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Crystal energy via charge in types A and C

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Abstract. The Ram–Yip formula for Macdonald polynomials (at $t = 0$) provides a statistic which we call charge. In types A and C it can be defined on tensor products of Kashiwara–Nakashima single column crystals. In this paper we show that the charge is equal to the (negative of the) energy function on affine crystals. The algorithm for computing charge is much simpler than the recursive definition of energy in terms of the combinatorial R-matrix.

Résumé. La formule de Ram et Yip pour les polynômes de Macdonald (à $t = 0$) fournit une statistique que nous appelons la charge. Dans les types A et C, elle peut être définie sur les produits tensoriels des cristaux pour les colonnes de Kashiwara–Nakashima. Dans ce papier, nous montrons que la charge est égale à (l'opposé de) la fonction d'énergie sur cristaux affines. L'algorithme pour calculer la charge est bien plus simple que la définition récursive de l'énergie en fonction de la R-matrice combinatoire.

Keywords: affine crystals, energy function, charge, Kashiwara-Nakashima tableaux, Macdonald polynomials

1 Introduction

The energy function of affine crystals is an important grading used in one-dimensional configuration sums [7, 8] and generalized Kostka polynomials [31, 33, 34]. It is defined by the action of the affine Kashiwara crystal operators through a local combinatorial rule and the R-matrix.

From a computational perspective, the definition of the energy is not very efficient, as it involves a recursive definition of a local energy, and also the combinatorial R-matrix, for which not in all cases efficient algorithms exist. This leads us to the role of the charge statistic, which can be calculated very efficiently, as it only involves the detection of descents and the computation of arm lengths of cells in Young diagrams.

Charge was originally defined in type A by Lascoux and Schützenberger [14] as a statistic on words with partition content. It is calculated by enumerating certain cycles in the given word, see Section 3. Lascoux and Schützenberger showed that the charge can also be defined as the grading in the so-called cyclage graph, and used it to express combinatorially the Kostka–Foulkes polynomials, or Lusztig's q-analogue of weight multiplicities [23], based on their Morris recurrence. In type A, Nakayashiki and Yamada [25]

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analyzed the subtle combinatorial relationship between charge and the R-matrix, showing that the energy coincides with the charge. In [13] it was observed that the cyclage is related to the action of the crystal operator f_0 on a tensor product of type A columns (by the Kyoto path model, the latter can be identified with an affine Demazure crystal). Thus, the results of Lascoux–Schützenberger and Nakayashiki–Yamada are rederived in a more conceptual way. See also the work of Shimozono [33, 34] for a more extensive discussion of the combinatorics involved in [13], in the more general context of tensor products of type A Kirillov–Reshetikhin (KR) crystals of arbitrary rectangular shapes, as opposed to only column shapes. Charge for KR crystals of rectangular shape (or Littlewood–Richardson tableaux) was also defined in [31] using cyclage.

Lecouvey [15, 16] extended two approaches to the Lascoux–Schützenberger charge, namely cyclage and catabolism, to types B, C, and D. He thus defined a charge statistic on the corresponding Kashiwara–Nakashima (KN) tableaux [12]. But he was only able to relate his charge to the corresponding Kostka–Foulkes polynomials in very special cases, as the original idea of Lascoux–Schützenberger based on the Morris recurrence, which he pursued, has limited applicability in this case.

In this paper we use a charge statistic coming from the Ram–Yip formula [28] for Macdonald polynomials $P_\mu(x;q,t)$ of arbitrary type [24] at $t=0$. The terms in this formula correspond to certain chains of Weyl group elements which come from the alcove walk model (this was defined in [2, 20, 21], and was then developed in subsequent papers, including [28]). The statistic is defined on the mentioned chains, and describes the powers of q. In [18] it is shown that, in types A and C, the chains are in bijection with elements in a tensor product of KR crystals of the form $B^{k,1}$. It is also shown that, under this bijection, the above statistic can be translated into a statistic on the elements of the mentioned crystal, which we call charge. Thus, we have

$$P_\mu(x;q,0) = \sum_{b \in B^{\mu'_1,1} \otimes B^{\mu'_2,1} \otimes \dots} q^{\mathrm{charge}(b)} x^{\mathrm{wt}(b)}. \tag{1}$$

In type A, one can rewrite this formula as an expansion of the Macdonald P-polynomials in terms of Schur polynomials $s_\lambda(x)$

$$P_\mu(x;q,0) = \sum_\lambda K_{\lambda'\mu'}(q)\; s_\lambda(x)\,; \tag{2}$$

here $K_{\lambda'\mu'}(q)$ is the Kostka–Foulkes polynomial and λ' denotes the transpose of the partition λ. A generalization of (2) to simply-laced types was given in [9]; in types A and D, this result is sharpened in [30, Section 9.2] by replacing the Kostka–Foulkes polynomials with the corresponding one-dimensional configuration sums (which are generating functions for the energy). Both (1) in type A and (2) are expressed combinatorially in terms of the Lascoux–Schützenberger charge, whereas the type C charge given by (1) is a new statistic. It is worth noting that the main ingredient in these charge constructions is the so-called quantum Bruhat graph [1], which first arose in connection to Chevalley multiplication formulas for the quantum cohomology of flag varieties.

