcal{X}$.

**output** $Y$ tableau of $L$ shape of the same length of $X$, and $Y_+, v \in \mathcal{X} \cup \emptyset$.

**procedure**

1. Define the subsets $w_{-k}, \ldots, w_{l+1}$ of $w$ by

$$w_t = \begin{cases} \{x \in w \mid x \le_r \min X_{-k}\} & (t = -k) \\ \{x \in w \mid \min X_{t-1} \le_c x \le_r \min X_i\} & (t = -k, \ldots, -1) \\ \{x \in w \mid \min X_{-1} \le_c x \le_c \min X_0\} & (t = 0) \\ \{x \in w \mid \min X_{t-1} \le_r x \le_c \min X_t\} & (t = 1, \ldots, l) \\ \{x \in w \mid \min X_l \le_r x\} & (t = l + 1) \end{cases}$$

2. Decompose $w_t$ into the subsets $\check{w}_t$ and $\hat{w}_t$ defined by

$$\hat{w}_t = \begin{cases} w_t & (t=-k) \\ \{\, x \in w_t \mid \max X_{t-1} \leq_c x \,\} & (t=-k+1,\ldots,0) \\ \{\, x \in w_t \mid \max X_{t-1} \leq_r x \,\} & (t=1,\ldots,l+1) \end{cases},$$

$\check{w}_t = w_t - \hat{w}_t$, for $t=-k,\ldots,l+1$,

3. Define $\check{u}_t$, $\hat{u}_k$, and $u_k$ $(t=-k,\ldots,l)$ by:

$$\check{u}_t = \begin{cases} \emptyset & (\text{if } \check{w}_{t+1}=\emptyset) \\ \{\, y \in X_t \mid y \not\leq_c \min \check{w}_{t+1} \,\} & (\text{if } t=-k,\ldots,-1 \text{ and } \check{w}_{t+1} \neq \emptyset) \\ \{\, y \in X_t \mid y \not\leq_r \min \check{w}_{t+1} \,\} & (\text{if } t=0,\ldots,l \text{ and } \check{w}_{t+1} \neq \emptyset) \end{cases},$$

$$\hat{u}_t = \begin{cases} \emptyset & (\text{if } \hat{w}_t=\emptyset) \\ X_t - \check{u}_t & (\text{if } \hat{w}_t \neq \emptyset) \end{cases}$$

$$u_t = \hat{u}_t \cup \check{u}_t \subset X_t.$$

4. Define $Y_t = (X_t - u_t) \cup w_t$ and $v_t = u_t$ for $t \neq 0$.
5. Let $(Y_0, v_0)$ be the pair obtained from the triple $(X_0, w_0, u_0)$ by Algorithm 4.1 if $l \geq 0$.
6. Let $Y = (Y_{-k},\ldots,Y_l)$, $Y_+ = w_{l+1}$, and $v = \bigcup_{t=-k}^{l} v_t$; and return $Y$, $Y_+$, $v$.

**Example 4.3** Let $X = (X_{-2}, X_{-1}; X_0; X_1, X_2, X_3)$ be

$$\boxed{13}\boxed{4'}\boxed{5}\boxed{56}\boxed{8}\boxed{9}.$$

Let us insert $w = 25'6'79'9 \in \mathcal{X}$ into $X$. Since the minimums in $X$ is

$$\boxed{1}\boxed{4'}\boxed{5}\boxed{5}\boxed{8}\boxed{9},$$

we have $(w_{-2},\ldots,w_4) = (\emptyset, 2, 5', \emptyset, 6'7, 9', 9)$. Since the maximums of $X$ is

$$\boxed{3}\boxed{4'}\boxed{5}\boxed{6}\boxed{8}\boxed{9},$$

we have

| $t$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|
| $\hat{w}_t$ | $\emptyset$ | $\emptyset$ | $5'$ | $\emptyset$ | $7$ | $9'$ | $9$ |
| $\check{w}_t$ | $\emptyset$ | $2$ | $\emptyset$ | $\emptyset$ | $6'$ | $\emptyset$ | $\emptyset$ |
| $\hat{u}_t$ | $\emptyset$ | $\emptyset$ | $5$ | $\emptyset$ | $8$ | $9$ | $-$ |
| $\check{u}_t$ | $3$ | $\emptyset$ | $\emptyset$ | $6$ | $\emptyset$ | $\emptyset$ | $-$ |

Finally we get

$$Y = \boxed{1}\boxed{24'}\boxed{5}\boxed{5}\boxed{6'7}\boxed{9'},\ Y_+ = \{\, 9 \,\},\ u = \{\, 3,5,6,8,9 \,\}.$$

We need to define the bumping algorithm applicable also when $X = L_t(T)$ is a pure column case, i.e. $t > \ell(\lambda)$. However, extension of the algorithm to the column case is straightforward, so we omit detailed description here.

