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Special Cases of the Parking Functions Conjecture and Upper-Triangular Matrices

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Abstract. We examine the $q = 1$ and $t = 0$ special cases of the parking functions conjecture. The parking functions conjecture states that the Hilbert series for the space of diagonal harmonics is equal to the bivariate generating function of $area$ and $dinv$ over the set of parking functions. Haglund recently proved that the Hilbert series for the space of diagonal harmonics is equal to a bivariate generating function over the set of Tesler matrices–upper-triangular matrices with every hook sum equal to one. We give a combinatorial interpretation of the Haglund generating function at $q = 1$ and prove the corresponding case of the parking functions conjecture (first proven by Garsia and Haiman). We also discuss a possible proof of the $t = 0$ case consistent with this combinatorial interpretation. We conclude by briefy discusing possible refinements of the parking functions conjecture arising from this research and point of view. **Note added in proof:** We have since found such a proof of the $t = 0$ case and conjectured more detailed refinements. This research will most likely be presented in full in a forthcoming article.

Résumé. On examine les cas spéciaux $q = 1$ et $t = 0$ de la conjecture des fonctions de stationnement. Cette conjecture déclare que la série de Hilbert pour l'espace des harmoniques diagonaux est égale à la fonction génératrice bivariée (paramètres $area$ et $dinv$) sur l'ensemble des fonctions de stationnement. Haglund a prouvé récemment que la série de Hilbert pour l'espace des harmoniques diagonaux est égale à une fonction génératrice bivariée sur l'ensemble des matrices de Tesler triangulaires supérieures dont la somme de chaque équerre vaut un. On donne une interprétation combinatoire de la fonction génératrice de Haglund pour $q = 1$ et on prouve le cas correspondant de la conjecture dans le cas des fonctions de stationnement (prouvé d'abord par Garsia et Haiman). On discute aussi d'une preuve possible du cas $t = 0$, cohérente avec cette interprétation combinatoire. On conclut en discutant brièvement les raffinements possibles de la conjecture des fonctions de stationnement de ce point de vue. **Note ajoutée sur épreuve:** j'ai trouvé depuis cet article une preuve du cas $t = 0$ et conjecturé des raffinements possibles. Ces résultats seront probablement présentés dans un article ultérieur.

Keywords: parking function, Hilbert series, diagonal harmonics

1 Introduction

A *parking function* of length n is a sequence $a_1 a_2 \cdots a_n$ such that, for all $1 \leq i \leq n$,

$$|f^{-1}(\{1, 2, \cdots, i\})| \geq i.$$

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Let P_n be the set of parking functions of length n. The parking functions conjecture (see [Hag08]) states that the Hilbert series for the space of diagonal harmonics is given by the following bivariate generating function over P_n.

$$\sum_{a_1a_2\cdots a_n \in P_n} q^{dinv(a_1a_2\cdots a_n)} t^{area(a_1a_2\cdots a_n)},$$

where $area$ and $dinv$ are statistics defined on parking functions.

Let the m-th *hook sum* of an upper-triangular matrix $A = [a_{i,j}]_{1 \le i \le j \le n}$ be given by

$$\sum_{r=m}^{n} a_{m,r} - \sum_{k=1}^{m-1} a_{k,m}.$$

Let A be a *Tesler matrix* if and only if it has non-negative integer entries and every hook sum is equal to one. Let T_n be the set of $n \times n$ Tesler matrices. The following matrix is an example of a member of T_5:

$$\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & & 2 & 0 \\ & & & & 2. \end{array}$$

Let $[n]_{q,t}$ be the polynomial $\sum_{i=0}^{n-1} q^i t^{n-1-i}$. If $A \in T_n$, let $extra(A)$ be n less than the number of nonzero entries of A. (By construction, A must have at least one nonzero entry per row.)

