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Closed paths whose steps are roots of unity

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Abstract. We give explicit formulas for the number $U_n(N)$ of closed polygonal paths of length N (starting from the origin) whose steps are n^{th} roots of unity, as well as asymptotic expressions for these numbers when $N \to \infty$. We also prove that the sequences $(U_n(N))_{N\geq 0}$ are P-recursive for each fixed $n \geq 1$ and leave open the problem of determining the values of N for which the *dual* sequences $(U_n(N))_{n\geq 1}$ are P-recursive.

Résumé. Nous donnons des formules explicites pour le nombre $U_n(N)$ de chemins polygonaux fermés de longueur N (débutant à l'origine) dont les pas sont des racines n-ièmes de l'unité, ainsi que des expressions asymptotiques pour ces nombres lorsque $N \to \infty$. Nous démontrons aussi que les suites $(U_n(N))_{N\geq 0}$ sont P-récursives pour chaque $n \geq 1$ fixé et laissons ouvert le problème de déterminer les valeurs de N pour lesquelles les suites *duales* $(U_n(N))_{n\geq 1}$ sont P-récursives.

Keywords: closed polygonal paths, roots of unity, P-recursive, asymptotics

1 Introduction

The subject of random walks is classical and appears in many areas of mathematics, physics and computer science (see, for example, $http://en.wikipedia.org/wiki/Random\_walks$). In this paper we combinatorially analyse a new type of closed random walks in the complex plane — a kind of restricted Brownian motion — whose steps are given by n^{th}-roots of unity. For $n \geq 1$, let $\Omega_n = \{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$ be the set of all n-th roots of unity, where $\omega_n = \exp(2\pi i/n) \in \mathbb{C}$. A polygonal path of length N, starting at the origin in the complex plane, whose steps are n-th roots of unity can be encoded by the sequence $w = [\omega_n^{k_1}, \ldots, \omega_n^{k_N}]$ of its successive steps, $\omega_n^{k_j} \in \Omega_n, j = 1, \ldots, N$. For $\nu = 0, \ldots, n-1$, let m_ν be the number of times that ω_n^ν appears in w. We call the sequence $\vec{m} = [m_0, \ldots, m_{n-1}]$ the *type* of w, and write $\vec{m} = \mathrm{type}(w)$. Of course, the path w is closed if and only if $\omega_n^{k_1} + \cdots + \omega_n^{k_N} = 0$ if and only if

$$m_0 + m_1\omega_n + m_2\omega_n^2 + \cdots + m_{n-1}\omega_n^{n-1} = 0. \tag{1.1}$$

We call a sequence $\vec{m} = [m_0, m_1, \ldots, m_{n-1}] \in \mathbb{N}^n$ *admissible* if (1.1) is satisfied. Figure 1 shows a closed pentagon made of 18-th roots of unity encoded by $[\omega_{18}^3, \omega_{18}^{11}, \omega_{18}^5, \omega_{18}^{12}, \omega_{18}^{17}]$ and a closed 11-gon made of 14-th roots of unity encoded by $[\omega_{14}^{12}, \omega_{14}, \omega_{14}^4, \omega_{14}^5, \omega_{14}^7, \omega_{14}^5\omega_{14}^{11}, \omega_{14}^{11}, \omega_{14}^9, \omega_{14}^3, \omega_{14}^{13}]$.

Clearly, the number of closed paths, of length N, with admissible type $\vec{m}$ is given by the multinomial coefficient $N!/m_0!m_1!\ldots m_{n-1}!$. This implies that the number $U_n(N)$ of closed polygonal paths of

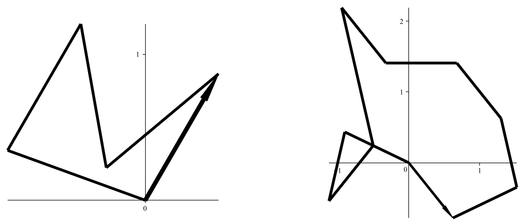

Fig. 1: Pentagon and 11-gon made of 18-th and 14-th roots of unity.

length N whose steps are n-th roots of unity is given by the formula

$$U_n(N) = \sum_{\substack{\vec{m}\,:\,\text{admissible}\\ m_0+\cdots+m_{n-1}=N}} \frac{N!}{m_0!m_1!\cdots m_{n-1}!}. \tag{1.2}$$

In Section 2, we characterize admissibility and express the numbers $U_n(N)$ as *constant term* extractions in suitable rational expressions. We also give a formula from which the computation of the numbers $U_n(N)$ can be reduced to the computation of the numbers $U_q(N')$, where $N' \leq N$ and q is a suitable divisor of n. Section 3 is devoted to an analysis of recursive and asymptotic properties of the numbers $U_n(N)$. Finally, some tables are given.

2 Constant term and reduction formulas

To take advantage of formula (1.2) for $U_n(N)$ on a symbolic algebra system, we state first a simple characterization of admissibility for a sequence $\vec{m} \in \mathbb{N}^n$. This is done using the classical cyclotomic polynomials $\Phi_n(z) = \prod(z-\omega)$, where ω runs through the primitive n-th roots of unity. Equivalently, this means that $\omega = \exp(2k\pi i/n)$, where $1 \leq k \leq n$ and $\mathrm{GCD}(n,k) = 1$. Since $z^n - 1 = \prod_{d|n} \Phi_d(z)$, Moebius inversion implies that $\Phi_n(z) = \prod_{d|n}(x^d - 1)^{\mu(n/d)}$, where μ denotes the Moebius function. This shows that $\Phi_n(z)$ is a monic polynomial in $\mathbb{Z}[z]$ of degree $\varphi(n)$, the Euler function of n. The following very easy, but basic lemma characterizes admissibility.