The goal of this paper is to show in an efficient, conceptual way that the charge in [18] coincides with the energy function on the corresponding tensor products of KR crystals. We focus on types A and C, and expect to extend this result to types B and D; for more details on the additional complexity in the latter types, we refer to [18, 22]. With M. Shimozono and S. Naito, we are also working on a uniform generalization to arbitrary types, based on the generalization of the alcove walk model in [19] and the statistic in the Ram–Yip formula mentioned above.

We use the recent reinterpretation in [30] of the (global) energy function as the affine grading on a tensor product of KR crystals under "Demazure arrows". In type A, KR crystals are perfect and hence, by the Kyoto path model [10, 11], can be realized as Demazure crystals. By the result of [30], "Demazure arrows" (see Definition 2.6) change energy by 1. Together with the result that charge is well-behaved under crystal operators, this proves the equality of energy and charge. For type C, we use the same approach, but in this case KR crystals are not perfect. There is still an embedding of a tensor product of KR crystals into an affine highest weight crystal (see Proposition 2.8) by analogy with the Kyoto path model, but now there are several highest weight components in the image, instead of just one. For each of these components we exhibit an explicit path from its highest weight (or ground state) to "type A elements" in the component, using only "Demazure arrows" (see Theorem 4.4). This additional result suffices to establish the equality of energy and charge in type C, based on the corresponding result in type A. Our main result can now be stated as follows.

Theorem 1.1 *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ be a tensor product of KR crystals in type $A_{n-1}^{(1)}$ or type $C_n^{(1)}$ with $r_N \geq r_{N-1} \geq \cdots \geq r_1 \geq 1$. Then for all $b \in B$ we have $D(b) = -\mathrm{charge}(b)$, where $D(b)$ is given in Definition* 2.4*, and* $\mathrm{charge}(b)$ *is defined in Section* 3*.*

From a theoretical point of view, the above result is not surprising due to work of Ion [9], which relates Macdonald polynomials at $t = 0$ and affine Demazure characters in simply-laced types. However, this result does not work in type C; in addition, it only gives an equality of polynomials (the generating functions for the statistics and the weights), not of individual terms.

To compare our work with the previous papers on charge and energy, let us first say that our results apply to arbitrary vertices in a tensor product of KN columns, not just to the highest weight elements (with respect to the nonzero arrows), that are used in the work involving Kostka–Foulkes polynomials. In type A, our approach via affine Demazure crystals comes closest to [13, 33, 34]. However, we do not use the so-called cyclage operation, which is based on the corresponding plactic relations, see [15]. These relations are the main cause of the complications in type C, in the work of Lecouvey [15, 16].

The paper is organized as follows. In Section 2 we review the necessary crystal theory and define the energy function. In Section 3 we give the definition of charge both in types A and C. The proof of Theorem 1.1 using the method of Demazure arrows is outlined in Section 4.

A long version of this paper containing all proofs has appeared [22].

2 Crystals and energy function

Crystal bases provide a combinatorial method to study representations of quantum algebras $U_q(\mathfrak{g})$. For a good review on crystal base theory see the book by Hong and Kang [6]. Here $\mathfrak{g}$ is a Lie algebra or affine Kac–Moody Lie algebra with index set I, weight lattice P, and simple roots α_i with $i \in I$. The set of dominant weights is denoted by P_+. For affine Kac–Moody (resp. finite Lie) algebras we denote the fundamental weights by Λ_i (resp. ω_i) for $i \in I$.

A $\mathfrak{g}$-*crystal* is a nonempty set B together with maps $e_i, f_i : B \to B \cup \{\emptyset\}$ for $i \in I$ and $\mathrm{wt} : B \to P$, where $f_i(x) = y$ if and only if $e_i(y) = x$. For $b \in B$, we set $\varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq \emptyset\}$, $\varphi_i(b) = \max\{k \mid f_i^k(b) \neq \emptyset\}$, $\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i$, and $\varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$. The beauty about crystal theory is that it is well-behaved with respect to taking tensor products. Let B_1 and B_2 be two $\mathfrak{g}$-crystals. As a set $B_1 \otimes B_2$ is the Cartesian product of the two sets. For $b = b_1 \otimes b_2 \in B_1 \otimes B_2$, the weight function is simply $\mathrm{wt}(b) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$. The crystal operators are given by $f_i(b_1 \otimes b_2) = f_i(b_1) \otimes b_2$ if $\varepsilon_i(b_1) \geq \varphi_i(b_2)$ and $b_1 \otimes f_i(b_2)$ otherwise. Similar formulas hold for $e_i(b)$.