Haglund [Hag] recently proved that the Hilbert series of diagonal harmonics is given by the following bivariate generating function over T_n:

$$\sum_{A=[a_{i,j}] \in T_n} ((q-1)(1-t))^{extra(A)} \prod_{a_{i,j}>0} [a_{i,j}]_{q,t}.$$

As a result of Haglund's proof, the parking functions conjecture can now be considered equivalent to the following identity of bivariate generating functions:

$$\sum_{a_1a_2\cdots a_n \in P_n} q^{dinv(a_1a_2\cdots a_n)} t^{area(a_1a_2\cdots a_n)} = \sum_{A=[a_{i,j}] \in T_n} ((q-1)(1-t))^{extra(A)} \prod_{a_{i,j}>0} [a_{i,j}]_{q,t}. \quad (1)$$

In the following extended abstract, we will prove the $q = 1$ special case of Equation (1) and give an outline of a proof of the $t = 0$ special case. The $q = 1$ special case was first proven by Garsia and Haiman [GH96].

2 The $q = 1$ special case

Let $T_n^* = \{A \in T_n : extra(A) = 0\}$. Alternately, let T_n^* be the set of $n \times n$ Tesler matrices with precisely one nonzero entry per row. Let $[n]_t = [n]_{q,t}|_{q=1} = \sum_{i=0}^{n-1} t^i$. For $a_1a_2\cdots a_n \in P_n$, let $area(a_1a_2\cdots a_n) = \binom{n+1}{2} - \sum_{i=1}^{n} a_i$.

When Equation (1) is evaluated at $q = 1$, the following equation results:

$$\sum_{a_1a_2\cdots a_n\in P_n} t^{area(a_1a_2\cdots a_n)} = \sum_{A=[a_{i,j}]\in T_n^*} \prod_{a_{i,j}>0} [a_{i,j}]_t. \tag{2}$$

To prove Equation (2), we will begin by constructing a surjective map ϕ from P_n to S_n. (**Note added in proof:** Haglund pointed out that this map sends each parking function to the inverse of the permutation it "parks" to. We will retain our original description.) We will then give a simple proof that $|T_n^*| = n!$. We will then construct a bijection between S_n and T_n^* by associating each permutation $\pi \in S_n$ to a unique Tesler matrix $C_\pi \in T_n^*$. Finally, we will prove that, for each $\pi \in S_n$,

$$\sum_{\phi(a_1a_2\cdots a_n)=\pi} t^{area(a_1a_2\cdots a_n)} = \prod_{C_\pi=[a_{i,j}],a_{i,j}>0} [a_{i,j}]_t.$$

This will prove Equation (2).

2.1 A surjective map from P_n to S_n.

Let the function ϕ on the set of n-sequences of positive integers be defined as follows: Given an n-sequence of positive integers $a_1a_2\cdots a_n$, let the sequence $b_1b_2\cdots b_n$ be defined recursively by

- $b_1 = a_1$, and
- For $i > 1$, let b_i be the smallest integer greater than or equal to a_i that is not a member of $\{b_1, b_2, \dots, b_{i-1}\}$.

Let $\phi(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$.

Lemma 1. *1. If $a_1a_2\cdots a_n \in S_n$, then $\phi(a_1a_2\cdots a_n) = a_1a_2\cdots a_n$.*

2. $\phi(a_1a_2\cdots a_n) \in S_n$ if and only if $a_1a_2\cdots a_n \in P_n$.
Therefore, ϕ is a surjective map from P_n to S_n.

Proof. 1. If $a_1a_2\cdots a_n \in S_n$ and $\phi(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$ as above, then $b_1 = a_1$. Assume inductively that $a_j = b_j$ for all $j \le i-1$. Then $a_i \notin \{b_1, b_2, \cdots, b_{i-1}\} = \{a_1, a_2, \cdots, a_{i-1}\}$, and a_i is obviously the smallest such integer greater than or equal to a_i. Therefore $a_i = b_i$ for all i, and $\phi(a_1a_2\cdots a_n) = a_1a_2\cdots a_n$.