Lemma 2.1 (criteria for admissibility). *For $n \geq 1$, the sequence $\vec{m} = [m_0, \ldots, m_{n-1}] \in \mathbb{N}^n$ is admissible if and only if the cyclotomic polynomial $\Phi_n(z)$ divides the polynomial*

$$P_{\vec{m}}(z) = m_0 + m_1 z + \cdots + m_{n-1} z^{n-1}.$$

Proof: Consider the euclidean division of $P_{\vec{m}}(z)$ by $\Phi_n(z)$ in the ring $\mathbb{Z}[z]$:

$$P_{\vec{m}}(z) = \Phi_n(z) Q_{\vec{m}}(z) + R_{\vec{m}}(z), \tag{2.1}$$

where $\deg R_{\vec{m}}(z) < \deg \Phi_n(z) = \varphi(n)$. Since $\Phi_n(\omega_n) = 0$ this shows that $\vec{m}$ is admissible if and only if $P_{\vec{m}}(\omega_n) = 0$ if and only if $R_{\vec{m}}(\omega_n) = 0$. But $R_{\vec{m}}(\omega_n) = 0$ if and only if $R_{\vec{m}}(z) = 0$ identically since $\Phi_n(z)$ is known to be the minimal polynomial of any of its roots and $\deg R_{\vec{m}} < \deg \Phi_n$. $\square$

Euclidean division shows that the coefficients of $R_{\vec{m}}(z)$ are $\mathbb{Z}$-linear combinations $l_k(m_0, \ldots, m_{n-1})$ of the m_i's. Hence, $\vec{m}$ is admissible if and only if $l_k(m_0, \ldots, m_{n-1}) = 0$ for $k = 0, \ldots, \varphi(n) - 1$. Table 1, made using the *rem* command in Maple gives the values of the l_k's for $n = 1, \ldots, 20$. For example, for $n = 6$, $\varphi(n) = 2$ and using Table 1, formula (1.2) takes the form

$$U_6(N) = \sum_{\substack{m_0+\cdots+m_5=N \\ m_0+m_5=m_2+m_3 \\ m_4+m_5=m_1+m_2}} \frac{N!}{m_0! \cdots m_5!}.$$

Note that, by the multinomial formula, this is equivalent to the following *constant term* formula

$$U_6(N) = \mathrm{CT}((t_1 + t_2 + \frac{t_1}{t_2} + \frac{t_2}{t_1} + t_1^{-1} + t_2^{-1})^N),$$

where $\mathrm{CT}(L(t_1, t_2, \ldots))$ denotes the constant term of the full expansion of L as a Laurent series in $t_1, t_2, \ldots$. This is generalized as follows.

Theorem 2.2 *There is a Laurent polynomial, $\Lambda_n(t_1, \ldots, t_{\varphi(n)})$, such that $U_n(N) = \mathrm{CT}(\Lambda_n(t_1, \ldots, t_{\varphi(n)})^N)$. Moreover, $\Lambda_n(t_1, \ldots, t_{\varphi(n)})$ is computed as follows. Let $m_0 + \cdots + m_{n-1}z^{n-1} = \Phi_n(z)Q(z) + R(z)$, where the remainder is $R(z) = \sum_{k=0}^{\varphi(n)-1} l_k(m_0, \ldots, m_{n-1})z^k$, with $l_k(m_0, \ldots, m_{n-1}) = \sum_{i=0}^{n-1} c_{k,i}m_i$, $c_{k,i} \in \mathbb{Z}$, $k = 0, \ldots, \varphi(n) - 1$. Then,*

$$\Lambda_n(t_1, \ldots, t_{\varphi(n)}) = \sum_{j=0}^{n-1} t_1^{c_{0,j}} t_2^{c_{1,j}} t_3^{c_{2,j}} \ldots t_{\varphi(n)}^{c_{\varphi(n)-1,j}}. \tag{2.2}$$

Proof: By the multinomial theorem,

$$\begin{aligned}
&\left(\sum_{j=0}^{n-1} t_1^{c_{0,j}} \ldots t_{\varphi(n)}^{c_{\varphi(n)-1,j}}\right)^N \\
&= \sum_{m_0+\cdots+m_{n-1}=N} \frac{N!}{m_0! \cdots m_{n-1}!} \left(t_1^{c_{0,0}} \ldots t_{\varphi(n)}^{c_{\varphi(n)-1,0}}\right)^{m_0} \cdots \left(t_1^{c_{0,n-1}} \ldots t_{\varphi(n)}^{c_{\varphi(n)-1,n-1}}\right)^{m_{n-1}} \\
&= \sum_{m_0+\cdots+m_{n-1}=N} \frac{N!}{m_0! \cdots m_{n-1}!} t_1^{l_0(m_0,\ldots,m_{n-1})} \ldots t_{\varphi(n)}^{l_{\varphi(n)-1}(m_0,\ldots,m_{n-1})}.
\end{aligned}$$

The result follows since the constant term is given by taking the sum of the terms corresponding to the exponents $l_k = 0$ for $k = 0, \ldots, \varphi(n) - 1$. $\square$

Table 2 gives the rational functions $\Lambda_n(t_1, \ldots, t_{\varphi(n)})$ for $n = 1, \ldots, 20$. Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be the canonical decomposition of the integer n. By definition, the *radical* of n is the square-free integer $q = \mathrm{rad}(n) = p_1 \cdots p_s$ consisting of the product of the p_i's. The computation of the cyclotomic polynomial $\Phi_n(z)$ is greatly simplified by making use of the well-known reduction formula

$$\Phi_n(z) = \Phi_q(z^{n/q}), \quad q = \mathrm{rad}(n). \tag{2.3}$$

This implies that the computation of the exponential generating function of the sequence $(U_n(N))_{N\geq 0}$ is reduced to that of $(U_q(N))_{N\geq 0}$ as follows.