## 4.5 *Insertion of $w$ into arbitrary tableau*

We define a procedure to insert an element $w \in \mathcal{X}$ into an arbitrary tableau $T$. In the procedure, we insert $w$ into the first L part of the tableaux. When some letters are bumped out, we insert them into the second L part of the tableau. Then, while some letters are bumped out, we try to insert them into the next L part of the tableux until no letters are bumped out.

**Algorithm 4.4**

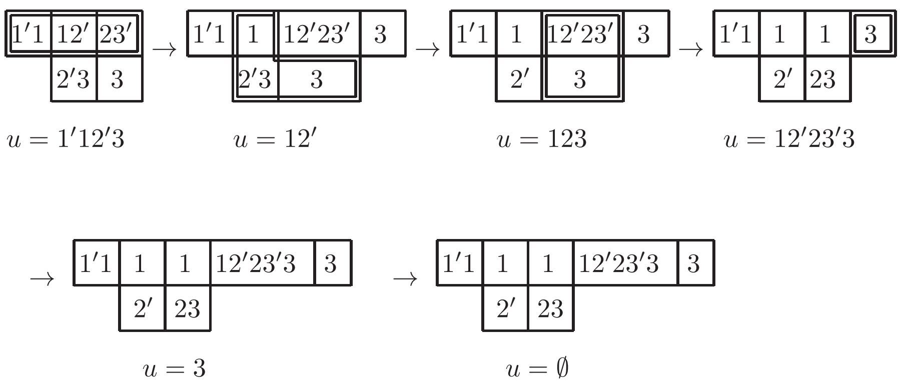
**input** $T \in \mathcal{T}(\lambda)$ and $w \in \mathcal{X}$.

**output** $U$, $S'$, $S$.

**procedure**

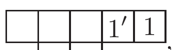
1. Let $u = w$, $U = T$, $S = \emptyset$ and $S' = \emptyset$.
2. While $u \neq \emptyset$, do the following:
   (a) Let $X$ be the $t$-th L part of $U$,
   (b) Let $(Y, Y_+, u)$ be the triple obtained from $(X, u)$ by Algorithm 4.2.
   (c) Let $U$ be the tableaux obtained from $U$ by replacing the $t$-th L part by $Y$.
   (d) If $Y_+ \neq \emptyset$, then do the following:
      i. Add a new box to the end of $t$-th L part of $U$, and insert $Y_+$ into the box.
      ii. If $X$ is a *pure column*, then add the new box to $S$, else add the new box to $S'$.
3. Return $U$, $S'$ and $S$.

**Example 4.5** Let $T$ be the leftmost tableau below. We insert $w = \{ 1', 1, 2', 3 \}$ into $T$ as follows.




For each step, the relevant part of modification is enclosed.

Sets $S'$ and $S$ are as follows:




where the box with entry $1'$ (resp. $1$) is $S'$ (resp. $S$).

### 4.6 Definition of the map $\phi$

In order to complete the description of the map $\phi$, we need one more combinatorial idea. Let $\theta$ be a 1-admissible strip. We define an involution $\varrho : \mathcal{C}(\theta) \to \mathcal{C}(\theta)$. A box $\alpha \in \theta$ is said to be *isolated* if $\alpha$ is not a diagonal box and there is no other box than $\alpha$ in the row and column where $\alpha$ presents. For each isolated box, apply its entry the obvious involution $1 \mapsto 1'$, $1' \mapsto 1$, while the non-isolated boxes are untouched. The resulting decomposition of $\theta$ is obviously admissible. For example, we have

$$\begin{array}{cccccccc}
\square & \square & \square & \square & \square & \square & \square & 1 \\
 & \square & \square & \square & \square & 1' & 1 & \\
 & & \square & \square & 1' & & & \\
 & & & 1 & & & &
\end{array}
\xrightarrow{\varrho}
\begin{array}{cccccccc}
\square & \square & \square & \square & \square & \square & \square & 1' \\
 & \square & \square & \square & \square & 1' & 1 & \\
 & & \square & \square & 1 & & & \\
 & & & 1 & & & &
\end{array}.$$

It is obvious that $\varrho$ is an involution.