A *highest weight crystal* $B(\lambda)$ of highest weight $\lambda \in P_+$ is a crystal with a unique element u_λ such that $e_i(u_\lambda) = \emptyset$ for all $i \in I$ and $\mathrm{wt}(u_\lambda) = \lambda$. On finite-dimensional highest weight crystals $B(\lambda)$ there exists an involution $S : B(\lambda) \to B(\lambda)$, called the *Lusztig involution*, which is a crystal isomorphism such that

$$S(f_i) = e_{i^*} \qquad \text{and} \qquad S(e_i) = f_{i^*}.$$

Here i^* is defined through the map $\alpha_i \mapsto \alpha_{i^*} := -w_0(\alpha_i)$ with w_0 the longest element in the Weyl group of $\mathfrak{g}$. Explicitly, we have $i^* = n - i$ for type A_{n-1} and $i^* = i$ for type C_n. Under S the highest weight element goes to the lowest weight element.

2.1 Kashiwara–Nakashima columns for type C

Kashiwara and Nakashima [12] developed a general tableau model for finite-dimensional highest weight crystals for all non-exceptional Lie algebras $\mathfrak{g}$. For type C_n, the Kashiwara–Nakashima (KN) columns [12] of height k index the vertices of the fundamental representation $V(\omega_k)$ of the symplectic algebra $\mathfrak{sp}_{2n}(\mathbb{C})$. These columns are filled with entries in $[\overline{n}] := \{1 < 2 < \cdots < n < \overline{n} < \overline{n-1} < \cdots < \overline{1}\}$.

Definition 2.1 *A column-strict filling $b = b(1) \dots b(k)$ with entries in $[\overline{n}]$ is a KN column if there is no pair $(z, \overline{z})$ of letters in b such that $z = b(p)$, $\overline{z} = b(q)$, $q - p \leq k - z$.*

We use a different definition of KN columns, which is equivalent to the one above by [32].

Definition 2.2 *Let b be a column and $I = \{z_1 > \cdots > z_r\}$ the set of unbarred letters z such that the pair $(z, \overline{z})$ occurs in b. The column b can be split when there exists a set of r unbarred letters $J = \{t_1 > \cdots > t_r\} \subset [n]$ such that:*

- *t_1 is the greatest letter in $[n]$ satisfying: $t_1 < z_1$, $t_1 \notin b$, and $\overline{t_1} \notin b$,*
- *for $i = 2, \dots, r$, the letter t_i is greatest in $[n]$ satisfying $t_i < \min(t_{i-1}, z_i)$, $t_i \notin b$, and $\overline{t_i} \notin b$.*

In this case we write the following, where the pair $b^L b^R$ is called a split column:

- *b^R for the column obtained by changing $\overline{z_i}$ into $\overline{t_i}$ in b for each letter $z_i \in I$, and by reordering if necessary,*
- *b^L for the column obtained by changing z_i into t_i in b for each letter $z_i \in I$, and by reordering if necessary.*

Example 2.3 The following is a KN column of height 5 in type C_n for $n \geq 5$, together with the corresponding split column, where we use the fact that $\{z_1 > z_2\} = \{5 > 4\}$, so $\{t_1 > t_2\} = \{2 > 1\}$:

$$b = \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline \overline{5} \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array}\,, \qquad b^L b^R = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline \overline{5} & \overline{3} \\ \hline \overline{4} & \overline{2} \\ \hline \overline{3} & \overline{1} \\ \hline \end{array}\,.$$

2.2 Kirillov–Reshetikhin crystals

For the definition of the crystal energy function, we need to endow the KN columns with an affine crystal structure. These finite-dimensional affine crystals are called *Kirillov–Reshetikhin (KR) crystals*. Combinatorial models for all non-exceptional types were provided in [3].

Here we only describe the KR crystals $B^{r,1}$ for types $A_{n-1}^{(1)}$ and $C_n^{(1)}$, where $r \in \{1,2,\ldots,n-1\}$ and $r \in \{1,2,\ldots,n\}$, respectively. As a classical type A_{n-1} (resp. C_n) crystal, the KR crystal is isomorphic to $B^{r,1} \cong B(\omega_r)$.

The crystal operator f_0 is given as follows. Let $b \in B^{k,1}$, represented by a one-column KN tableau. In type A_{n-1}, if b contains the letter n and no 1, $f_0(b)$ is the obtained from b by removing n and adding 1 to the column, leaving all letters in strictly increasing order. Otherwise $f_0(b) = \emptyset$. In type C_n, if b contains the letter $\overline{1}$, then $f_0(b)$ is obtained from b by removing the $\overline{1}$ and adding the letter 1, arranging all letters again in strictly increasing order. Otherwise $f_0(b) = \emptyset$. Note that if b contains $\overline{1}$, then it cannot contain 1 by the KN condition of Definition 2.1.