Proposition 2.3 (reduction formula for $U_n(N)$). *Let* $n \geq 1$ *and* $q = \mathrm{rad}(n)$. *Then,*

$$\sum_{N\geq 0} U_n(N)\frac{X^N}{N!} = \left(\sum_{N\geq 0} U_q(N)\frac{X^N}{N!}\right)^{n/q}. \tag{2.4}$$

Proof: Using the remainder function, we have by linearity,

$$R_{\vec{m}}(z) = \mathrm{rem}(P_{\vec{m}}(z), \Phi_n(z)) = \sum_{k=0}^{n-1} m_k \mathrm{rem}(z^k, \Phi_n(z)). \tag{2.5}$$

Now, for $0 \leq \nu \leq q-1$, consider the euclidean division

$$z^\nu = \Phi_q(z)Q_\nu(z) + \rho_\nu(z), \tag{2.6}$$

where $\rho_\nu(z) = \mathrm{rem}(z^\nu, \Phi_q(z))$. The substitution $z \to z^{n/q}$ in (2.6) followed by a multiplication by z^r gives, using (2.3), $z^{\nu n/q+r} = \Phi_q(z^{n/q})z^r Q_\nu(z^{n/q}) + z^r\rho_\nu(z^{n/q}) = \Phi_n(z)z^r Q_\nu(z^{n/q}) + z^r\rho_\nu(z^{n/q})$. Let $k = \nu n/q + r$, where $0 \leq r < n/q$. Then,

$$\deg z^r\rho_\nu(z^{n/q}) = r + \frac{n}{q}\deg\rho_\nu(z) \leq r + \frac{n}{q}(\varphi(q)-1) = r + \varphi(n) - \frac{n}{q} < \varphi(n).$$

This implies that $\mathrm{rem}(z^k, \Phi_n(z)) = z^r\rho_\nu(z^{n/q})$. Substituting this into (2.5) and collecting terms, we find that the $\varphi(n)$ conditions for admissibility, $[l_k(m_0, m_1, \ldots, m_{n-1}) = 0]_{0\leq k\leq\varphi(n)-1}$, split into n/q blocks of $\varphi(q)$ conditions, $[l_i(m_j, m_{\frac{n}{q}+j}, m_{2\frac{n}{q}+j}, \ldots, m_{(q-1)\frac{n}{q}+j}) = 0]_{0\leq i\leq\varphi(q)-1}$, $0 \leq j \leq \frac{n}{q}-1$, from which (2.4) follows. □

Table 3 gives the numerical values of $U_n(N)$ for $1 \leq n \leq 20$ and $0 \leq N \leq 20$.

3 Analysis of the sequences

Let us say that a path is *normalized* if its first step is the complex number 1 (i.e. the path starts *horizontally* along the positive real axis). Each normalized path $[1, \omega_n^{\nu_2}, \ldots, \omega_n^{\nu_N}]$ generates, by rotation, n distinct paths $\omega_n^k[1, \omega_n^{\nu_2}, \ldots, \omega_n^{\nu_N}] = [\omega_n^k, \omega_n^{k+\nu_2}, \ldots, \omega_n^{k+\nu_N}]$, $k = 0, 1, \ldots, n-1$. This implies that n divides $U_n(N)$ for every $n \geq 1$ and $N \geq 1$. As Tables 1 and 2 indicate, the structure of the sequence $(U_n(N))_{N\geq 0}$ heavily depend on the arithmetical nature of n. For example, let $n = p$ be a prime number. Then for such values of n, admissibility for a vector $\vec{m} \in \mathbb{N}^p$ means that $m_0 = m_1 = \cdots = m_{p-1}$ since, in this case, $\Phi_p(z) = 1 + z + \cdots + z^{p-1}$ and $R_{\vec{m}}(z) = (m_0 - m_{p-1}) + (m_1 - m_{p-1})z + \cdots + (m_{p-2} - m_{p-1})z^{p-2}$, (see Table 1, for example). Formula (1.2) then takes the form

$$U_p(N) = \frac{N!}{(\frac{N}{p})!^p} \quad \text{if } p|N, \ 0 \text{ otherwise}. \tag{3.1}$$

Note that when $p = 2$, (3.1) corresponds to the classical central binomial coefficients enumerating one-dimensional closed lattice paths of length N. When $p = 3$, (3.1) corresponds to the De Bruijn numbers (sequence A006480 in Sloane-Plouffe encyclopedia [Sloane(2010)]). For prime powers $n = p^\alpha$, we have by Proposition 2.3,

$$\sum_{N\geq 0} U_{p^\alpha}(N)\frac{X^N}{N!} = \left(\sum_{k\geq 0}\frac{X^{kp}}{k!^p}\right)^{p^{\alpha-1}} \tag{3.2}$$

since, in this case $q = p$. Note that when $n = 8 = 2^3$, then $U_8(N)$ is the number of 4-dimensional closed lattice paths in $\mathbb{Z}^4$ of length N starting at the origin (see sequence A039699 in Sloane). The reader can check that, more generally, $U_{2^\alpha}(N)$ is the number of closed lattice paths in $\mathbb{Z}^{2^{\alpha-1}}$ of length N starting at the origin. Interestingly enough, for any other dimension $d \neq 2^{\alpha-1}$, such a connection betweens lattice paths in $\mathbb{Z}^d$ and plane paths whose steps are roots of unity does not exist.

When n is not a prime power, the situation is more delicate. For example, if $n = 6$, then, using the Maple package GFUN [Salvy and Zimmermann(1994)], it can be seen that $(U_n(N))_{N\geq 0}$ satisfies the following linear recurrence with polynomial coefficients,

$$(N+3)^2U_6(N+3) = (N+2)(N+3)U_6(N+2)+24(N+2)^2U_6(N+1)+36(N+1)(N+2)U_6(N) \tag{3.3}$$

with initial conditions $U_6(0) = 1, U_6(1) = 0, U_6(2) = 6$. Such sequences are called polynomially recursive (P-recursive for short) and are characterized by the fact that their (ordinary or exponential) generating series are D-finite (i.e. satisfy a linear differential equation with polynomial coefficients). As a consequence, P-recursive sequences are closed under many operations including linear combinations, pointwise and Cauchy products [Stanley(1980)]. Moreover their asymptotic estimates, as $N \to \infty$, are well behaved. In our context, the general situation is summarized by Theorem 3.2. below. We need first the following technical lemma.