2. If $a_1a_2\cdots a_n \notin P_n$ and $\phi(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$ as above, then there must be an integer j such that $|\{i : a_i \le j\}| < j$. Since $b_i \ge a_i$ for all i, $b_i \le j$ only if $a_i \le j$. Therefore there are strictly fewer than k integers i such that $b_i \le j$. Therefore there are strictly fewer than j integers i such that $b_i \le j$. Therefore $b_1b_2\cdots b_n \notin S_n$. Therefore $b_1b_2\cdots b_n \in S_n$ only if $a_1a_2\cdots a_n \in P_n$.

If $a_1a_2\cdots a_n \in P_n$ and $\phi(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$, then $b_1b_2\cdots b_n \notin S_n$ if and only if, for some $R \in [n]$, there does not exist an integer i such that $b_i = R$. Assume R is minimal with this property. Therefore $[R-1] \subset \{b_1, b_2, \cdots, b_n\}$.

Because $a_1a_2\cdots a_n$ is a parking function, there must be at least R integers i such that $a_i \le R$. Let $c_1 < c_2 < \cdots < c_P$ be the increasing rearrangement of these integers, so $P \ge R$. For each $1 \le j \le P$, $a_{c_j} \le R$ and $b_{c_j} \ne R$. If $b_{c_j} > R$, then R would be an integer less than b_{c_j} that is greater than a_{c_j} and not in $\{b_1, b_2, \cdots, b_{c_j-1}\}$. This contradicts the definition of b_{c_j}. Since $b_{c_j} \neq R$ by definition, $b_{c_j} < R$ for all j. Therefore $b_{c_1}, b_{c_2}, \ldots, b_{c_P}$ is a sequence of P distinct integers strictly less than R with $P \geq R$. This is impossible, and therefore $b_1 b_2 \cdots b_n \in S_n$.

Therefore ϕ is a surjective map from P_n to S_n. $\square$

Let $\pi_1\pi_2\cdots\pi_n$ be a parking function of length n. Let the functions $g_\pi(i)$ be defined by $g_\pi(i) = min(\pi_i - \pi_j : i < j, \pi_i > \pi_j)$, with $g_\pi(i) = \pi_i$ if there is no such π_j. For example, if $\pi = 24153$, then $(g_\pi(1), g_\pi(2), g_\pi(3), g_\pi(4), g_\pi(5)) = (1, 1, 1, 2, 3)$. Alternately, $g_\pi(i)$ is the largest integer such that $\pi_i - 1, \pi_i - 2, \cdots, \pi_i - g_\pi(i) + 1$ all appear before π_i in π.

Theorem 2. *Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in S_n.*

1. $\phi^{-1}(\pi_1\pi_2\cdots\pi_n) = \{a_1a_2\cdots a_n : \pi_i \geq a_i \geq \pi_i - g_\pi(i) + 1\}$.
2. $|\phi^{-1}(\pi_1\pi_2\cdots\pi_n)| = \prod_{i=1}^n g_\pi(i)$.
3. $$(n+1)^{n-1} = \sum_{\pi \in S_n} \prod_{i=1}^n g_\pi(i).$$
4. $$\sum_{\phi(a_1a_2\cdots a_n)=\pi} t^{area(a_1a_2\cdots a_n)} = \prod_{i=1}^n [g_\pi(i)]_t.$$

Proof. (sketch) Consider $\phi^{-1}(24153)$. The parking function $23153 \notin \phi^{-1}(24153)$, since ϕ will keep the second entry equal to 3, as another 3 has not yet appeared. In fact, $\phi(23153) = 23154$. However, $24143 \in \phi^{-1}(24153)$, since ϕ will change the fourth entry from 4 to 5, since another 4 will have appeared in the second entry. Similarly, $344215 \notin \phi^{-1}(346215)$, since ϕ will not change the third entry from a 4 to a 6. Instead, because the integer 5 between 4 and 6 has not yet appeared, ϕ will change the third entry from 4 to 5, and $\phi(344215) = 345216$. In general, π_i can be replaced by an integer k and still have ϕ change k back to π_i if and only if k, and all integers between k and π_i, appear before π_i. Alternately, π_i can be replaced by k if and only if $\pi_i \geq k \geq \pi_i - g_\pi(i) + 1$. The rest of the theorem follows directly or from the definition $area(a_1a_2\cdots a_n) = \binom{n+1}{2} - \sum_{i=1}^n a_i$. $\square$

2.2 A weight-preserving bijection between T_n^* and S_n.

Recall that T_n^* is the set of Tesler matrices with precisely one nonzero entry per row.

