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Stable rigged configurations and Littlewood–Richardson tableaux

Masato Okado[1†] and Reiho Sakamoto[2‡]

[1] *Department of Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka, Japan*
[2] *Department of Physics, Tokyo University of Science, Kagurazaka, Shinjuku-ku, Tokyo, Japan*

Abstract. For an affine algebra of nonexceptional type in the large rank we show the fermionic formula depends only on the attachment of the node 0 of the Dynkin diagram to the rest, and the fermionic formula of not type A can be expressed as a sum of that of type A with Littlewood–Richardson coefficients. Combining this result with theorems of Kirillov–Schilling–Shimozono and Lecouvey–Okado–Shimozono, we settle the $X = M$ conjecture under the large rank hypothesis.

Résumé. Pour une algèbre affine de type nonexceptionnel de grand rang nous prouvons que la formule fermionique dépend seulement du voisinage du noeud 0 dans le diagramme de Dynkin, et également que la formule fermionique en type autre que A peut être exprimée comme combinaison de celles de type A avec des coefficients de Littlewood–Richardson. Combinant ce résultat avec des théorèmes de Kirillov–Schilling–Shimozono et de Lecouvey–Okado–Shimozono, nous résolvons la conjecture $X = M$ lorsque le rang est grand.

Keywords: affine crystals, rigged configurations, Littlewood–Richardson tableaux, fermionic formula

1 Introduction

Let $\mathfrak{g}$ be an affine Lie algebra and I the index set of its Dynkin nodes. Let $\mathfrak{g}_0$ be the classical subalgebra of $\mathfrak{g}$, namely, the finite-dimensional simple Lie algebra whose Dynkin nodes are given by $I_0 := I\setminus\{0\}$ where the node 0 is taken as in [10]. Let $U'_q(\mathfrak{g})$ be the quantized enveloping algebra associated to $\mathfrak{g}$ without the degree operator. Among finite-dimensional $U'_q(\mathfrak{g})$-modules there is a distinguished family called Kirillov-Reshetikhin (KR) modules, which have nice properties such as $T(Q, Y)$-systems, fermionic character formulas, and so on. See for instance [1, 9, 14, 20] and references therein. In [7, 6], assuming the existence of the crystal basis $B^{r,s}$ ($r \in I_0, s \in \mathbb{Z}_{>0}$) of a KR module we defined the one-dimensional (1-d) sum

$$X_{\lambda,B}(q) = \sum_{b\in B} q^{D(b)}$$

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where the sum is over I_0-highest weight vectors in $B = B^{r_1,s_1} \otimes \cdots \otimes B^{r_m,s_m}$ with weight λ and D is a certain $\mathbb{Z}$-valued function on B called the energy function (see e.g. (3.9) of [6]), and conjectured that X has an explicit expression M called the fermionic formula ($X = M$ conjecture). This conjecture is settled in full generality if $\mathfrak{g} = A_n^{(1)}$ [13], when $r_j = 1$ for all j if $\mathfrak{g}$ is of nonexceptional affine types [27], and when $s_j = 1$ for all j if $\mathfrak{g} = D_n^{(1)}$ [26]. It should also be noted that recently the existence of KR crystals for nonexceptional affine types was settled [21, 23] and their combinatorial structures were clarified [2].

Another interesting equality related to X is the $X = K$ conjecture by Shimozono and Zabrocki [29, 28] that originated from the study of certain q-deformed operators on the ring of symmetric functions. Suppose $\mathfrak{g}$ is of nonexceptional type. If the rank of $\mathfrak{g}$ is sufficiently large, X does not depend on $\mathfrak{g}$ itself, but only on the attachment of the affine Dynkin node 0 to the rest of the Dynkin diagram. See Table 1. Let $X^{\diamondsuit}_{\lambda,B}(q)$ ($\diamondsuit = \varnothing, \Box, \Box\!\Box, \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$) denote the 1-d sum for $\mathfrak{g}$ of kind $\diamondsuit$. Then the $X = K$ conjecture, which

Tab. 1:

$\diamondsuit$	$\mathfrak{g}$ of kind $\diamondsuit$
$\varnothing$	$A_n^{(1)}$
$\Box$	$D_{n+1}^{(2)}, A_{2n}^{(2)}$
$\Box\!\Box$	$C_n^{(1)}$
$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}$

has been settled in [28, 18, 19], states that if $\diamondsuit \neq \varnothing$, the following equality holds.

$$X^{\diamondsuit}_{\lambda,B}(q) = q^{-\frac{|B|-|\lambda|}{|\diamondsuit|}} \sum_{\mu \in \mathcal{P}^{\diamondsuit}_{|B|-|\lambda|}, \eta \in \mathcal{P}^{\Box}_{|B|}} c^{\eta}_{\lambda\mu} X^{\varnothing}_{\eta,B}(q^{\frac{2}{|\diamondsuit|}}) \tag{1}$$

Here $|B| = \sum_{i=1}^m r_i s_i$, $\mathcal{P}^{\diamondsuit}_N$ is the set of partitions of N whose diagrams can be tiled by $\diamondsuit$, and $c^{\eta}_{\lambda\mu}$ is the Littlewood–Richardson coefficient. Note also that $\overline{X}^{\diamondsuit}_{\lambda,B}(q)$ in [19] is related to our $X^{\diamondsuit}_{\lambda,B}(q)$ by $\overline{X}^{\diamondsuit}_{\lambda,B}(q) = X^{\diamondsuit}_{\lambda,B}(q^{-1})$.