2.3 The D function

Let B_1, B_2 be two affine crystals with generators v_1 and v_2, respectively, such that $B_1 \otimes B_2$ is connected and $v_1 \otimes v_2$ lies in a one-dimensional weight space. By [17, Proposition 3.8], this holds for any two KR crystals. The generator $v_{r,s}$ for the KR crystal $B^{r,s}$ is the unique element of classical weight $s\omega_r$.

The *combinatorial R-matrix* [10, Section 4] is the unique crystal isomorphism $\sigma : B_2 \otimes B_1 \to B_1 \otimes B_2$. By weight considerations, this must satisfy $\sigma(v_2 \otimes v_1) = v_1 \otimes v_2$. In [10] and [27, Theorem 2.4], the *local energy function $H = H_{B_2,B_1} : B_2 \otimes B_1 \to \mathbb{Z}$* is defined recursively in terms of the crystal operators e_i and the combinatorial R-matrix.

Definition 2.4 *For $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A_{n-1}^{(1)}$ or $C_n^{(1)}$, set $H_{j,i}^R := H_i \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{j-1}$ and $H_{j,i}^L := H_{j-1} \sigma_{j-2} \sigma_{j-3} \cdots \sigma_i$, where σ_j and H_j act on the j-th and $(j+1)$-st tensor factors. We define a right and left energy function $D_B^R, D_B^L : B \to \mathbb{Z}$ as*

$$D_B^R := \sum_{N \geq j > i \geq 1} H_{j,i}^R \quad \text{and} \quad D_B^L := \sum_{N \geq j > i \geq 1} H_{j,i}^L. \tag{3}$$

We focus on the energy $D_B := D_B^L$ and, when there is no confusion, we shorten D_B to simply D.

There is a precise relationship between D^R and D^L using the Lusztig involution. To state it, let us introduce the map $\tau : B_N \otimes \cdots \otimes B_1 \to B_1 \otimes \cdots \otimes B_N$ with $\tau(b_N \otimes \cdots \otimes b_1) = S(b_1) \otimes \cdots \otimes S(b_N)$.

For types $A_{n-1}^{(1)}$ and $C_n^{(1)}$ and $B_i = B^{r_i,1}$, the KR crystal B_i is connected as a classical crystal and under S the classically highest weight element u_i^{highest} maps to the lowest weight element u_i^{lowest}. It is not hard to show from the explicit description of S, e_0 and f_0 in this case, that the following relations hold:

$$f_0 \circ S = S \circ e_0 \qquad \text{and} \qquad f_0 \circ \tau = \tau \circ e_0.$$

This shows in particular that the crystal commutor of Henriques and Kamnitzer [5] is lifted to an affine crystal isomorphism in these cases and hence must coincide with the combinatorial R-matrix σ.

Proposition 2.5 *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A_{n-1}^{(1)}$ or $C_n^{(1)}$ and $b \in B$. Then $D_B^R(b) = D_B^L(\tau(b))$.*

2.4 D energy as affine grading

As suggested in [26, Section 2.5] and proven in [30], the energy D^R is the same as the affine degree grading in the associated highest weight affine crystals up to an overall shift. We will explain this now since it plays a crucial role in the proof of the equality between charge and energy.

We begin with the definition of "Demazure arrows". For this we need constants c_r for $r \in I \setminus \{0\}$ as for example defined in [4]. In the cases of concern to us here, we have $c_r = 1$ for all r in type $A_{n-1}^{(1)}$ and $c_r = 2$ for $1 \le r < n$ and $c_n = 1$ in type $C_n^{(1)}$.

Definition 2.6 *Let $B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$ be a tensor product of KR crystals and fix an integer ℓ such that $\ell \ge \lceil s_k/c_k \rceil$ for all $1 \le k \le N$. We call such a tensor product a composite KR crystal of level bounded by ℓ. An arrow f_i is called an ℓ-Demazure arrow on $b \in B$ if $\varphi_i(b) > 0$ and either $i \in I \setminus \{0\}$ or $i = 0$ and $\varepsilon_0(b) \ge \ell$.*

In the setting of this paper, we are only concerned with tensor products of types $A_{n-1}^{(1)}$ and $C_n^{(1)}$ of the form $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$. In this case one can pick $\ell = 1$ and a *Demazure arrow* for B is a 1-Demazure arrow.