Lemma 3.1 *Let $\vec{t} = (t_1, \ldots, t_{\varphi(n)}) \in \mathbb{C}^{\varphi(n)}$. Then the Laurent polynomial Λ_n satisfies*

$$\max_{\substack{|t_\nu|=1\\ 1\leq\nu\leq\varphi(n)}} |\Lambda_n(\vec{t})| = n. \tag{3.4}$$

Moreover, if $n = p^\alpha$, a prime power, then the maximum value (3.4) *is attained precisely at the p distinct points $(e^{2\pi i\nu/p}, \ldots, e^{2\pi i\nu/p})$, $\nu = 0, \ldots, p-1$ and we have $\Lambda_n(e^{2\pi i\nu/p}, \ldots, e^{2\pi i\nu/p}) = ne^{2\pi i\nu/p}$. If n is not a prime power, then the maximum value* (3.4) *is attained only at the point $(1, \ldots, 1)$ and we have $\Lambda_n(1, \ldots, 1) = n$.*

Proof: By Theorem 2.2, Λ_n can be written as a sum of n terms,

$$\Lambda_n(\vec{t}) = t_1 + \cdots + t_{\varphi(n)} + \Gamma_n(\vec{t}), \tag{3.5}$$

where Γ_n is a sum of $n - \varphi(n)$ unitary Laurent monomials in $t_1, \ldots, t_{\varphi(n)}$. Each of the n terms in Λ_n has modulus 1 when $|t_\nu| = 1$, $\nu = 1, \ldots, \varphi(n)$. Hence (3.4) follows from the triangular inequality and the fact that $\Lambda_n(1, \ldots, 1) = n$. Note that the maximum value in (3.4) is attained only at points $\vec{t^*} = (t_1^*, \ldots, t_{\varphi(n)}^*)$ for which the n monomials take a common value, $e^{i\theta^*}$, say. In particular, from (3.5), we must have $t_1^* = t_2^* = \cdots = t_{\varphi(n)}^* = e^{i\theta^*}$. We consider two cases:

(i) if $n = p^\alpha$, then it can be checked that each term in Γ_n has total degree $-(p-1)$. This implies that $e^{i\theta^*} = e^{-i(p-1)\theta^*}$. That is, $e^{i\theta^*}$ is a p-th root of unity: $e^{2\pi i\nu/p}$, $\nu = 0, \ldots, p-1$;

(ii) if $n \neq p^\alpha$, the situation is more delicate. If we can show that at least one of the terms in Γ_n has total degree 0, then the maximal value in (3.4) will be attained only at the point $(1,\ldots,1)$, since this would imply that $e^{i\theta^*} = (e^{i\theta^*})^0 = 1$. The existence of such a 0-degree term is proved as follows. By (2.2), the general term $t_1^{c_{0,j}} t_2^{c_{1,j}} \cdots t_{\varphi(n)}^{c_{\varphi(n)-1,j}}$ has total degree $\sum_{k=0}^{\varphi(n)-1} c_{k,j}$. When $j = \varphi(n)$, this total degree is 0. To see this, note that $\sum_{k=0}^{\varphi(n)-1} c_{k,j} z^k = \mathrm{rem}(z^j, \Phi_n(z))$. Taking $j = \varphi(n)$, $z = 1$, this gives $\sum_{k=0}^{\varphi(n)-1} c_{k,\varphi(n)} = \mathrm{rem}(z^{\varphi(n)}, \Phi_n(z))|_{z=1} = (z^{\varphi(n)} - \Phi_n(z))|_{z=1} = 0$, since $\Phi_n(1) = 1$ when $n \neq p^\alpha$.

□

Theorem 3.2 *For any $n > 1$, we have an asymptotic estimate of the form*

$$U_n(N) \sim a_n \frac{n^N}{N^{\frac{1}{2}\varphi(n)}} \left(1 + \frac{b_{1,n}}{N} + \frac{b_{2,n}}{N^2} + \ldots\right), \quad \text{as } N \to \infty, \tag{3.6}$$

where a_n, $b_{j,n}$ are independent of N. When $n = p^\alpha$ is a prime power, then N must be a multiple of p as it goes to infinity in (3.6). *More explicitly, the leading coefficient a_n is given by*

$$a_n = \begin{cases} (n/2\pi)^{\frac{1}{2}\varphi(n)} / \sqrt{\prod_{p|n} p^{\varphi(n)/(p-1)}} & \text{if } n \text{ is not a prime power,} \\ p \cdot (n/2\pi)^{\frac{1}{2}\varphi(n)} / \sqrt{\prod_{p|n} p^{\varphi(n)/(p-1)}} & \text{if } n = p^\alpha \text{ is a prime power.} \end{cases}$$

For each $n \geq 1$, the sequence $(U_n(N))_{N \geq 0}$ is P-recursive but is not algebraic when $n > 2$.

Proof: In order to establish the asymptotic estimate (3.6), first note that the constant term extraction $U_n(N) = \mathrm{CT}(\Lambda_n(t_1,\ldots,t_{\varphi(n)})^N)$ can be expressed as the multiple integral

$$U_n(N) = \frac{1}{(2\pi)^{\varphi(n)}} \int \cdots \int_{(-\pi,\pi]^{\varphi(n)}} \Lambda_n(e^{iu_1},\ldots,e^{iu_{\varphi(n)}})^N du_1 \cdots du_{\varphi(n)} \tag{3.7}$$

which is the average value of Λ_n^N over the $\varphi(n)$-dimensional torus $\{(t_1,\ldots,t_{\varphi(n)}) \in \mathbb{C}^{\varphi(n)} \,|\, |t_\nu| = 1, \nu = 1,\ldots,\varphi(n)\}$. Now by Theorem 2.2,

$$L_n(\vec{u}) := \Lambda_n(e^{iu_1},\ldots,e^{iu_{\varphi(n)}}) = \sum_{j=0}^{n-1} e^{i\lambda_j(\vec{u})}, \tag{3.8}$$

where $\lambda_j(\vec{u}) = \sum_{k=0}^{\varphi(n)-1} c_{k,j} u_{k+1}$ is a real-valued linear combination of $u_1,\ldots,u_k$, $0 \leq j \leq \varphi(n)-1$. By the triangular inequality, $|L_n(\vec{u})| \leq n$ for every $\vec{u} \in (-\pi,\pi]^{\varphi(n)}$. To obtain the asymptotic estimate of (3.6) it suffices to approximate (3.7) by a gaussian distribution around each point $\vec{u^*} = (u_1^*,\ldots,u_{\varphi(n)}^*) \in (-\pi,\pi]^{\varphi(n)}$ for which the maximum value $|L_n(\vec{u^*})| = |ne^{i\theta^*}| = n$ is attained. This is Laplace's method [De Bruijn(1981)]. By Lemma 3.1,

(i) if $n \neq p^\alpha$, then $\vec{u^*} = \vec{0}$ is the only point in $(-\pi,\pi]^{\varphi(n)}$ for which $|L_n(\vec{u^*})| = n$. In fact $\theta^* = 0$;

(ii) if $n = p^\alpha$, then there are exactly p possible values of u^* for which $|L_n(\vec{u^*})| = n$. In fact $\theta^* = 2\nu\pi/p \mod 2\pi \in (-\pi,\pi], \nu = 0,\ldots,p-1$.