If we believe the $X = M$ conjecture, we have the right to expect exactly the same relation on the M side under the same assumption of the rank. This is what we wish to clarify in this paper. Namely, if $\mathfrak{g}$ is one of nonexceptional affine type and the rank is sufficiently large, we show the fermionic formula depends only on the symbol $\diamondsuit$, denoted by $M^{\diamondsuit}(\lambda, \mathbf{L}; q)$, and if $\diamondsuit \neq \varnothing$ we have

$$M^{\diamondsuit}(\lambda, \mathbf{L}; q) = q^{-\frac{|\mathbf{L}|-|\lambda|}{|\diamondsuit|}} \sum_{\mu \in \mathcal{P}^{\diamondsuit}_{|\mathbf{L}|-|\lambda|}, \eta \in \mathcal{P}^{\Box}_{|\mathbf{L}|}} c^{\eta}_{\lambda\mu} M^{\varnothing}(\eta, \mathbf{L}; q^{\frac{2}{|\diamondsuit|}}). \tag{2}$$

Here $\mathbf{L} = (L_i^{(a)})_{a \in I_0, i \in \mathbb{Z}_{>0}}$ is a datum such that $L_i^{(a)}$ counts the number of $B^{a,i}$ in B and $|\mathbf{L}| = \sum_{a \in I_0, i \in \mathbb{Z}_{>0}} a i L_i^{(a)}$.

The proof of (2) proceeds as follows. We first rewrite the fermionic formula as

$$M^{\diamondsuit}(\lambda, \mathbf{L}; q) = \sum_{(\nu^{\bullet}, J^{\bullet}) \in \mathrm{RC}^{\diamondsuit}(\lambda, \mathbf{L})} q^{c(\nu^{\bullet}, J^{\bullet})}$$

by introducing the notion of stable rigged configurations. c is a certain bilinear form on the rigged configurations called charge (see (2.11) of [22]). We then construct for $\diamondsuit \neq \varnothing$ a bijection

$$\Psi : \mathrm{RC}^{\diamondsuit}(\lambda, \mathbf{L}) \longrightarrow \bigsqcup_{\mu \in \mathcal{P}^{\diamondsuit}_{|\mathbf{L}|-|\lambda|}, \eta \in \mathcal{P}^{\square}_{|\mathbf{L}|}} \mathrm{RC}^{\varnothing}(\eta, \mathbf{L}) \times LR^{\eta}_{\lambda\mu},$$

where $LR^{\eta}_{\lambda\mu}$ is the set of Littlewood–Richardson skew tableaux of shape η/λ and weight μ (see, e.g., [4]). Roughly speaking, the bijection Ψ proceeds as follows. When the rank is sufficiently large, there exists k such that the a-th configuration $\nu^{(a)}$ is the same for $a = k, k+1, \ldots$. As opposed to the KKR algorithm [11] that removes a box from $\nu^{(a)}$ starting from $a = 1$, we perform a similar algorithm starting from the largest a. If we continue this procedure until all boxes are removed from $\nu^{(a)}$ for sufficiently large a, we can regard this as a rigged configuration of type A. Reflecting this sequence of procedures we can also define a recording tableau, that is shown to be a Littlewood–Richardson skew tableau. This map can be reversed at each step, and therefore defines a bijection.

Finally we show

$$c(\nu^{\bullet}, J^{\bullet}) = c(\nu'^{\bullet}, J'^{\bullet}) - \frac{|\mathbf{L}| - |\lambda|}{|\diamondsuit|}$$

where $(\nu'^{\bullet}, J'^{\bullet})$ is the first component of the image of $(\nu^{\bullet}, J^{\bullet})$ by Ψ. We note that the two equalities (1) and (2) together with the result of [13] imply

$$X^{\diamondsuit}_{\lambda,B}(q) = M^{\diamondsuit}(\lambda, \mathbf{L}; q)$$

for $\diamondsuit \neq \varnothing$ and therefore settle the $X = M$ when $\mathfrak{g}$ is of nonexceptional type and the rank is sufficiently large.