Lemma 3. *A Tesler matrix in T_n^* is uniquely determined by the locations of its nonzero entries. In particular, $|T_n^*| = n!$.*

Proof. (sketch) There are n choices about where to put the nonzero entry in the first row, which must equal 1. There are then $n-1$ choices about where to put the nonzero entry in the second row, which must equal 1 unless the nonzero entry in the first row was in the second column, in which case it must equal 2. Continuing in this fashion from top to bottom it is clear that the condition that every hook sum is equal to one uniquely determines the entries once their location is determined. $\square$

(This lemma was realized by the author along with Mirkó Visontai).

We will now define a map from S_n to T_n^*. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, let $h_\pi(i)$ be the smallest integer such that $i < h_\pi(i)$, $\pi_i < \pi_{h_\pi(i)}$, and, if $\pi_i < k < \pi_{h_\pi(i)}$, then $k \in \{\pi_1, \pi_2, \ldots, \pi_{i-1}\}$. Alternately, $h_\pi(i)$ is the location of the first entry to the right of π_i that is larger than π_i but such that every integer in between it and π_i appears to the left of π_i. (If there is no such integer let $h_\pi(i) = i$.) For example, if $\pi = 24153$, then $(h_\pi(1), h_\pi(2), h_\pi(3), h_\pi(4), h_\pi(5)) = (5, 4, 5, 4, 5)$.

By construction, $i \leq h_\pi(i) \leq n$ for all i. Therefore, let $C_\pi \in T_n^*$ be the unique Tesler matrix with nonzero entries precisely at $(i, h_\pi(i))$ for all $1 \leq i \leq n$. For example, if $\pi = 24153$, then

$$C_\pi = \begin{matrix} 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & & 2 & 0 \\ & & & & 3. \end{matrix}$$

We will refer to the following inductive construction of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ as the *bumping algorithm*. Given a fixed integer $1 \leq m_1 \leq n$ and permutation $\pi^* = \pi_1^*\pi_2^*\cdots\pi_{n-1}^*$, let $\pi_1\pi_2\cdots\pi_n$ be defined as follows: If $m_1 = 1$, then $\pi_1\pi_2\cdots\pi_n = (n+1)\pi_1^*\pi_2^*\cdots\pi_{n-1}^*$. If $m_1 > 1$, then

- $\pi_1 = \pi_{m_1-1}^*$.
- $\pi_i = \pi_{i-1}^*$ if $\pi_{i-1}^* < \pi_{m_1-1}^*$.
- $\pi_i = \pi_{i-1}^* + 1$ if $\pi_{i-1}^* \geq \pi_{m_1-1}^*$.

For example, if $\pi^* = 412635$ and $m_1 = 4$, then the bumping algorithm results in $\pi = 2513746$.

Theorem 4. *1. If π is constructed by the bumping algorithm from m_1 and π^*, then $h_\pi(1) = m_1$ and $h_\pi(i) = h_{\pi^*}(i-1) + 1$ for $i > 2$.*

2. Therefore the map $\pi \to C_\pi$ is a bijection between S_n and T_n^*.

Proof. (sketch) The first part follows from the definition of h_π and inductively demonstrates that for every sequence $m_1m_2\cdots m_n$ such that $i \leq m_i \leq n$ for all i, there is at least one permutation $\pi = \pi_1\pi_2\cdots\pi_n$ with $h_\pi(i) = m_i$ for all i. This proves the second part. □