**Proposition 4.6** *Let $\lambda \in \mathbb{S}$, $T \in \mathcal{T}(\lambda)$, and $w \in \mathcal{X} = \mathcal{T}(1)$. We have by Algorithm 4.4 a tableau $U = (T \hookleftarrow w) \in \mathcal{T}(\mu)$ for some $\mu \in \mathbb{S}$ such that $\mu \supset \lambda$ and a decomposition $(S', S)$ of $\theta = \mu/\lambda$. We have $(S', S) \in \mathcal{C}(\theta)$, and therefore $\theta$ is a 1-admissible strip.*

Let $T \in \mathcal{T}(\lambda)$ and $w \in \mathcal{X}$ as in the above proposition. We define $\phi(T, w)$ to be $(U, \varrho(S', S)) \in \mathcal{T}(\mu) \times \mathcal{C}(\mu/\lambda)$.

### 4.7 Proof of Prop. 3.2

To show that $\phi$ is a bijection, we construct its inverse map. See [INN] for details.

## 5 Robinson–Schensted type correspondence

### 5.1 Quasi-standard tableaux

We will define a notion of "recording" tableaux in our setting. The resulting object is an analogue of a standard tableau, which we will call a *quasi-standard* tableau.

For $T \in \mathcal{T}(\lambda)$ and $w \in \mathcal{X}$ we denote by $T \hookleftarrow w$ the tableau given in Prop. 3.2. Let $T \in \mathcal{T}(\lambda)$ and $(w_1, \ldots, w_m) \in \mathcal{X}^m$. By the consecutive insertions

$$T^{(i)} = (\cdots((T \hookleftarrow w_1) \hookleftarrow w_2) \cdots \hookleftarrow w_i)$$

we have a tableaux $T^{(i)} \in \mathcal{T}(\nu^{(i)})$ for some shifted diagram $\nu^{(i)}$ and an element of $\mathcal{C}(\nu^{(i)}/\nu^{(i-1)})$ given by Proposition 3.2. Thus we have a nested sequence of shifted diagrams

$$\lambda = \nu^{(0)} \subset \nu^{(1)} \subset \nu^{(2)} \subset \cdots \subset \nu^{(m)} = \mu, \tag{4}$$

and also 1-admissible decompositions $(C_i', C_i)$ of $\theta^{(i)} = \nu^{(i)}/\nu^{(i-1)}$. These objects are expressed as a tableau like

$$\begin{array}{ccccccccccc}
\square & \square & \square & \square & \square & \square & \square & 1' & 1 & 2' & 4 \\
 & \square & \square & \square & 1' & 2' & 2 & 3' & 4' & 4 & \\
 & & \square & 1' & 1 & 3' & 3 & 4' & 4 & & \\
 & & & 1 & 2' & 3 & 4 & & & & \\
 & & & & 2 & & & & & &
\end{array},$$

where the boxes filled with $i$ (resp. $i'$) are $C_i$ (resp. $C'_i$).

We call such a tableau a *quasi-standard* tableau of degree $m$. The precise definition is the following.

**Definition 5.1** *A map $U : \mu/\lambda \longrightarrow \mathcal{B}_m := \{1', 1, \ldots, m', m\}$ is a quasi-standard tableau of degree $m$, if $U$ is semistandard in the sense of Def. 2.1 and for any $1 \le i \le m$, $U^{-1}(\{i, i'\})$ is a 1-admissible strip with admissible decomposition given by $(U^{-1}(i'), U^{-1}(i))$.*

Let $\mathcal{S}_m(\mu/\lambda)$ denote the set of quasi-standard tableaux of degree $m$ on $\mu/\lambda$.

*Remark.* By the construction, $\mathcal{S}_1(\mu/\lambda)$ is non-empty if and only if $\theta = \mu/\lambda$ is an 1-admissible strip. Then we have $\mathcal{S}_1(\theta) = \mathcal{C}(\theta) = \mathcal{C}_1(\theta)$. For an $m$-admissible strip $\theta$, the set $\mathcal{C}_m(\theta)$ is a subset of $\mathcal{S}_m(\theta)$.

## 5.2 Robinson–Schensted correspondence

The following result is an immediate consequence of Prop. 3.2.