We conclude by estimating (3.7) by a sum of moments of gaussian distributions in the following way:

$$U_n(N) \sim \frac{n^N}{(2\pi)^{\varphi(n)}} \sum_{L_n(\vec{u^*})=ne^{i\theta^*}} e^{iN\theta^*} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{N}{2n}Q^*(\vec{u}-\vec{u^*})} H^*(\vec{u}-\vec{u^*})^N du_1 \dots du_{\varphi(n)},$$

where, for each $\vec{u^*}$ such that $L_n(\vec{u^*}) = ne^{i\theta^*}$,

$$\frac{1}{n}L_n(\vec{u}) = e^{i\theta^*}\left(1 - \frac{1}{2n}Q^*(\vec{u}-\vec{u^*}) + O(\| \vec{u}-\vec{u^*} \|^3)\right) = e^{i\theta^*}e^{-\frac{1}{2n}Q^*(\vec{u}-\vec{u^*})}H^*(\vec{u}-\vec{u^*}), \quad (3.9)$$

where $Q^*(\vec{v})$ is the positive definite quadratic form associated to the symmetric $\varphi(n) \times \varphi(n)$ matrix $K = CC^T$ in which $C = [c_{k,j}]_{0\leq k\leq\varphi(n)-1, 0\leq j\leq n-1}$, where the $c_{k,j}$'s are defined by (2.2) and $H^*(\vec{v}) = 1 + O(\| \vec{v} \|^3)$. It turns out that $\det(K) = \prod_{p|n} p^{\varphi(n)/(p-1)}$, which is a consequence of the known fact that the absolute value of the discriminant of $\Phi_n(z)$ is equal to $n^{\varphi(n)} \prod_{p|n} p^{\varphi(n)/(p-1)}$, for $n > 2$.

The P-recursivity of $(U_n(N))_{N\geq 0}$ is established as follows. Fix $n \geq 1$ and let $k = \varphi(n)$. We shall show that the series

$$\sum_{N\geq 0} U_n(N)X^N = \mathrm{CT}_{t_1,\dots,t_k} \frac{1}{1 - X\Lambda_n(t_1,\dots,t_k)} \quad (3.10)$$

is D-finite in X where $\mathrm{CT}_{t_1,\dots,t_k}$ means constant term extraction relative to the variables $t_1,\dots,t_k$. First, fix integers $m_1 > 0,\dots,m_k > 0$ in such a way that $t_1^{m_1}\dots t_k^{m_k}\Lambda_n(t_1,\dots,t_k)$ is a polynomial in $t_1,\dots,t_k$. The rational function

$$f(t_1,\dots,t_k,X) = \frac{1}{1 - t_1^{m_1}\dots t_k^{m_k}X\Lambda_n(t_1,\dots,t_k)} = \sum_{n_1,\dots,n_k,N\geq 0} a(n_1,\dots,n_k,N)t_1^{n_1}\dots t_k^{n_k}X^N \quad (3.11)$$

is obviously D-finite in the variables $t_1,\dots,t_k,X$. By Theorem 2.2, the numbers $U_n(N)$ can be expressed as the following coefficient extraction in $f(t_1,\dots,t_k,X)$:

$$U_n(N) = [t_1^{m_1N}\dots t_k^{m_kN}X^N]f(t_1,\dots,t_k,X).$$

Hence, by (3.10),

$$\sum_{N\geq 0} U_n(N)X^N = \sum_{N\geq 0} a(m_1N,\dots,m_kN,N)X^N. \quad (3.12)$$

Consider now the algebraic, hence D-finite, series

$$g(t_1,\dots,t_k,X) = \sum_{n_1,\dots,n_k,N\geq 0} b(n_1,\dots,n_k,N)t_1^{n_1}\dots t_k^{n_k}X^N,$$

where $b(n_1,\dots,n_k,N) = a(m_1n_1,\dots,m_kn_k,N)$. Formula (3.12) shows that

$$\sum_{N\geq 0} U_n(N)X^N = \sum_{N\geq 0} b(N,\dots,N,N)X^N$$

which is a (full) diagonal of $g(t_1,\dots,t_k,X)$. We conclude using the fact that any diagonal of a D-finite series is also D-finite, a result due to Lipshitz [Lipshitz(1988)]. The non algebraicity of $(U_n(N))_{N\geq 0}$, for each $n > 2$, follows from the fact that $\varphi(n)$ is even and the dominant term of the asymptotic formula contains $N^{-\text{positive integer}}$. This is incoherent with Puiseux expansion around an algebraic singularity. $\square$

A better control of the coefficients $b_{j,n}$ can be achieved by a smooth local change of variables, $\vec{u} = \vec{u^*} + \vec{g}(\vec{w})$, $\vec{g}(\vec{0}) = \vec{0}$ in (3.9) such that $\frac{1}{n} L_n(\vec{u}) = e^{i\theta^*} e^{-\frac{1}{2n} Q^*(\vec{w})}$. This is always possible by Morse Lemma [Morse(1925)]. The first terms of the asymptotic estimates of Theorem 3.2 are given in Table 4 for $n = 1, \ldots, 20$.