Given a permutation $\pi \in S_n$, let D_π be the increasing forest on $[n]$ constructed inductively as follows: Given a permutation $\pi^* = \pi_1^*\pi_2^*\cdots\pi_{n-1}^*$ and integer $1 \leq m_1 \leq n$, increase the label of every node of D_{π^*} by one. Add 1 as a leaf of the node now labelled m_1. If $m_1 = 1$, leave 1 as an isolated node. Let D_π be the resulting increasing forest, where π is constructed by the bumping algorithm from π^* and m_1. (Increasing forests are well-known to be equinumerous with permutations, see [Sta97].)

Theorem 5. *For each $i \in [n]$, the parent of the node labelled i in the increasing forest D_π is the node labelled $h_\pi(i)$.*

Proof. The parent of the node labelled 1 is the node labelled m_1. For $i > 1$, assume inductively that the parent of the node labelled $i-1$ in D_{π^*} is the node labelled $h_{\pi^*}(i-1)$. Then the parent of the node labelled i in D_π is the node labelled $h_{\pi^*}(i-1) + 1 = h_\pi(i)$. □

Theorem 6. *1. Given $\pi \in S_n$, the Tesler matrix $C_\pi \in T_n^*$ has the entry $g_\pi(i)$ at $(i, h_\pi(i))$. Equivalently, if an upper-triangular matrix has only the nonzero entries $g_\pi(i)$ at $(i, h_\pi(i))$, then it is a Tesler matrix.*

2.
$$\prod_{C_\pi=[a_{i,j}],a_{i,j}>0} [a_{i,j}]_t = \prod_{i=1}^{n} [g_\pi(i)]_t = \sum_{\phi(a_1 a_2 \cdots a_n)=\pi} t^{area(a_1 a_2 \cdots a_n)}.$$

Proof. (sketch) From the definitions of g_π and h_π and taking $h_\pi^0(j) = j$, it can be seen that

$$g_\pi(i) = |\, \{j : h_\pi^r(j) = i\} \,|.$$

Alternately, $g_\pi(i)$ is the number of integers eventually sent to i by h_π. Using Theorem (5), h_π can be seen as a function that sends an integer labelling a node in the increasing forest D_π to the integer labelling the parent of that node. Therefore $g_\pi(i)$ is the number of descendents in D_π of the node labelled i, including the node itself.

If an upper-triangular matrix has only the nonzero entries $g_\pi(i)$ at $(i, h_\pi(i))$, then the i-th hook sum is $g_\pi(i) - \sum_{j:h_\pi(j)=i} g_\pi(j)$. This is equal to the number of descendents in D_π of the node labelled i (including itself) minus the number of descendents of the nodes with i as a parent (including themselves) This must equal 1. Therefore the upper-triangular matrix with only the nonzero entries $g_\pi(i)$ at $(i, h_\pi(i))$ is the unique Tesler matrix $C_\pi \in T_n^*$. ☐

This proves Equation (2).

3 The $t = 0$ special case

Note added in proof: The version of this extended abstract originally submitted described our efforts to find a proof of Equation (3). We have since completed a proof of Equation (3), which we will give the broad outlines of below. A more complete description of both proofs may likely be in a forthcoming article.

Recall that, if $a_1 a_2 \cdots a_n$ is a parking function of length n, then $area(a_1 a_2 \cdots a_n) = \binom{n+1}{2} - \sum_{i=1}^{n} a_i$.