Lemma 2.7 *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A_{n-1}^{(1)}$ or $C_n^{(1)}$ and $b \in B$. Then*
1. *$\varepsilon_0(b) \ge 1$ implies $D^R(f_0(b)) = D^R(b) + 1$;*
2. *$\varphi_0(b) \ge 1$ implies $D^L(e_0(b)) = D^L(b) + 1$.*

The proof of the following result essentially appeared in [10, Proof of Theorem 4.4.1] and was spelled out in this precise form in [30, Proposition 8.1]. Here $P_\ell^+ = \{\lambda \in P^+ \mid \mathrm{lev}(\lambda) = \ell\}$, where $\mathrm{lev}(\lambda) := \lambda(c)$ is the level of λ and c is the central element $c = \sum_{i \in I} a_i^\vee \alpha_i^\vee$.

Proposition 2.8 *For B a composite KR crystal of level bounded by ℓ, $B \otimes B(\ell\Lambda_0) \cong \bigoplus_{\Lambda'} B(\Lambda')$, where the sum is over a finite collection of (not necessarily distinct) $\Lambda' \in P_\ell^+$.*

In Section 4.1, we discuss the implications of this proposition in more detail for type $C_n^{(1)}$ and $\ell = 1$.

3 The charge construction

3.1 The classical charge

Let us start by recalling the construction of the classical charge of a word, which is due to Lascoux and Schützenberger [14]. Assume that w is a word with letters in the alphabet $[n] := \{1, \ldots, n\}$ which has partition content, i.e., the number of j's is greater than or equal to the number of $j+1$'s, for each $j = 1, \ldots, n-1$. The statistic $\mathrm{charge}(w)$ is calculated as a sum based on the following algorithm. Scan the word starting from its right end, and select the numbers $1, 2, \ldots$ in this order, up to the largest possible k. We always pick the first available entry $j+1$ to the left of the previous entry j. Whenever there is no such entry, we pick the rightmost entry $j+1$, so we start scanning the word from its right end once again; in this case, we also add $k-j$ to the sum that computes $\mathrm{charge}(w)$. At the end of this process, we remove the selected numbers and repeat the whole procedure until the word becomes empty.

Example 3.1 Consider the word $w = 11\mathbf{3}\mathbf{2}2\mathbf{1}\mathbf{4}323$, where the first group of selected numbers is shown in bold. The corresponding contribution to the charge is 1. After removing the bold numbers and another round of selections (again shown in bold), we have $1\mathbf{1}2\mathbf{3}\mathbf{2}3$, so the contribution to the charge is 2. We are left with the word 123, whose contribution to the charge is $2+1 = 3$. So $\mathrm{charge}(w) = 1+2+(2+1) = 6$.

We now reinterpret the classical charge as a statistic on a tensor product of type $A_{n-1}^{(1)}$ KR crystals. Such a crystal indexed by a column of height k is traditionally denoted $B^{k,1}$, and its vertices are indexed by increasing fillings of the mentioned column with integers in $[n]$. Given a partition μ (i.e., a dominant weight in the root system), let $B_\mu := \bigotimes_{i=1}^{\mu_1} B^{\mu'_i,1}$, where μ' is the conjugate partition to μ. This is simply the set of column-strict fillings of the Young diagram μ with integers in $[n]$. Note that unlike in Section 2, the tensor factors are ordered in weakly decreasing order.

Fix a filling b in B_μ written as a concatenation of columns $b_1 \dots b_{\mu_1}$. We attach to it a filling $c :=$ circ-ord$(b) = c_1 \dots c_{\mu_1}$ according to the following algorithm, which is based on the circular order $\prec_i$ on $[n]$ starting at i, namely $i \prec_i i+1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i-1$.

Algorithm 3.2
let $c_1 := b_1$;
for j *from* 2 *to* μ_1 *do*
  for i *from* 1 *to* μ'_j *do*
    let $c_j(i) := \min\left(b_j \setminus \{c_j(1), \dots, c_j(i-1)\},\ \prec_{c_{j-1}(i)}\right)$
  end do;
end do;
return $c := c_1 \dots c_{\mu_1}$.

Example 3.3 Algorithm 3.2 constructs the filling c from the filling b below. The bold entries in c are only relevant in Example 3.5 below:

$$b = \begin{array}{|c|c|c|c|}\hline 3&2&1&2\\\hline 5&3&2\\\cline{1-3} 6&4&4\\\cline{1-3}\end{array} \qquad \text{and} \qquad c = \begin{array}{|c|c|c|c|}\hline 3&3&\mathbf{4}&2\\\hline \mathbf{5}&2&2\\\cline{1-3} \mathbf{6}&\mathbf{4}&1\\\cline{1-3}\end{array}\ . \tag{4}$$