Corollary 3.3 *If n is not a prime power, then $\exists N_0 = N_0(n)$ such that $U_n(N) > 0$ for $N \geq N_0$.* $\square$

The sequences $(U_n(N))_{N \geq 0}$, $n = 1, 2, \ldots$, can be considered in a *dual* way: for each fixed N, one can consider the sequence $(U_n(N))_{n \geq 1}$ by reading each column of Table 3. The first five of these dual sequences, $(U_n(0))_{n \geq 1}, (U_n(1))_{n \geq 1}, \ldots, (U_n(4))_{n \geq 1}$, are P-recursive. The fifth one, $(U_n(4))_{n \geq 1}$, can be described as follows: $U_n(4) = 3n(n-1)\chi(2|n)$, where $\chi(T(n)) = 1$ if $T(n)$ is true and 0 otherwise. This can be checked by noting that closed paths of length 4 whose steps are n^{th} roots of unity are (possibly degenerated and non-convex) rhombuses. Following extensive computations we conjecture that $(U_n(5))_{n \geq 1}$ is also P-recursive and is of the form $U_n(5) = 24n\chi(5|n) + 20n(n-3)\chi(6|n)$. We leave open the problem of determining the values of N for which $(U_n(N))_{n \geq 1}$ is P-recursive.

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References

[De Bruijn(1981)] N. G. De Bruijn. *Asymptotic methods in analysis.* New York Dover Publications, 1981. ISBN 0486642216.

[Lipshitz(1988)] L. Lipshitz. The diagonal of a D-finite power series is D-finite. *Journal of Algebra*, 113: 373–378, 1988.

[Morse(1925)] M. Morse. Relations between the critical points of a real function of n independent variables. *Trans. Amer. Math. Soc.*, 27:345–396, 1925.

[Salvy and Zimmermann(1994)] B. Salvy and P. Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994.

[Sloane(2010)] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences.* Enlarged edition, 2010. Published electronically at http://oeis.org/classic/index.html.

[Stanley(1980)] R. P. Stanley. Differentiably finite power series. *Europ. J. Combinatorics*, 1:175–188, 1980.

n	*Linear combinations for admissibility*
1	$[m_0]$
2	$[m_0 - m_1]$
3	$[m_0 - m_2, m_1 - m_2]$
4	$[m_0 - m_2, m_1 - m_3]$
5	$[m_0 - m_4, m_1 - m_4, m_2 - m_4, m_3 - m_4]$
6	$[m_0 - m_2 - m_3 + m_5, m_1 + m_2 - m_4 - m_5]$
7	$[m_0 - m_6, m_1 - m_6, m_2 - m_6, m_3 - m_6, m_4 - m_6, m_5 - m_6]$
8	$[m_0 - m_4, m_1 - m_5, m_2 - m_6, m_3 - m_7]$
9	$[m_0 - m_6, m_1 - m_7, m_2 - m_8, m_3 - m_6, m_4 - m_7, m_5 - m_8]$
10	$[m_0 - m_4 - m_5 + m_9, m_1 + m_4 - m_6 - m_9, m_2 - m_4 - m_7 + m_9, m_3 + m_4 - m_8 - m_9]$
11	$[m_0 - m_{10}, m_1 - m_{10}, m_2 - m_{10}, m_3 - m_{10}, m_4 - m_{10}, m_5 - m_{10}, m_6 - m_{10}, m_7 - m_{10},$ $m_8 - m_{10}, m_9 - m_{10}]$
12	$[m_0 - m_4 - m_6 + m_{10}, m_1 - m_5 - m_7 + m_{11}, m_2 + m_4 - m_8 - m_{10}, m_3 + m_5 - m_9 - m_{11}]$
13	$[m_0 - m_{12}, m_1 - m_{12}, m_2 - m_{12}, m_3 - m_{12}, m_4 - m_{12}, m_5 - m_{12}, m_6 - m_{12}, m_7 - m_{12},$ $m_8 - m_{12}, m_9 - m_{12}, m_{10} - m_{12}, m_{11} - m_{12}]$
14	$[m_0 - m_6 - m_7 + m_{13}, m_1 + m_6 - m_8 - m_{13}, m_2 - m_6 - m_9 + m_{13}, m_3 + m_6 - m_{10} - m_{13},$ $m_4 - m_6 - m_{11} + m_{13}, m_5 + m_6 - m_{12} - m_{13}]$
15	$[m_0 - m_8 - m_9 - m_{10} + m_{13} + m_{14}, m_1 + m_8 - m_{11} - m_{13}, m_2 + m_9 - m_{12} - m_{14},$ $m_3 - m_8 - m_9 + m_{14}, m_4 + m_8 - m_{13} - m_{14}, m_5 - m_8 - m_{10} + m_{13},$ $m_6 - m_9 - m_{11} + m_{14}, m_7 + m_8 + m_9 - m_{12} - m_{13} - m_{14}]$
16	$[m_0 - m_8, m_1 - m_9, m_2 - m_{10}, m_3 - m_{11}, m_4 - m_{12}, m_5 - m_{13}, m_6 - m_{14}, m_7 - m_{15}]$
17	$[m_0 - m_{16}, m_1 - m_{16}, m_2 - m_{16}, m_3 - m_{16}, m_4 - m_{16}, m_5 - m_{16}, m_6 - m_{16}, m_7 - m_{16},$ $m_8 - m_{16}, m_9 - m_{16}, m_{10} - m_{16}, m_{11} - m_{16}, m_{12} - m_{16}, m_{13} - m_{16}, m_{14} - m_{16}, m_{15} - m_{16}]$
18	$[m_0 - m_6 - m_9 + m_{15}, m_1 - m_7 - m_{10} + m_{16}, m_2 - m_8 - m_{11} + m_{17}, m_3 + m_6 - m_{12} - m_{15},$ $m_4 + m_7 - m_{13} - m_{16}, m_5 + m_8 - m_{14} - m_{17}]$
19	$[m_0 - m_{18}, m_1 - m_{18}, m_2 - m_{18}, m_3 - m_{18}, m_4 - m_{18}, m_5 - m_{18}, m_6 - m_{18}, m_7 - m_{18}, m_8 - m_{18},$ $m_9 - m_{18}, m_{10} - m_{18}, m_{11} - m_{18}, m_{12} - m_{18}, m_{13} - m_{18}, m_{14} - m_{18}, m_{15} - m_{18}, m_{16} - m_{18},$ $m_{17} - m_{18}]$
20	$[m_0 - m_8 - m_{10} + m_{18}, m_1 - m_9 - m_{11} + m_{19}, m_2 + m_8 - m_{12} - m_{18}, m_3 + m_9 - m_{13} - m_{19},$ $m_4 - m_8 - m_{14} + m_{18}, m_5 - m_9 - m_{15} + m_{19}, m_6 + m_8 - m_{16} - m_{18}, m_7 + m_9 - m_{17} - m_{19}]$