Let us summarize the combinatorial bijections that are relevant to our paper as the following schematic diagram:

$$\begin{tikzcd}
\{\text{type } \mathfrak{g} \text{ path}\} \arrow[d, leftrightarrow, "\text{(b)}"'] & \{\text{type } A_n^{(1)} \text{ path}\} \times \mathrm{LR} \arrow[d, leftrightarrow, "\text{(a)}"] \\
\{\text{type } \mathfrak{g} \text{ RC}\} \arrow[r, leftrightarrow, "\Psi"] & \{\text{type } A_n^{(1)} \text{ RC}\} \times \mathrm{LR}
\end{tikzcd}$$

Here "path" stands for the highest weight elements of $\bigotimes_i B^{r_i,s_i}$ and "RC" stands for the rigged configurations. Our bijection Ψ, that exists when the rank is large, corresponds to the bottom edge. Bijection (a), which we call type $A_n^{(1)}$ RC-bijection, is established in full generality in the papers [11, 12, 13]. Algorithms for bijection (b) are known explicitly in the following cases:

- $(B^{1,1})^{\otimes L}$ type paths for all nonexceptional algebras $\mathfrak{g}$ [24],
- $\bigotimes B^{r_i,1}$ type paths for $\mathfrak{g} = D_n^{(1)}$ [26],
- $\bigotimes B^{1,s_i}$ type paths for all nonexceptional algebras $\mathfrak{g}$ [27].

For the cases that the bijection (b) is established, our bijection Ψ thus gives the combinatorial bijection between the set of type $\mathfrak{g}$ paths and the product set of the type $A_n^{(1)}$ paths and the Littlewood–Richardson skew tableaux. We refer to [28] for related combinatorial problems.

We expect that the bijection (b) exists in full generality even without the large rank hypothesis. It will give a combinatorial proof of the $X = M$ conjecture. Furthermore, it also gives an essential tool for the study of a tropical integrable system known as the box-ball system (see e.g., [3, 5, 8]) which is a soliton system defined on the paths and is supposed to give a physical background for the $X = M$ identities. More precisely, the rigged configurations are identified with the complete set of the action and angle variables for the type $A_n^{(1)}$ box-ball system [15] (see [17] for a generalization to type $D_n^{(1)}$). It is also interesting to note that by introducing a tropical analogue of the tau functions in terms of the charge $c(\nu^\bullet, J^\bullet)$, the initial value problem for the type $A_n^{(1)}$ box-ball systems is solved in [16, 25]. Therefore the construction of the bijection (b) in full generality will be a very important future problem.

Organization of the present paper is as follows. In section 2 we recall minimal facts about the rigged configurations. In section 3 we define the algorithm. In section 4 we describe main properties of the algorithm. Section 5 is devoted for a nontrivial example.

This paper is an extended abstract of the original paper [22].

2 Stable rigged configurations

In order to define the algorithm, we prepare minimal facts from the rigged configurations for nonexceptional algebras of rank n. The rigged configurations are the following set of data: $\mathbf{L} = (L_i^{(a)})_{a\in I_0, i\in\mathbb{Z}_{>0}}$ that appears in introduction, together with

$$(\nu^\bullet, J^\bullet) = \{(\nu^{(1)}, J^{(1)}), (\nu^{(2)}, J^{(2)}), \cdots (\nu^{(n)}, J^{(n)})\}$$

where $\nu^{(a)} = (\nu_1^{(a)}, \nu_2^{(a)}, \ldots, \nu_{l^a}^{(a)})$ $(1 \le a \le n)$ is positive integer sequence (called configuration) and $J^{(a)} = (J_1^{(a)}, J_2^{(a)}, \ldots, J_{l^a}^{(a)})$ is integer sequence associated with each entry of $\nu^{(a)}$ (called riggings). Here we have to impose some conditions on these sets that depend on the specific choice of the algebra. However we do not need to prepare full version of the definition. In fact, it is shown in [22] that the rigged configurations for algebras of sufficiently large rank takes a simplified structure. Let us assume that the rank n of the algebra is very large. Then we can show that there is some large $N (\ll n)$ such that there exists $N' \ll N$ with the property $\nu^{(N')} = \nu^{(N'+1)} = \cdots = \nu^{(N)}$ holds. According to [22], we can ignore details of $(\nu^{(a)}, J^{(a)})$ $(N < a)$ and we have to only think about the rest of the rigged configurations. The vacancy number $p_i^{(a)}$ $(a \le N)$ is defined as

$$p_i^{(a)} = \sum_{k\in\mathbb{Z}_{>0}} L_k^{(a)} \min(i,k) + Q_i^{(a-1)} - 2Q_i^{(a)} + Q_i^{(a+1)}$$

where $Q_i^{(a)} = \sum_j \min(i, \nu_j^{(a)})$. In our setting, we have $\nu^{(a)} \in \mathcal{P}^\diamondsuit$ and $p_i^{(a)} = 0$ for $N \le a$. For $a \le N$, we require the following inequalities:

$$0 \le J_i^{(a)} \le p_{\nu_i^{(a)}}^{(a)}, \qquad (\forall a, i).$$

We call such $(\nu^\bullet, J^\bullet)$ under large rank limit *stable rigged configurations*. In the original arguments in [22], we make precise estimates on the rank n such that our procedure is possible. In the present note, we will rely on the result of such estimate and forget about any technical difficulties related with $(\nu^{(a)}, J^{(a)})$ for $a \approx n$. For a stable rigged configuration $(\nu^\bullet, J^\bullet)$ one can define the weight λ as

$$\lambda_a = \sum_{b \geq a, i \in \mathbb{Z}_{>0}} iL_i^{(b)} + |\nu^{(a-1)}| - |\nu^{(a)}|,$$

where λ_a is the length of the a-th row of λ when identified with the Young diagram. For $(\nu^\bullet, J^\bullet)$ we denote it by $\mathrm{wt}\,(\nu^\bullet, J^\bullet)$. The stable rigged configurations depend only on the choice of $\diamondsuit$ (if we ignore the information near n). We will denote the set of the stable rigged configurations as $\mathrm{RC}^\diamondsuit(\lambda, \mathbf{L})$.

3 The bijection

The goal of this section is to give definitions of our main algorithms Ψ and its inverse $\tilde{\Psi}$. Roughly speaking, the algorithms consist of two parts: the one is box removing or adding procedure on the rigged configurations, and the other one is to create a kind of recording tableau T which eventually generates the LR tableaux. We will divide the definition according to this distinction. During definition, we choose a large integer N as in the previous section. We remark that more precise estimate on the rank n is possible (see [22]).

Definition 1 *The map δ_l*

$$\delta_l : (\nu^\bullet, J^\bullet) \longmapsto \{(\nu'^\bullet, J'^\bullet), k\},$$

is defined by the following algorithm. Here l is one of lengths of rows of $\nu^{(N)}$.

(i) Choose one of length l rows of $\nu^{(N)}$. Then choose rows of $\nu^{(a)}$ $(a < N)$ recursively as follows. Suppose that we have chosen a row of $\nu^{(a)}$. Find the shortest singular rows of $\nu^{(a-1)}$ whose length is equal to or longer than the chosen row of $\nu^{(a)}$. If there is no such row, set $k = a$ and stop. Otherwise choose one of such singular rows and continue. If the process does not stop, set $k = 1$.

(ii) $\nu'^\bullet$ is obtained by removing one box from the right end of each chosen row at Step (i).

(iii) The new riggings $J'^\bullet$ are defined as follows. For the rows that are not changed in Step (ii), take the same riggings as before. Otherwise set the new riggings equal to the corresponding vacancy numbers computed by using $\nu'^\bullet$.

Definition 2 *The map Ψ*

$$\Psi : (\nu^\bullet, J^\bullet) \longmapsto \{(\nu'^\bullet, J'^\bullet), T\}$$

is defined as follows. As the initial condition, set T = Young diagram that represents the weight of $(\nu^\bullet, J^\bullet)$. Let h_i denote the height of the i-th column (counting from left) of the partition $\nu^{(N)}$ and let $l = \nu_1^{(N)}$.

(i) We will apply δ_l for h_l times. Each time when we apply δ_l, we recursively redefine $(\nu^\bullet, J^\bullet)$ and T as follows. Assume that we have done δ_l^{i-1} and obtained $\{(\nu^\bullet, J^\bullet), T\}$. Let us apply δ_l one more time:

$$\delta_l : (\nu^\bullet, J^\bullet) \longmapsto \{(\nu'^\bullet, J'^\bullet), k\},$$

Using the output, do the following. Define new $(\nu^\bullet, J^\bullet)$ to be $(\nu'^\bullet, J'^\bullet)$. Define new T by putting i on the right of the k-th row of the previous T.

(ii) *Recursively apply* $\delta_{l-1}^{h_{l-1}}, \ldots, \delta_2^{h_2}, \delta_1^{h_1}$ *by the same procedure as in Step (i). Then the final outputs* $(\nu'^{\bullet}, J'^{\bullet})$ *and* T *give the image of* Ψ.

Now we are going to give the description of the algorithm $\tilde{\Psi}$ which will be shown to be the inverse of Ψ. Again we shall forget about the procedures near n (ignore information for $N < a$).

Definition 3 *The map* $\tilde{\delta}_k$

$$\tilde{\delta}_k : (\nu^{\bullet}, J^{\bullet}) \longmapsto (\nu'^{\bullet}, J'^{\bullet})$$

is defined by the following algorithm. Here the integer k *should satisfy* $k \leq N$.

(i) *Starting from* $\nu^{(k)}$, *choose rows of* $\nu^{(a)}$ $(k < a)$ *recursively as follows. To initialize the process, let us tentatively assume that we have chosen an infinitely long row of* $\nu^{(k-1)}$. *Suppose that we have chosen a row of* $\nu^{(a-1)}$. *Find the longest singular rows of* $\nu^{(a)}$ *whose length does not exceed the length of the chosen row of* $\nu^{(a-1)}$. *If there is no such row, suppose that we have chosen a length 0 row of* $\nu^{(a)}$ *and continue. Otherwise choose one of such singular rows and continue.*

(ii) $\nu'^{\bullet}$ *is obtained by adding one box to each chosen row in Step (i). If the length of the chosen row is 0, create a new row at the bottom of the corresponding partition* $\nu^{(a)}$.