Lemma 7. *A parking function $a_1 a_2 \cdots a_n \in P_n$ has $area$ equal to zero if and only if it is a permutation in S_n.*

Therefore, when Equation (1) is evaluated at $t = 0$, the following equation results:

$$\sum_{\pi_1\pi_2\cdots\pi_n \in S_n} q^{dinv(\pi_1\pi_2\cdots\pi_n)} = \sum_{A=[a_{i,j}]\in T_n} (q-1)^{extra(A)} q^{\sum_{a_{i,j}>0}(a_{i,j}-1)}. \tag{3}$$

The $dinv$ statistic is complicated to define on general parking functions (see [Hag08]). However, it is easier to define on permutations: For $\pi_1\pi_2\cdots\pi_n \in S_n$, let

$$dinv(\pi_1\pi_2\cdots\pi_n) = |\, \{(i,j) : i < j, \pi_i < \pi_j\} \,|.$$

Alternately, $dinv$ gives the number of pairs of entries of $\pi_1\pi_2\cdots\pi_n$ that are not inversions. It is well known that $dinv$ is a *Mahonian* statistic when applied to permutations, so

$$\sum_{\pi_1\pi_2\cdots\pi_n\in S_n} q^{dinv(\pi_1\pi_2\cdots\pi_n)} = \prod_{i=1}^{n}[i]_q.$$

We will sketch a proof of Equation (3). In particular, we will partition the set of $n\times n$ Tesler matrices T_n into $n!$ disjoint subsets T_π indexed by the permutations $\pi\in S_n$. These subsets T_π are such that, for each $\pi\in S_n$,

$$\sum_{A=[a_{i,j}]\in T_\pi} (q-1)^{extra(A)} q^{\sum_{a_{i,j}>0}(a_{i,j}-1)} = q^{dinv(\pi)}.$$

This will prove Equaiton (3).

3.1 Decoding Tesler Matrices

Given a Tesler matrix $A=[a_{i,j}]\in T_n$, let A^* be the multiset of ordered pairs and integers such that $(i,j)\in A^*$ with multiplicity k if and only if $a_{i,j}=k$ for $i<j$ and $i\in A^*$ with multiplicity k if and only if $a_{i,i}=k$. For example, if

$$A=\begin{matrix}0&1&0&0&0\\ &0&2&0&0\\ &&1&1&1,\\ &&&1&1\\ &&&&3\end{matrix}$$

then $A^*=\{(1,2),(2,3),(2,3),3,(3,4),(3,5),4,(4,5),5,5,5\}$. From each such A^*, we can construct a list of n vertically-arranged sets, each written in increasing order, such that:

- a is immediately before b in a set if and only if $(a,b)\in A^*$.
- The last entries in each set are the single integers in A^*, weakly decreasing from the top set to the bottom set.

In particular, we want to construct such a list of n vertically-arranged sets, each written in increasing order, such that:

- The last entry in each set must be greater than or equal to any entry in a set below.
- The first entry in each set does not appear in any set below, and is the only entry in that set where this is the case.
- If a is immediately before b in a set and c is immediately before b in a set above, then $c\geq b$.

For example, one list of sets resulting from $\{(1,2),(2,3),(2,3),3,(3,4),(3,5),4,(4,5),5,5,5\}$ satisfying the above rules is

$$\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
 & & 2 & 3 & 5 \\
 & & & & 5 \\
 & & & & 4 \\
 & & & & 3
\end{array}$$

We will omit the proof of the following theorem stating that this construction is bijective.

Theorem 8. *1. There is precisely one list of sets satisfying the above rules for each matrix $A \in T_n$.*

2. If a list of sets satisfies the above rules, then the first entry of each set, read from top to bottom, gives a permutation π.

For a fixed $\pi \in S_n$, let T_π be the set of Tesler matrices $A \in T_n$ such that the unique list of sets satisfying the above rules gives the permutation π. For example, T_{31254} consists of the following Tesler matrices (each paired with the corresponding list of sets)

$$\begin{array}{ccc}
 & 3 & 5 \\
1 & 2 & 5 \\
 & 2 & 5 \\
 & & 5 \\
 & & 4
\end{array}
\quad
\begin{array}{ccccc}
0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 2 \\
 & & 0 & 0 & 1 \\
 & & & 1 & 0 \\
 & & & & 4
\end{array}
\qquad
\begin{array}{ccc}
3 & 4 & 5 \\
1 & 2 & 5 \\
 & 2 & 5 \\
 & & 5 \\
 & & 4
\end{array}
\quad
\begin{array}{ccccc}
0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 2 \\
 & & 0 & 1 & 0 \\
 & & & 1 & 1 \\
 & & & & 4
\end{array}$$