We introduce some terminology in order to reinterpret the classical charge in terms of a statistic on B_μ. Given the considered filling b in B_μ, we define its *charge word* as the biword $\mathrm{cw}(b)$ containing a biletter $\binom{k}{j}$ for each entry k in the column b_j of b. We order the biletters in the decreasing order of the k's, and for equal k's, in the decreasing order of j's. The obtained word formed by the lower letters j will be denoted by $\mathrm{cw}_2(b)$. We refer to Example 3.5 for an illustration of the charge word. On the other hand, given the filling $c = c_1 \dots c_{\mu_1}$ constructed by Algorithm 3.2, we say that the cell γ in column c_j and row i is a descent if $c_j(i) > c_{j+1}(i)$, assuming that $c_{j+1}(i)$ is defined. Let $\mathrm{Des}(c)$ denote the set of descents in c. As usual, we define the arm length $\mathrm{arm}(\gamma)$ of a cell γ as the number of cells to its right.

It is not hard to see that Algorithm 3.2 for constructing c from b translates precisely into the selection algorithm which computes $\mathrm{charge}(\mathrm{cw}_2(b))$. More precisely, consider the ith sequence $1, 2, \dots$ extracted from $\mathrm{cw}_2(b)$ (which turns out to have length μ_i), and the letter j in this sequence; then the top letter paired with the mentioned letter j in $\mathrm{cw}(b)$ is precisely the entry $c_j(i)$ in row i and column j of the filling c. In particular, the steps to the right in the ith iteration of the charge computation correspond precisely to the descents in the ith row of c, while the corresponding charge contributions and arm lengths coincide. We conclude that

$$\sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \mathrm{charge}(\mathrm{cw}_2(b))\,. \tag{5}$$

For simplicity, we set $\mathrm{charge}(b) := \mathrm{charge}(\mathrm{cw}_2(b))$.

Remark 3.4 In [18] we showed that the charge statistic on B_μ can be derived from the Ram–Yip formula [28] for the corresponding Macdonald polynomial at $t = 0$. In fact, we showed that Algorithm 3.2 is closely related to the corresponding quantum Bruhat graph (see, e.g., [1]). So we can conclude that this graph explains the charge construction itself. The mentioned idea was extended to type C, and it led to the definition of a type C charge, that we describe in Section 3.2.

Example 3.5 Note that $\mathrm{cw}_2(b)$ for b in Example 3.3 is precisely the word w in Example 3.1. In fact, the full biword $\mathrm{cw}(b)$ is shown below, using the order on the biletters specified above. The index attached to a lower letter is the number of the iteration in which the given letter is selected in the process of computing $\mathrm{charge}(b)$:

$$\mathrm{cw}(b) = \left(\begin{array}{cccccccccc} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1_3 & 1_2 & 3_1 & 2_3 & 2_1 & 1_1 & 4_1 & 3_2 & 2_2 & 3_3 \end{array} \right) .$$

One can note the parallel between the mentioned selection process and the construction of c from b in Example 3.3. The entries in the cells of $\mathrm{Des}(c)$ are shown in bold in (4).

3.2 The type C charge

In this section we recall from [18] the construction of the type C charge. We start by fixing a dominant weight μ in the root system of type C_n. Let $B_\mu := \bigotimes_{i=1}^{\mu_1} B^{\mu'_i,1}$, where $B^{k,1}$ is the type $C_n^{(1)}$ KR crystal indexed by a column of height k. Note that B_μ is the set of fillings $b = b_1 \ldots b_{\mu_1}$ of the shape μ with integers in $[\overline{n}]$ whose columns b_j are KN columns; indeed, the KN columns of height k label the vertices of $B^{k,1}$. As mentioned above, it will be more useful to represent b_j in the split form $b_j^L b_j^R$; in this case, b becomes a filling $b_1^L b_1^R \ldots b_{\mu_1}^L b_{\mu_1}^R$ of the shape 2μ.

Now fix a filling b in B_μ, represented with split columns, which are labeled from left to right by $1, 1', 2, 2', \ldots$. We can apply a slight modification of Algorithm 3.2 to b and obtain a filling $c = c_1^L c_1^R \ldots c_{\mu_1}^L c_{\mu_1}^R =$ circ-ord(b) of 2μ; namely, we start by setting $c_1^L := b_1^L$, and then consider the (doubled) columns of b from left to right. We use the circular order on $[\overline{n}]$ starting at various values i, which we still denote by $\prec_i$.