Tab. 1: The linear combinations $(l_k(\vec{m}))_{0 \le k \le \varphi(n)-1}$ for admissibility, $n = 1, \ldots, 20$.

n	$\Lambda_n(t_1,\ldots,t_{\varphi(n)})$
1	t_1
2	$(t_1+t_1^{-1})$
3	$\left(t_1+t_2+\frac{1}{t_1t_2}\right)$
4	$(t_1+t_2+t_1^{-1}+t_2^{-1})$
5	$\left(t_1+t_2+t_3+t_4+\frac{1}{t_1t_2t_3t_4}\right)$
6	$\left(t_1+t_2+\frac{t_1}{t_2}+\frac{t_2}{t_1}+t_1^{-1}+t_2^{-1}\right)$
7	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{1}{t_1t_2t_3t_4t_5t_6}\right)$
8	$(t_1+t_2+t_3+t_4+t_1^{-1}+t_2^{-1}+t_3^{-1}+t_4^{-1})$
9	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{1}{t_1t_4}+\frac{1}{t_2t_5}+\frac{1}{t_3t_6}\right)$
10	$\left(t_1+t_2+t_3+t_4+\frac{t_1t_3}{t_2t_4}+\frac{t_2t_4}{t_1t_3}+t_1^{-1}+t_2^{-1}+t_3^{-1}+t_4^{-1}\right)$
11	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+t_9+t_{10}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}}\right)$
12	$\left(t_1+t_2+t_3+t_4+\frac{t_1}{t_3}+\frac{t_3}{t_1}+\frac{t_2}{t_4}+\frac{t_4}{t_2}+t_1^{-1}+t_2^{-1}+t_3^{-1}+t_4^{-1}\right)$
13	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+t_9+t_{10}+t_{11}+t_{12}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}t_{11}t_{12}}\right)$
14	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{t_1t_3t_5}{t_2t_4t_6}+\frac{t_2t_4t_6}{t_1t_3t_5}+t_1^{-1}+t_2^{-1}+t_3^{-1}+t_4^{-1}+t_5^{-1}+t_6^{-1}\right)$
15	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+\frac{t_1t_4t_7}{t_3t_5t_8}+\frac{t_2t_5t_8}{t_1t_4t_6}+\frac{t_1t_6}{t_2t_5t_8}+\frac{t_3t_8}{t_1t_4t_7}+\frac{1}{t_1t_6}+\frac{1}{t_2t_7}+\frac{1}{t_3t_8}\right)$
16	$(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+t_1^{-1}+t_2^{-1}+t_3^{-1}+t_4^{-1}+t_5^{-1}+t_6^{-1}+t_7^{-1}+t_8^{-1})$
17	$\left(t_1+t_2+\cdots+t_{16}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}t_{11}t_{12}t_{13}t_{14}t_{15}t_{16}}\right)$
18	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{t_1}{t_4}+\frac{t_4}{t_1}+\frac{t_2}{t_5}+\frac{t_5}{t_2}+\frac{t_3}{t_6}+\frac{t_6}{t_3}+t_1^{-1}+t_2^{-1}+t_3^{-1}+t_4^{-1}+t_5^{-1}+t_6^{-1}\right)$
19	$\left(t_1+t_2+\cdots+t_{18}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}t_{11}t_{12}t_{13}t_{14}t_{15}t_{16}t_{17}t_{18}}\right)$
20	$\left(t_1+t_2+\cdots+t_8+\frac{t_1t_5}{t_3t_7}+\frac{t_3t_7}{t_1t_5}+\frac{t_2t_6}{t_4t_8}+\frac{t_4t_8}{t_2t_6}+t_1^{-1}+t_2^{-1}+t_3^{-1}+t_4^{-1}+t_5^{-1}+t_6^{-1}+t_7^{-1}+t_8^{-1}\right)$

Tab. 2: The Laurent polynomials Λ_n for $n=1,\cdots,20$.