$$\begin{array}{cccc}
 & 3 & 4 & 5 \\
1 & 2 & 4 & 5 \\
 & & 2 & 5 \\
 & & & 5 \\
 & & & 4
\end{array}
\quad
\begin{array}{ccccc}
0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 1 \\
 & & 0 & 1 & 0 \\
 & & & 1 & 2 \\
 & & & & 4
\end{array}
\qquad
\begin{array}{ccc}
3 & 4 & 5 \\
1 & 4 & 5 \\
 & 2 & 5 \\
 & & 5 \\
 & & 4
\end{array}
\quad
\begin{array}{ccccc}
0 & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 1 \\
 & & 0 & 1 & 0 \\
 & & & 1 & 2 \\
 & & & & 4
\end{array}$$

$$\begin{array}{cccc}
 & 3 & 4 & 5 \\
1 & 2 & 4 & 5 \\
 & 2 & 4 & 5 \\
 & & & 5 \\
 & & & 4
\end{array}
\quad
\begin{array}{ccccc}
0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 2 & 0 \\
 & & 0 & 1 & 0. \\
 & & & 1 & 3 \\
 & & & & 4
\end{array}$$

Note that the Tesler matrix C_{31254} is in T_{31254}, and that the sum of $(q-1)^{extra(A)} q^{\sum_{a_{i,j}>0}(a_{i,j}-1)}$ over T_{31254} is equal to $q^7 = q^{dinv(31254)}$. Finally, note that the term of q^7 comes from the term of the generating function corresponding to the last Tesler matrix. In general, for $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, T_π will always contain precisely one Tesler matrix such that the first set in the corresponding list of sets consists of π_1 followed by everything after and larger than π_1, the second set consists of π_2 followed by everything after and larger than π_2, and so on. This Tesler matrix will always give a term of $q^{dinv(\pi)}$. It is possible to define an involution that cancels out the other terms, but we will not describe the involution in this extended abstract.

Theorem 9. *The following equation holds due to an involution:*

$$\sum_{A=[a_{i,j}] \in T_\pi} (q-1)^{extra(A)} q^{\sum_{a_{i,j}>0}(a_{i,j}-1)} = q^{dinv(\pi)}.$$

3.2 A non-Mahonian statistic on S_n

Having sketched a combinatorial proof of Equation (3), we will now discuss how it relates to the combinatorial proof presented earlier of Equation (2).

We can define a statistic akin to $dinv$ by replacing the t-analogues in Equation (2) with q,t-analogues. For $a_1a_2\cdots a_n \in P_n$ with $\phi(a_1a_2\cdots a_n) = \pi_1\pi_2\cdots\pi_n$, let $pdinv(a_1a_2\cdots a_n)$ be defined by

$$pdinv(a_1a_2\cdots a_n) = \sum_{i=1}^{n}(a_i - \pi_i + g_\pi(i) + 1).$$

Alternately, $pdinv(a_1a_2\cdots a_n)$ is the sum of how much there is left that can be subtracted from each entry and still leave a parking function in $\phi^{-1}(\pi_1\pi_2\cdots\pi_n)$.

Lemma 10. *1. $pdinv$ satisfies the following generating function identity:*

$$\sum_{a_1a_2\cdots a_n\in P_n} q^{pdinv(a_1a_2\cdots a_n)}t^{area(a_1a_2\cdots a_n)} = \sum_{A=[a_{i,j}]\in T_n^*}\prod_{a_{i,j}>0}[a_{i,j}]_{q,t}.$$

2. $pdinv$ is equidistributed with $area$ over P_n and is therefore equidistributed with $dinv$ over P_n.