(iv) *The new riggings* $J'^{\bullet}$ *are defined as follows. For the rows that are not changed in Step (ii), take the same riggings as before. Otherwise set the new riggings equal to the corresponding vacancy numbers computed by using* $\nu'^{\bullet}$.

Definition 4 *The map* $\tilde{\Psi}$

$$\tilde{\Psi} : \{(\nu^{\bullet}, J^{\bullet}), T\} \longmapsto (\nu'^{\bullet}, J'^{\bullet})$$

is defined as follows.

(i) *Let* h_1 *be the largest integer contained in* T. *For* h_1 *do the following procedure. Among all* h_1, *find the rightmost one and fix. Repeat the same procedure for* $h_1 - 1, h_1 - 2, \ldots, 2, 1$. *Call these fixed* h_1 *integers of* T *the first group. Remove all members of the first group from* T *and do the same procedure for the new* T. *Call the integers that are fixed this time the second group. Repeat the same procedure recursively until all integers of* T *are grouped. Let the total number of groups be* l, *the cardinality of the* i*-th group be* h_i *and the position of the letter* j *contained in the* i*-th group be the* $k_{i,j}$*-th row (counting from top of* T*).*

(ii) *The output of* $\tilde{\Psi}$ *is defined as follows:*

$$(\nu'^{\bullet}, J'^{\bullet}) = \tilde{\delta}_{k_{l,1}} \cdot \cdots \cdot \tilde{\delta}_{k_{2,1}} \tilde{\delta}_{k_{2,2}} \cdots \tilde{\delta}_{k_{2,h_2}} \tilde{\delta}_{k_{1,1}} \tilde{\delta}_{k_{1,2}} \cdots \tilde{\delta}_{k_{1,h_1}} (\nu^{\bullet}, J^{\bullet}).$$

4 Main properties

The crux of the combinatorics is contained in the following two theorems on the well-definedness of both maps Ψ and $\tilde{\Psi}$, which are proved in [22].

Theorem 1 *Assume that* $(\nu^\bullet, J^\bullet) \in \mathrm{RC}^\diamondsuit$. *Suppose that the rank* n *is sufficiently large. Then the map* Ψ

$$\Psi : (\nu^\bullet, J^\bullet) \longmapsto \{(\nu'^\bullet, J'^\bullet), T\}$$

is well-defined. More precisely, $(\nu'^\bullet, J'^\bullet) \in \mathrm{RC}^\varnothing$ *and the LR tableau* $T \in LR^\eta_{\lambda\mu}$ *satisfy the following properties:*

$$\lambda = \mathrm{wt}(\nu^\bullet, J^\bullet), \qquad \mu = \nu^{(N)}, \qquad \eta = \mathrm{wt}(\nu'^\bullet, J'^\bullet).$$

Theorem 2 *Assume that* $(\nu^\bullet, J^\bullet) \in \mathrm{RC}^\varnothing$ *and* T *is the LR tableau that satisfy the following three properties:* $T \in LR^\eta_{\lambda\mu}$ *where* $\lambda, \mu \in \mathcal{P}^\diamondsuit$ *and* $\eta = \mathrm{wt}(\nu^\bullet, J^\bullet)$. *Then the map* $\tilde{\Psi}$;

$$\tilde{\Psi} : \{(\nu^\bullet, J^\bullet), T\} \longmapsto (\nu'^\bullet, J'^\bullet),$$

is well-defined. More precisely, we have $(\nu'^\bullet, J'^\bullet) \in \mathrm{RC}^\diamondsuit$, $\mathrm{wt}(\nu'^\bullet, J'^\bullet) = \lambda$ *and* $\nu'^{(N)} = \mu$.

By construction, δ and $\tilde{\delta}$ are mutually inverse procedure. Therefore the above theorems imply the following main theorem.

Theorem 3 *Assume that* $(\nu^\bullet, J^\bullet) \in \mathrm{RC}^\diamondsuit$. *Suppose that the rank* n *is sufficiently large. Then* Ψ *gives a bijection between the* $\mathrm{RC}^\diamondsuit$ *and the product set of* $\mathrm{RC}^\varnothing$ *and the LR tableaux as follows:*

$$\begin{aligned}\Psi :& (\nu^\bullet, J^\bullet) \longmapsto \{(\nu'^\bullet, J'^\bullet), T\},\\ &(\nu^\bullet, J^\bullet) \in \mathrm{RC}^\diamondsuit(\lambda, \mathbf{L}), \quad \{(\nu'^\bullet, J'^\bullet), T\} \in \mathrm{RC}^\varnothing(\eta, \mathbf{L}) \times LR^\eta_{\lambda\mu},\end{aligned}$$

where λ, μ, η *satisfy the following properties:*

$$\lambda = \mathrm{wt}(\nu^\bullet, J^\bullet), \qquad \mu = \nu^{(N)}, \qquad \eta = \mathrm{wt}(\nu'^\bullet, J'^\bullet).$$