Example 3.6 Consider the following tensor product of KN columns:

$$\begin{array}{|c|}\hline \overline{5} \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \overline{1} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \overline{3} \\ \hline \end{array} .$$

This is represented with split columns as the following filling b of the shape $2\mu = (6, 6, 6, 2)$, where the top row consists of the column labels; the corresponding filling c is also shown:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1' & 2 & 2' & 3 & 3' \\ \hline \end{array}$$

$$b = \begin{array}{|c|c|c|c|c|c|}\hline \overline{5} & \overline{5} & 2 & 3 & 1 & 1 \\ \hline \overline{3} & \overline{3} & \overline{4} & \overline{4} & 2 & 3 \\ \hline \overline{2} & \overline{2} & \overline{3} & \overline{2} & \overline{3} & \overline{2} \\ \hline \overline{1} & \overline{1} \\ \cline{1-2} \end{array}, \qquad c = \begin{array}{|c|c|c|c|c|c|}\hline \overline{5} & \overline{5} & \overline{4} & \overline{4} & \overline{3} & \overline{2} \\ \hline \overline{3} & \overline{3} & \overline{3} & \overline{\mathbf{2}} & 1 & 1 \\ \hline \overline{2} & \overline{\mathbf{2}} & 2 & \mathbf{3} & 2 & 3 \\ \hline \overline{1} & \overline{1} \\ \cline{1-2} \end{array} .$$

Define the charge word $\mathrm{cw}(b)$ of b by analogy with type A, as the biword containing a biletter $\binom{k}{j}$ for each entry k in column j of b; here j and k belong to the alphabets $\{1 < 1' < 2 < 2' < \ldots\}$ and $[\overline{n}]$, respectively. We order the biletters as in the type A case (in the decreasing order of the k's, and for equal

k's, in the decreasing order of j's), and define $\mathrm{cw}_2(b)$ in the same way (as the word formed by the lower letters j).

The modification of Algorithm 3.2 for constructing c from b can be rephrased in terms of $\mathrm{cw}(b)$, as explained below; we will refer to this rephrasing as the charge algorithm. We start by scanning $\mathrm{cw}_2(b)$ from right to left and by selecting the entries $1, 1', 2, 2', \ldots, \mu_1, (\mu_1)'$ in this order, according to the following rule: always pick the first available entry to the left, but if the desired entry is not available then scan the word from its right end once again. As in type A, we can see that the sequence of top letters paired with $1, 1', 2, 2', \ldots, \mu_1, (\mu_1)'$ is the first row of the filling c (read from right to left). We then remove the selected entries from $\mathrm{cw}_2(b)$ and repeat the above procedure, which will now give the other rows of c, from top to bottom. It was shown in [18] that we always go left from j to j', but we can go right from j' to $j + 1$.

Example 3.7 This is a continuation of Example 3.6. The charge word $\mathrm{cw}(b)$, with the order on the biletters indicated above, is

$$\begin{pmatrix} \overline{1} & \overline{1} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{4} & \overline{4} & \overline{5} & \overline{5} & 3 & 3 & 2 & 2 & 1 & 1 \\ 1'_4 & 1_4 & 3'_1 & 2'_2 & 1'_3 & 1_3 & 3_1 & 2_2 & 1'_2 & 1_2 & 2'_1 & 2_1 & 1'_1 & 1_1 & 3'_3 & 2'_3 & 3_3 & 2_3 & 3'_2 & 3_2 \end{pmatrix}.$$

The index attached to a lower letter is the number of the iteration in which the given letter is selected by the charge algorithm.

Descents are defined as usual, cf. Section 3.1. It is easy to see that the descents in c correspond to the steps to the right in the charge algorithm applied to $\mathrm{cw}_2(b)$. By an observation made above, we only have descents of the form $c_j^R(i) > c_{j+1}^L(i)$. We are led to the following definition of the type C charge.

Definition 3.8 *Consider a word w with letters in the alphabet $1, 1', 2, 2', \ldots$, containing as many letters j as j', and at least as many letters j as $j + 1$. Apply the charge algorithm to w, and assume that a selected entry j' is always to the left of the previously selected j. Let $\mathrm{charge}(w)$ be the sum of $k - j$ for each selected entry $j + 1$ to the right of the previously selected j', where the selected entries in the given iteration are $1, 1', \ldots, k, k'$.*

The above discussion leads to the following result:

$$\frac{1}{2} \sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \mathrm{charge}(\mathrm{cw}_2(b))\,. \tag{6}$$

For simplicity, we again set $\mathrm{charge}(b) := \mathrm{charge}(\mathrm{cw}_2(b))$.

Example 3.9 This is still a continuation of Example 3.6. The entries in the descents of c are shown above in bold. Correspondingly, the charge algorithm applied to $\mathrm{cw}_2(b)$ makes one step to the right in the second iteration (from $2'$ to 3), and two steps to the right in the third iteration (from $1'$ to 2 and from $2'$ to 3). Thus, $\mathrm{charge}(b) = 1 + (2 + 1) = 4$.

4 Energy and charge in types A and C

Let us first rederive the result of Nakayashiki–Yamada [25] showing the equality of the energy function and charge in type A_{n-1}. We do this in a more conceptual way, by using the method of "Demazure arrows" (the proof in [25] is based on subtle combinatorics of Young tableaux). Furthermore, we work with all the crystal vertices in a tensor product of columns, not just the highest weight vertices considered in [25].