**Proposition 5.2** *Let $T \in \mathcal{T}(\lambda)$ and $(w_1, \ldots, w_m) \in \mathcal{X}^m$. By consecutive insertions*

$$T' = (\cdots((T \hookleftarrow w_1) \hookleftarrow w_2) \cdots \hookleftarrow w_m)$$

*we have a tableaux $T' \in \mathcal{T}(\mu)$ for some shifted diagram $\mu \supset \lambda$ and the recording tableau U. Then we have $U \in \mathcal{S}_m(\mu/\lambda)$. By this correspondence we have a weight preserving bijection*

$$\phi_m : \mathcal{T}(\lambda) \times \mathcal{X}^m \longrightarrow \bigsqcup_{\mu} \mathcal{T}(\mu) \times \mathcal{S}_m(\mu/\lambda), \tag{5}$$

*where the sum runs for shifted diagrams $\mu$ such that $\mathcal{S}_m(\mu/\lambda) \neq \emptyset$.*

Then we have immediately the following:

**Corollary 5.3** *We have*

$$GQ_\lambda(x) \cdot GQ_1(x)^m = \sum_{\mu} \beta^{|\mu/\lambda| - m} \# \mathcal{S}_m(\mu/\lambda) \times GQ_\mu(x),$$

*where the sum runs for shifted diagrams $\mu$ such that $\mathcal{S}_m(\mu/\lambda) \neq \emptyset$.*

As a special case of $\lambda = \emptyset$, we have the following.

**Corollary 5.4 (Robinson–Schensted correspondence)** *There is a weight preserving bijection*

$$\mathcal{X}^m \longrightarrow \bigsqcup_{\lambda} \mathcal{T}(\lambda) \times \mathcal{S}_m(\lambda).$$

This bijection is a set-valued extension of the results in [Sa] and [Wo].

**Example 5.5** Let $(w_1, w_2, w_3) = (2'3, 12'2, 134)$. By the correspondence in Cor. 5.3 we have pair of tableaux

$$\left( \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 34 \\ \hline \multicolumn{1}{c|}{} & 23' & \multicolumn{3}{c}{} \\ \cline{2-2} \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2' & 2 & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{3}{c}{} \\ \cline{2-2} \end{array} \right),$$

as a result of bumping process:

$$\emptyset \overset{w_1}{\rightsquigarrow} \begin{array}{|c|} \hline 23' \\ \hline \end{array} \overset{w_2}{\rightsquigarrow} \begin{array}{|c|c|c|} \hline 12 & 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 3' & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \overset{w_3}{\rightsquigarrow} \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 34 \\ \hline \multicolumn{1}{c|}{} & 23' & \multicolumn{3}{c}{} \\ \cline{2-2} \end{array}.$$

# 6 Outline of proof of Thm 3.4

Now we have the bijection $\phi_m$ in Prop. 5.2. Since a tableau in $\mathcal{T}(m)$ is a sequence in $\mathcal{X}$ such that

$$X_1 \leq_r \cdots \leq_r X_m,$$

we can think of $\mathcal{T}(m)$ as a subset of $\mathcal{X}^m$. Thus we only need to determine the image of $\mathcal{T}(\lambda) \times \mathcal{T}(m)$ under the map $\phi_m$. The case $m = 1$ is obvious since $\mathcal{T}(1) = \mathcal{X}$. The case $m = 2$ is crucial.

**Lemma 6.1** *Let $T \in \mathcal{T}(\lambda)$ and $w = (w_1, w_2) \in \mathcal{X}^2$, and*

$$\phi_2(T, w) = (T', (C_1', C_1), (C_2', C_2)).$$

*Then the following are equivalent:*

1. $w_1 \leq_r w_2$.
2. $(C_1', C_1) \lhd (C_2', C_2)$.

It is easy to see that the lemma leads to a proof of Thm 3.4. We show this lemma by an argument using "bumping routes". Details are given in [INN].

# References

[Bu] A. S. Buch, A Littlewood-Richardson rule for $K$-theory of Grassmannians, Acta. Math. **189** (2002), 37-78.

[BR] A. S. Buch, V. Ravikumar, Pieri rules for the $K$-theory of cominuscule Grassmannians, arXiv:1005.2605.

[CTY] E. Clifford, H. Thomas, A. Yong, $K$-theoretic Schubert calculus for $OG(n, 2n + 1)$ and jeu de taquin for shifted increasing tableaux, arXiv:1002.1664v2

[IN] T. Ikeda, H. Naruse, $K$-theoretic analogue of Schur $Q$-functions and maximal isotropic Grassmannians, in preparation.

[INN] T. Ikeda, H. Naruse, Y. Numata, Bumping algorithm for shifted set-valued tableaux, in preparation.

[Sa] B. E. Sagan, Shifted tableaux, Schur $Q$-functions, and a conjecture of R. Stanley. J. Comb. Theory, Ser. A 45(1), 62-103, 1987.

[Wo] D. Worley, A theory of shifted Young tableaux, Ph. D. Thesis, Massachusetts Institute of Technology, 1984.