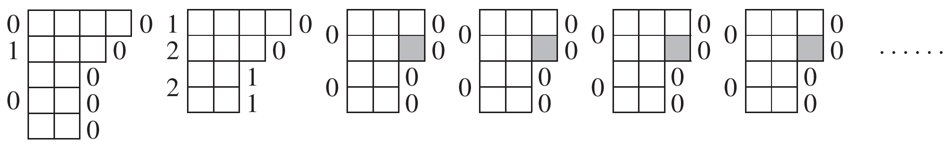
The inverse procedure is given by $\Psi^{-1} = \tilde{\Psi}$.

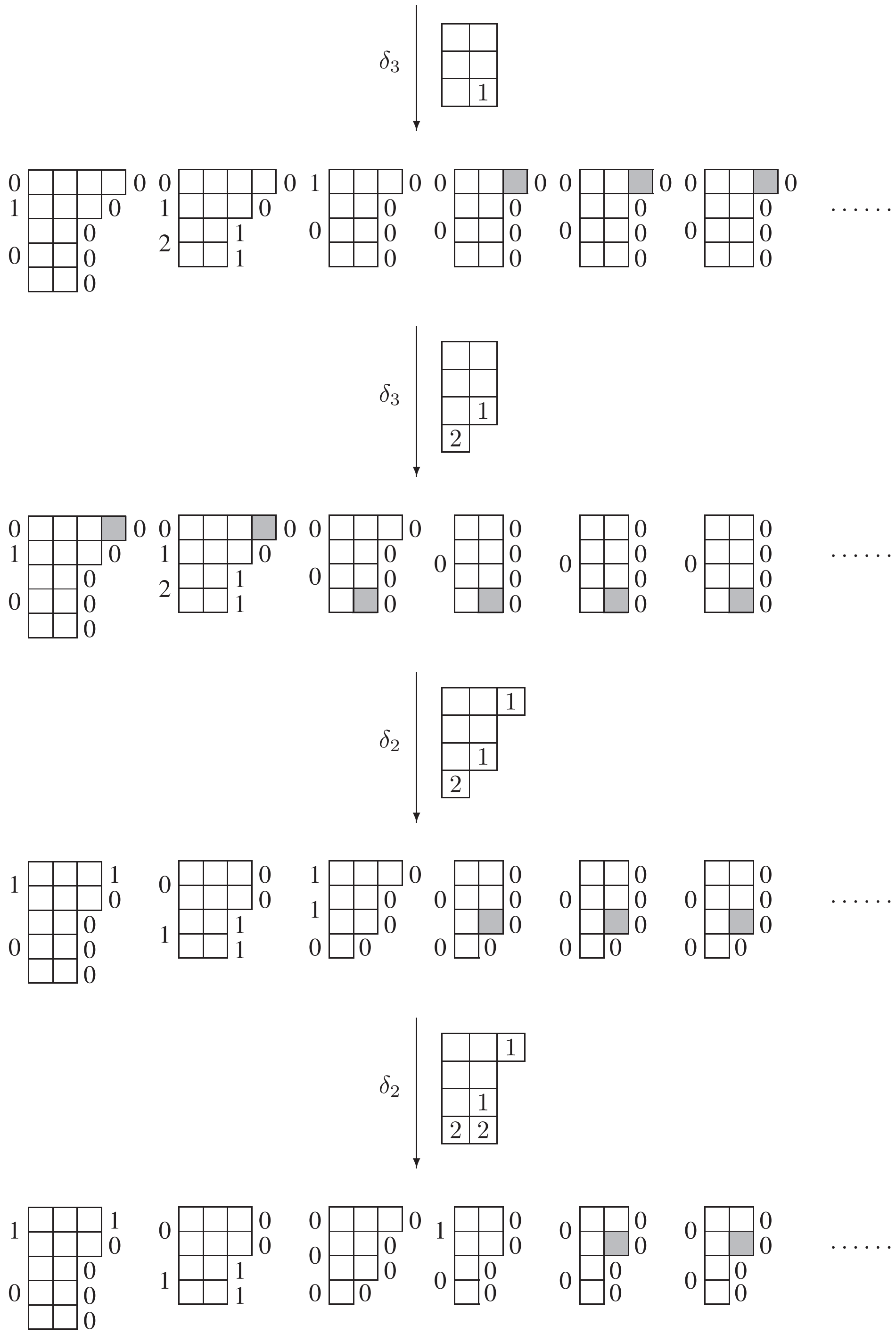
5 Example

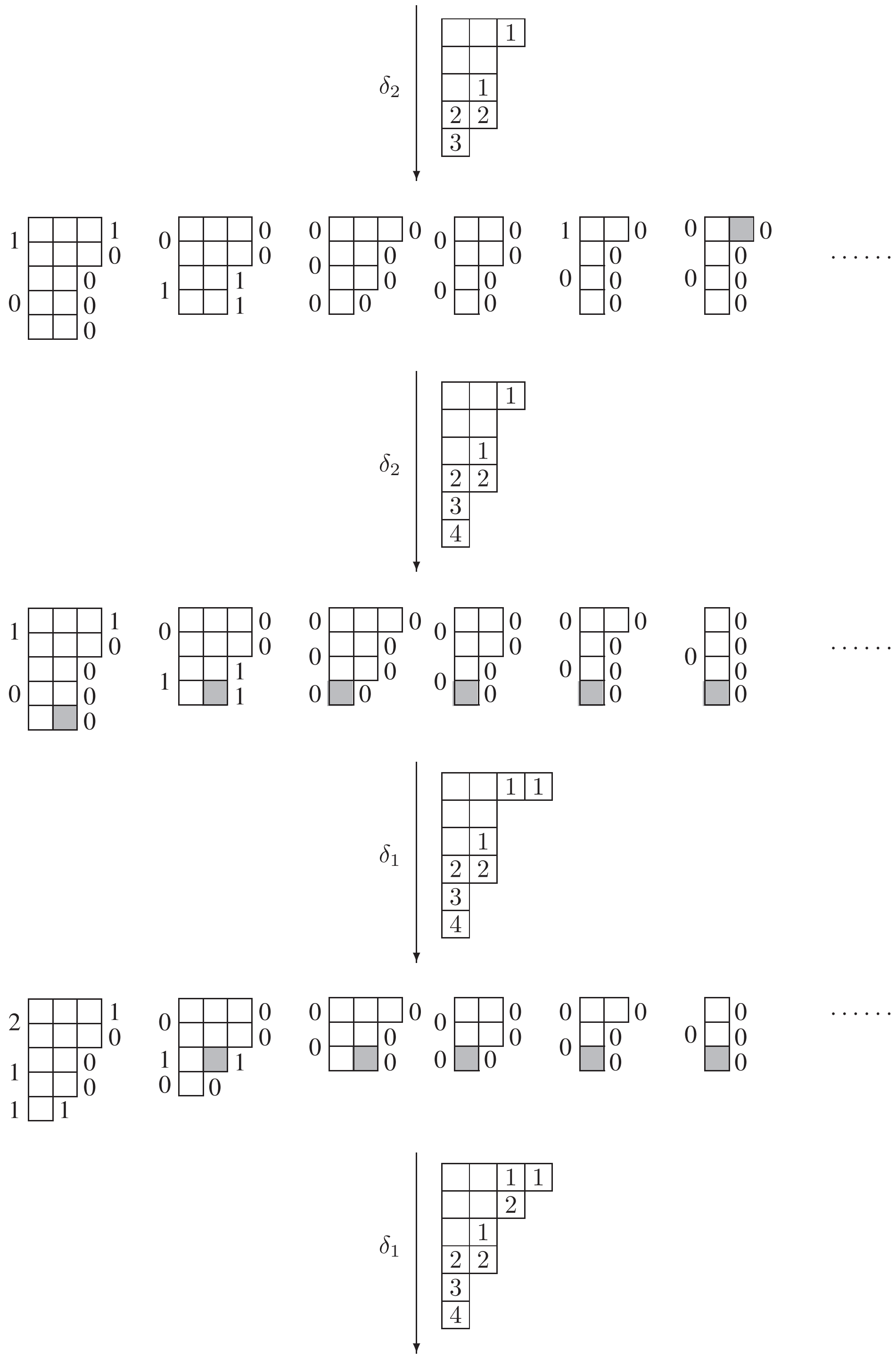
Let us consider the special case of the bijection Ψ where the bijection [24, 27] between the rigged configurations and the tensor products of crystals is also available. Consider the following element of the tensor product $(B^{1,3})^{\otimes 3} \otimes (B^{1,2})^{\otimes 2} \otimes (B^{1,1})^{\otimes 2}$ of type $D^{(1)}_n$ $(n \geq 8)$ crystals:

$$p = \boxed{1\,|\,1\,|\,1} \otimes \boxed{2\,|\,\bar{1}\,|\,\bar{1}} \otimes \boxed{1\,|\,2\,|\,\bar{2}} \otimes \boxed{2\,|\,3} \otimes \boxed{2\,|\,\bar{2}} \otimes \boxed{\bar{2}} \otimes \boxed{2}.$$

Due to Theorem 8.6 of [27] all the isomorphic elements under the combinatorial R-matrices correspond to the same rigged configuration. Then the map Ψ proceeds as follows. In the following diagrams, the first rigged configuration corresponds to the above p. Here, we put the vacancy numbers (resp. riggings) on the left (resp. right) of the corresponding rows. The gray boxes represent the boxes to be removed by each δ indicated on the left of each arrow. The corresponding recording tableau T is given on the right of each arrow.