Proof. The first part of the lemma is clear from the alternate description of $pdinv$. The equidistribution of $area$ and $pdinv$ follows from the fact that each $[a_{i,j}]_{q,t}$ is symmetric in q and t. The equidistribution $dinv$ and $pdinv$ then follows from the equidistribution of $area$ and $dinv$. □

However, $(pdinv, area)$ is *not* joint-equidistributed with $(dinv, area)$ over P_n. If it were, then there would be no need for the terms of Equation (1) corresponding to Tesler matrices with more than one entry per row. For example, the sum of $q^{pdinv(a_1a_2a_3)}$ over S_3, the set of parking functions of length 3 with zero area, is equal to $1 + 3q + q^2 + q^3$, while the sum of $q^{dinv(a_1a_2a_3)}$ over S_3 is equal to $1 + 2q + 2q^2 + q^3$. ($dinv$ being Mahonian over S_3.)

We can therefore think of the Tesler matrices with more than one entry per row as somehow de-coupling and re-coupling $pdinv$ and $area$ in some parking functions to obtain (conjecturally) the $(dinv, area)$ joint distribution.

As with $dinv$, $pdinv$ is difficult to interpret combinatorially as a statistic on P_n but easier to interpret combinatorially as a statistic on S_n. In particular, $pdinv$ enumerates a subset of non-inversion pairs.

Theorem 11. *For $\pi_1\pi_2\cdots\pi_n \in S_n$, $pdinv(\pi_1\pi_2\cdots\pi_n)$ is the number of pairs (i,j) such that $i < j$, $\pi_i < \pi_j$ and, for all $i < j < k$, $\pi_k \notin [\pi_i, \pi_j]$. Alternately, $pdinv(\pi_1\pi_2\cdots\pi_n)$ is the number of pairs (i,j) such that π_j forms a non-inversion with π_i and there is no integer after it in between π_i and π_j in magnitude.*

Proof. (sketch) By definition, $pdinv(\pi_1\pi_2\cdots\pi_n) = -n + \sum_{i=1}^{n} g_\pi(i)$. Recall that $g_\pi(i)$ is the largest integer such that $\pi_i - 1, \pi_i - 2, \cdots, \pi_i - g_\pi(i) + 1$ all appear before π_i in π. π_i will form a non-inversion with each of these integers, and for each integer there can be no integer after π_i in between it and π_i in magnitude. Since there are $g_\pi(i) - 1$ such integers, there will be $-n + \sum_{i=1}^{n} g_\pi(i) = pdinv(\pi_1\pi_2\cdots\pi_n)$ such pairs in all. □

Looking only at the $t = 0$ special case, we can therefore think of the effect of the matrices in T_π as adding these "missing" anti-inversions back to $pdinv(\pi)$ to result in $dinv(\pi)$. Each of these missing anti-inversions comes from the $pdinv$ of another parking function, and so on.

4 Further Directions and Conjectures

Note added in proof: This section has been updated to account for our current work and conjectures.

The proof of the $t = 0$ special case, Equation (3), shows that the Tesler matrices can be divided into disjoint subsets T_π indexed by the permutations $\pi \in S_n$ such that the Haglund generating function is q-positive at $t = 0$ when summed over each subset. The full Haglund generating function is not q, t-positive when summed over each T_π. However, we have verified for $n \leq 7$ that the Haglund generating function is q, t-positive when summed over the union of T_π over naturally-defined subsets of S_n. This suggests that the parking functions conjecture might be refinable. Unfortunately, the resulting positive generating functions do not match the generating function of $dinv$ and $area$ over the corresponding parking functions. This suggests that some alternate statistic f, such that $(f, area)$ is joint-equidistributed with $(dinv, area)$, might better illuminate the combinatorics of the parking functions conjecture.

Also, we are unfamiliar with the work surrounding the Kreveras theorem, which states that parking functions with area k are equinumerous with spanning forests with k inversions, but it apparently lacks a "nice" bijective proof, and it is possible our proof of Equation (2) could yield one through our use of spanning increasing forests. See Pak [Pak] and Kreveras [Kre80].

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