On the degree-chromatic polynomial of a tree

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Abstract. The *degree chromatic polynomial* $P_m(G,k)$ of a graph G counts the number of k-colorings in which no vertex has m adjacent vertices of its same color. We prove Humpert and Martin's conjecture on the leading terms of the degree chromatic polynomial of a tree.

Résumé. Le *polynôme degré chromatique* $P_m(G,k)$ d'un graphe G compte le nombre de k-colorations dans lesquelles aucun sommet n'a m sommets adjacents de sa même couleur. On démontre la conjecture de Humpert et Martin sur les coefficients principaux du polynôme degré chromatique d'un arbre.

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George David Birkhoff defined the chromatic polynomial of a graph to attack the renowned four color problem. The chromatic polynomial $P(G,k)$ counts the k-colorings of a graph G in which no two adjacent vertices have the same color [Read(1968)].

Given a graph G, Humpert and Martin defined its m-chromatic polynomial $P_m(G,k)$ to be the number of k-colorings of G such that no vertex has m adjacent vertices of its same color. They proved this is indeed a polynomial. When $m=1$, we recover the usual chromatic polynomial of the graph.

The chromatic polynomial is of the form

$$P(G,k) = k^n - ek^{n-1} + o(k^{n-1}),$$

where n is the number of vertices and e the number of edges of G. For $m>1$ the formula is no longer true, but Humpert and Martin conjectured the following formula which we now prove:

Theorem 1 ([Humpert and Martin(2010), Humpert and Martin(2011)], Conjecture) *Let T be a tree with n vertices and let m be an integer with $1<m<n$, then the following equation holds.*

$$P_m(T,k) = k^n - \sum_{v\in V(T)} \binom{d(v)}{m} k^{n-m} + o(k^{n-m})$$

Proof: For a given coloring of T, say vertices v_1 and v_2 are "friends" if they are adjacent and have the same color. For each v, let A_v be the set of colorings such that v has at least m friends. We want to find the number of colorings which are not in any A_v, and we will use the inclusion-exclusion principle. As the total number of k-colorings is k^n, we have

$$P_m(T,k) = k^n - \sum_{v\in V} |A_v| + \sum_{v_1,v_2\in V} |A_{v_1}\cap A_{v_2}| - \dots$$

We first show that $|A_v| = \binom{d(v)}{m}k^{n-m} + o(k^{n-m})$. Let $A_v^{(l)}$ be the set of k-colorings such that v has exactly l friends. In order to obtain a coloring in $A_v^{(l)}$, we may choose the l friends in $\binom{d(v)}{l}$ ways, the color of v and its friends in k ways, the color of the remaining adjacent vertices to v in $(k-1)^{d(v)-l}$ ways, and the color of the rest of the vertices in $k^{n-1-d(v)}$ ways. Then

$$\begin{aligned}|A_v| = \sum_{l=m}^{n-1} |A_v^{(l)}| &= \sum_{l=m}^{n-1} \binom{d(v)}{l} k^{n-d(v)}(k-1)^{d(v)-l} \\ &= \binom{d(v)}{m} k^{n-m} + o(k^{n-m}).\end{aligned}$$

To complete the proof, it is sufficient to see that for any set S of at least 2 vertices $|\bigcap_{v\in S} A_v| = o(k^{n-m})$; clearly we may assume $S = \{v_1, v_2\}$. Consider the following cases:

Case 1 *v_1 and v_2 are not adjacent.*

Split A_{v_1} into equivalence classes with the equivalence relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \neq v_2.$$

Note that each equivalence class C consists of k colorings, which only differ in the color of v_2. In addition, for each C at most $\frac{d(v_2)}{m}$ of its colorings are in A_{v_2}, as if $\sigma \in A_{v_2}$ there must be m vertices adjacent to v_2 with the color $\sigma(v_2)$. Therefore

$$|A_{v_1} \cap A_{v_2}| = \sum_C |C \cap A_{v_2}| \leq \sum_C \frac{d(v_2)}{m} = \frac{|A_{v_1}|}{k} \cdot \frac{d(v_2)}{m}.$$

It follows that $\frac{|A_{v_1} \cap A_{v_2}|}{|A_{v_1}|}$ goes to 0 as k goes to infinity, so $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$.

Case 2 *v_1 and v_2 are adjacent.*

Let W be the set of adjacent vertices to v_2 other than v_1. They are not adjacent to v_1 as T has no cycles. Split A_{v_1} into equivalence classes with the equivalence relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \notin W.$$

Each equivalence class Cconsists of $k^{|W|}$colorings, which may only differ in the colors of the vertices in W. If $v_1$and $v_2$are friends in the colorings of C, then a coloring in $|C \cap A_{v_2}|$must contain at least $m-1$vertices in $W$of the same color as v_2. Therefore

$$|C \cap A_{v_2}| = \sum_{l=m-1}^{|W|} \binom{|W|}{l} (k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l} k^{|W|-1} = 2^{|W|}k^{|W|-1}.$$

Notice that here we are using $m \geq 2$so that $l \geq 1$. Otherwise, if v_1 and v_2 are not friends in the colorings of C, then

$$|C \cap A_{v_2}| = \sum_{l=m}^{|W|} \binom{|W|}{l} (k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l} k^{|W|-1} = 2^{|W|} k^{|W|-1}.$$

Therefore

$$\begin{aligned} |A_{v_1} \cap A_{v_2}| = \sum_C |C \cap A_{v_2}| &< \sum_C 2^{|W|} k^{|W|-1} \\ &= \frac{|A_{v_1}|}{k^{|W|}} \cdot 2^{|W|} k^{|W|-1} = \frac{|A_{v_1}| \cdot 2^{|W|}}{k} \end{aligned}$$

and $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$ follows as in the first case.

This completes the proof of the theorem. □

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