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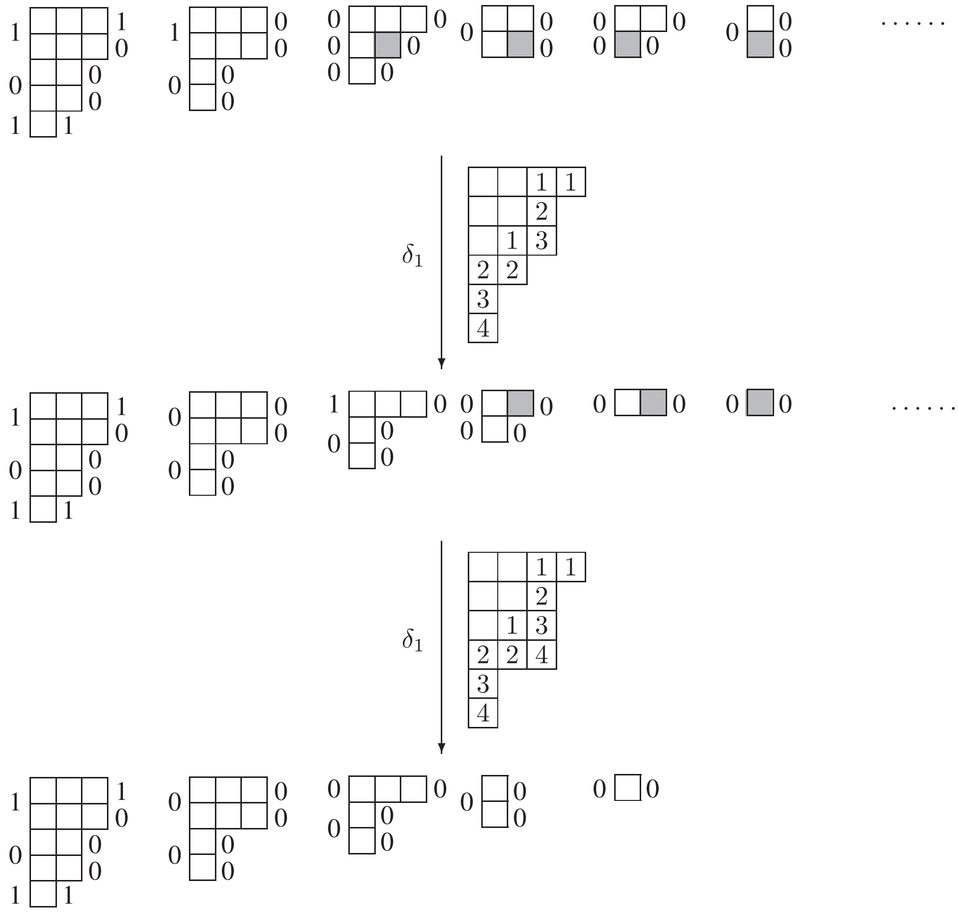

The final rigged configuration and T of the above diagrams give the image of Ψ. Under the bijection [12] the final rigged configuration corresponds to the following element:

$$p' = \boxed{1}\boxed{1}\boxed{1} \otimes \boxed{2}\boxed{2}\boxed{2} \otimes \boxed{1}\boxed{3}\boxed{3} \otimes \boxed{4}\boxed{4} \otimes \boxed{3}\boxed{5} \otimes \boxed{4} \otimes \boxed{6}.$$

Remark 1 *As for an example of $\tilde{\Psi}$, one should read the above example in the reverse order. More precisely, reverse all arrows and apply $\tilde{\delta}_4$, $\tilde{\delta}_3$, $\tilde{\delta}_2$, $\tilde{\delta}_1$, $\tilde{\delta}_6$, $\tilde{\delta}_5$, $\tilde{\delta}_4$, $\tilde{\delta}_1$, $\tilde{\delta}_4$, $\tilde{\delta}_3$ in this order.*

Remark 2 *Let p and p' as in the example in this section and consider them as elements of $D_8^{(1)}$. If we apply the involution σ at Section 5.3 of [19], we have*

$$\sigma(p) = \boxed{\bar{8}}\boxed{\bar{8}}\boxed{\bar{8}} \otimes \boxed{8}\boxed{8}\boxed{\bar{7}} \otimes \boxed{6}\boxed{\bar{8}}\boxed{\bar{6}} \otimes \boxed{\bar{7}}\boxed{\bar{6}} \otimes \boxed{6}\boxed{\bar{6}} \otimes \boxed{7} \otimes \boxed{\bar{7}}.$$

Then p' coincides with the I_0-highest element corresponding to $\sigma(p)$. We expect that the same relation holds for arbitrary image of Ψ.

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