The proof of Theorem 1.1 in type A immediately follows from the two results below. These describe the behavior of the type A charge with respect to the classical crystal operators and the crystal operator f_0, respectively.

Proposition 4.1 *The type A_{n-1} charge is preserved by the crystal operators $f_1, \ldots, f_{n-1}$.*

Proposition 4.2 *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ be of type $A_{n-1}^{(1)}$ with $r_N \geq r_{N-1} \geq \cdots \geq r_1$ and $b \in B$. If $\varphi_0(b) \geq 1$ and $\varepsilon_0(b) \geq 1$, then the type A_{n-1} charge satisfies* $\mathrm{charge}(e_0(b)) = \mathrm{charge}(b) - 1$.

4.1 Kyoto path model for nonperfect type C

Proposition 2.8 can be made more explicit in the case of $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ and $\ell = 1$ for type $C_n^{(1)}$, by providing a correspondence between highest weight elements (or ground states) in $B \otimes B(\Lambda_0)$ and elements in $B(\Lambda')$ in the sum on the right hand side of the equation in Proposition 2.8, which are of type A. This will help in the next section to prove Theorem 1.1 for type $C_n^{(1)}$.

We call the highest weight elements in $B \otimes B(\Lambda_0)$ *ground state paths*. There is a recursive construction for them, which starts by listing all elements $b_1 \in B^{r_1,1}$ such that $\varepsilon(b_1) = \Lambda_0$. Suppose $b_k \otimes \cdots \otimes b_1 \in B^{r_k,1} \otimes \cdots \otimes B^{r_1,1}$ are already constructed. Then $b_{k+1} \in B^{r_{k+1},1}$ can be any of the elements such that $\varepsilon(b_{k+1}) = \varphi(b_k)$. The weight of the ground state is $\varphi(b_N)$, which is some fundamental weight Λ_h. For perfect crystals there are unique elements $b_N, \ldots, b_1$ with the described properties. However, in type $C_n^{(1)}$ the crystals $B^{r,1}$ are not perfect and the above construction gives a tree of ground state elements.

Example 4.3 Take $B = B^{1,1} \otimes B^{2,1} \otimes B^{2,1} \otimes B^{3,1}$ of type $C_3^{(1)}$. Then b_1 is the column 321 and b_2 the column $\overline{2}\overline{3}$. For b_3 there are two choices, namely the columns 32 or $\overline{2}2$. In the first case b_4 is $\overline{3}$, and in the second case b_4 can be 2 or $\overline{1}$. In summary the three ground states of weights $\Lambda_2, \Lambda_2, \Lambda_0$, respectively, are

$$\boxed{\overline{3}} \otimes \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \overline{3} \\ \hline \overline{2} \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \otimes u_{\Lambda_0} \qquad \boxed{2} \otimes \begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \overline{3} \\ \hline \overline{2} \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \otimes u_{\Lambda_0} \qquad \boxed{\overline{1}} \otimes \begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \overline{3} \\ \hline \overline{2} \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \otimes u_{\Lambda_0} .$$

Theorem 4.4 *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $C_n^{(1)}$. From each ground state $u \otimes u_{\Lambda_0} \in B \otimes B(\Lambda_0)$ there exists a sequence of Demazure arrows f_i (see Definition 2.6), which ends at an element $b \otimes u_{\Lambda_0}$ such that b does not contain any barred letter.*

In order to provide a proof of Theorem 4.4, we describe the explicit sequence of f_i satisfying the required conditions. For details, we refer to [22].

4.2 Energy and charge in type C

To provide a proof of Theorem 1.1 for type C, we first reduce certain special cases in type C to type A.

Proposition 4.5 *If b is a tensor product of columns with all entries in $[n]$, then the type A_{n-1} and C_n energies of b coincide. Furthermore, if the columns have weakly decreasing heights, they equal the (type A_{n-1} or C_n) charge of b.*

By analogy with the approach in type A, the proof of Theorem 1.1 in type C now immediately follows from the two results below, which describe the behavior of the type C charge with respect to the crystal operators. Note that we also need the results in Theorem 4.4 and Proposition 4.5.

Proposition 4.6 [19] *The type C_n charge is preserved by the crystal operators $f_1, \ldots, f_n$.*

Proposition 4.7 *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ be of type C_n with $r_N \geq r_{N-1} \geq \cdots \geq r_1$ and $b \in B$. If $\varphi_0(b) \geq 1$ and $\varepsilon_0(b) \geq 1$, then the type C_n charge satisfies* $\operatorname{charge}(e_0(b)) = \operatorname{charge}(b) - 1$.

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