n	$U_n(N),\ N = 0, \dots, 20$
1	$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0$
2	$1, 0, 2, 0, 6, 0, 20, 0, 70, 0, 252, 0, 924, 0, 3432, 0, 12870, 0, 48620, 0, 184756$
3	$1, 0, 0, 6, 0, 0, 90, 0, 0, 1680, 0, 0, 34650, 0, 0, 756756, 0, 0, 17153136, 0, 0$
4	$1, 0, 4, 0, 36, 0, 400, 0, 4900, 0, 63504, 0, 853776, 0, 11778624, 0, 165636900, 0, 2363904400, 0, 34134779536$
5	$1, 0, 0, 0, 0, 120, 0, 0, 0, 0, 113400, 0, 0, 0, 0, 168168000, 0, 0, 0, 0, 305540235000$
6	$1, 0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, 8676360, 47977776, 266378112, 1488801600,$ $8355739392, 47104393050, 266482019232, 1512589408044, 8610448069080, 49144928795820$
7	$1, 0, 0, 0, 0, 0, 0, 5040, 0, 0, 0, 0, 0, 0, 681080400, 0, 0, 0, 0, 0, 0$
8	$1, 0, 8, 0, 168, 0, 5120, 0, 190120, 0, 7939008, 0, 357713664, 0, 16993726464, 0, 839358285480, 0,$ $42714450658880, 0, 2225741588095168$
9	$1, 0, 0, 18, 0, 0, 2430, 0, 0, 640080, 0, 0, 215488350, 0, 0, 84569753268, 0, 0, 36905812607664, 0, 0$
10	$1, 0, 10, 0, 270, 240, 10900, 25200, 551950, 2116800, 32458860, 169092000, 2120787900, 13427013600$ $149506414200, 1075081207200, 11143223412750, 87198375264000, 865743970019500, 7171730187336000,$ 69416724049550020
11	$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 39916800, 0, 0, 0, 0, 0, 0, 0, 0, 0$
12	$1, 0, 12, 24, 396, 2160, 23160, 186480, 1845900, 17213280, 171575712, 1703560320, 17365421304,$ $178323713568, 1856554560432, 19487791106784, 206411964321420, 2201711191213248, 23642813637773616$ $255355132936441824, 2772650461148938656$
13	$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6227020800, 0, 0, 0, 0, 0, 0, 0$
14	$1, 0, 14, 0, 546, 0, 32900, 10080, 2570050, 2540160, 238935564, 465696000, 25142196156, 76886409600,$ $2900343069624, 12211317518400, 359067702643650, 1915829643087360, 47006105030584700,$ $300455419743198720, 6437718469449262996$
15	$1, 0, 0, 30, 0, 360, 7650, 0, 302400, 4544400, 11226600, 324324000, 4310633250, 24324300000, 437404968000,$ $5634178329780, 45972927000000, 697866761592000, 8962716395833200, 88725951057744000,$ 1258898645656852200
16	$1, 0, 16, 0, 720, 0, 50560, 0, 4649680, 0, 514031616, 0, 64941883776, 0, 9071319628800, 0, 1369263687414480,$ $0, 219705672931613440, 0, 37024402443528248320$
17	$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 355687428096000, 0, 0, 0$
18	$1, 0, 18, 36, 918, 5400, 82800, 801360, 10907190, 132053040, 1802041668, 24199809480, 340640607384,$ $4834708246368, 70229958125184, 1032223723667136, 15391538570569590, 231935110984687968,$ $3531542904056225916, 54244559313713885688, 839979883121036697468$
19	$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 121645100408832000, 0$
20	$1, 0, 20, 0, 1140, 480, 102800, 151200, 12310900, 38707200, 1812247920, 9574488000, 313983978000,$ $2391608419200, 62051403928800, 611744666332800, 13627749414064500, 160896284989440000,$ $3253345101771050000, 43527416858084016000, 829176006298475046640$

Tab. 3: The sequences $(U_n(N))_{0\leq N\leq 20}$ for $n = 1, \dots, 20$.

n	*Asymptotic estimate of* $U_n(N)$ *as* $N \to \infty$	*Extra condition*
1	0	*NIL*
2	$\frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{2^N}{\sqrt{N}} \left(1 - \frac{1}{4N} + \frac{1}{32N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
3	$\frac{3\sqrt{3}}{2\pi} \cdot \frac{3^N}{N} \left(1 - \frac{2}{3N} + \frac{2}{9N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 3$
4	$\frac{2}{\pi} \cdot \frac{4^N}{N} \left(1 - \frac{1}{2N} + \frac{1}{8N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
5	$\frac{25\sqrt{5}}{4\pi^2} \cdot \frac{5^N}{N^2} \left(1 - \frac{2}{N} + \frac{2}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 5$
6	$\frac{\sqrt{3}}{2\pi} \cdot \frac{6^N}{N} \left(1 - \frac{1}{2N} + \frac{1}{12N^2} + O(\frac{1}{N^3})\right)$	*NIL*
7	$\frac{343\sqrt{7}}{8\pi^3} \cdot \frac{7^N}{N^3} \left(1 - \frac{4}{N} + \frac{8}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 7$
8	$\frac{8}{\pi^2} \cdot \frac{8^N}{N^2} \left(1 - \frac{1}{N} + \frac{1}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
9	$\frac{243\sqrt{3}}{8\pi^3} \cdot \frac{9^N}{N^3} \left(1 - \frac{3}{N} + \frac{4}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 3$
10	$\frac{5\sqrt{5}}{4\pi^2} \cdot \frac{10^N}{N^2} \left(1 - \frac{1}{N} + \frac{3}{4N^2} + O(\frac{1}{N^3})\right)$	*NIL*
11	$\frac{161051\sqrt{11}}{32\pi^5} \cdot \frac{11^N}{N^5} \left(1 - \frac{10}{N} + \frac{50}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{11}$
12	$\frac{3}{\pi^2} \cdot \frac{12^N}{N^2} \left(1 - \frac{1}{N} + \frac{2}{3N^2} + O(\frac{1}{N^3})\right)$	*NIL*
13	$\frac{4826809\sqrt{13}}{64\pi^6} \cdot \frac{13^N}{N^6} \left(1 - \frac{14}{N} + \frac{98}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{13}$
14	$\frac{49\sqrt{7}}{8\pi^3} \cdot \frac{14^N}{N^3} \left(1 - \frac{3}{2N} + \frac{3}{N^2} + O(\frac{1}{N^3})\right)$	*NIL*
15	$\frac{1125}{16\pi^4} \cdot \frac{15^N}{N^4} \left(1 - \frac{4}{N} + \frac{25}{3N^2} + O(\frac{1}{N^3})\right)$	*NIL*
16	$\frac{512}{\pi^4} \cdot \frac{16^N}{N^4} \left(1 - \frac{2}{N} + \frac{9}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
17	$\frac{6975757441\sqrt{17}}{256\pi^8} \cdot \frac{17^N}{N^8} \left(1 - \frac{24}{N} + \frac{288}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{17}$
18	$\frac{81\sqrt{3}}{8\pi^3} \cdot \frac{18^N}{N^3} \left(1 - \frac{3}{2N} + \frac{5}{2N^2} + O(\frac{1}{N^3})\right)$	*NIL*
19	$\frac{322687697779\sqrt{19}}{512\pi^9} \cdot \frac{19^N}{N^9} \left(1 - \frac{30}{N} + \frac{450}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{19}$
20	$\frac{125}{\pi^4} \cdot \frac{20^N}{N^4} \left(1 - \frac{2}{N} + \frac{7}{N^2} + O(\frac{1}{N^3})\right)$	*NIL*

Tab. 4: Asymptotic estimates of $U_n(N)$ as $N \to \infty$, for $n = 1, \ldots, 20$.