Smooth Fano polytopes whose Ehrhart polynomial has a root with large real part (extended abstract)

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Abstract. The symmetric edge polytopes of odd cycles (del Pezzo polytopes) are known as smooth Fano polytopes. In this extended abstract, we show that if the length of the cycle is 127, then the Ehrhart polynomial has a root whose real part is greater than the dimension. As a result, we have a smooth Fano polytope that is a counterexample to the two conjectures on the roots of Ehrhart polynomials.

Résumé. Les polytopes d'arêtes symétriques de cycles impaires (del Pezzo polytopes) sont connus sous le nom de polytopes de Fano lisses. Dans ce résumé étendu, nous montrons que si la longueur du cycle est 127, alors le polynôme d'Ehrhart a une racine dont la partie réele est plus grande que la dimension. En conséquence, nous avons un polytope de Fano lisse qui est un contre exemple à deux conjectures sur les racines de polynômes d'Ehrhart.

Keywords: Ehrhart polynomials, Gröbner bases, Gorenstein Fano polytopes.

1 Introduction

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope (i.e., a convex polytope all of whose vertices have integer coordinates) of dimension D and $\partial\mathcal{P}$ its boundary. Let

$$i(\mathcal{P}, m) = |m\mathcal{P} \cap \mathbb{Z}^d|, \qquad m = 1, 2, \ldots$$

where $m\mathcal{P} = \{m\alpha \mid \alpha \in \mathcal{P}\}$. It is known that $i(\mathcal{P}, m)$ is a polynomial in m of degree D with $i(\mathcal{P}, 0) = 1$. We call $i(\mathcal{P}, m)$ the *Ehrhart polynomial* of $\mathcal{P}$. A lot of facts are known for the coefficients of the Ehrhart polynomial. The most fundamental fact is that the leading coefficient of $i(\mathcal{P}, m)$ equals the volume of $\mathcal{P}$. Using some linear inequalities for the coefficient of $i(\mathcal{P}, m)$, the following is proved:

Proposition 1.1 ([Beck et al.(2005)]) *Let $\mathcal{P}$ be a D-dimensional lattice polytope. Then*

(a) The roots of $i(\mathcal{P}, m)$ are bounded in norm by $1 + (D + 1)!$.

(b) All real roots of $i(\mathcal{P}, m)$ lie in the half-open interval $[-D, \lfloor D/2 \rfloor)$.

Moreover the following conjecture is given in [Beck et al.(2005)]:

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Conjecture 1.2 ([Beck et al.(2005)]) All roots α of Ehrhart polynomials of D dimensional lattice polytopes satisfy $-D \leq \mathrm{Re}(\alpha) \leq D-1$.

It is known that Conjecture 1.2 is true when

- $\alpha \in \mathbb{R}$ (by Proposition 1.1);
- $D = 2$ ([Beck et al.(2005)]);
- $D = 3, 4, 5$ ([Braun and Develin(2008)]).

In addtition, [Braun(2008)] showed that the norm bound of roots of the Ehrhart polynomial is $O(D^2)$.

The *h-vector* $h(\mathcal{P}) = (h_0, h_1, \ldots, h_D) \in \mathbb{Z}^{D+1}$ of $\mathcal{P}$ is defined by

$$\sum_{m=0}^{\infty} i(\mathcal{P}, m)\lambda^m = \frac{h_0 + h_1\lambda + \cdots + h_D\lambda^D}{(1-\lambda)^{D+1}}.$$

Then $h_0 = 1$ and $h_1 = |\mathcal{P} \cap \mathbb{Z}^d| - (D+1)$. It is known that each h_j is nonnegative. A *Fano polytope* is an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d such that the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior $\mathcal{P} \setminus \partial\mathcal{P}$ of $\mathcal{P}$. Recall that the dual polytope $\mathcal{P}^\vee$ of a Fano polytope $\mathcal{P}$ is the convex polytope

$$\mathcal{P}^\vee = \{x \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } y \in \mathcal{P}\},$$

where $\langle x, y \rangle$ is the usual inner product of $\mathbb{R}^d$. A Fano polytope is said to be *Gorenstein* if its dual polytope is integral. A Gorenstein Fano polytope is often said to be a *reflexive polytope*. It is known that the following conditions are equivalent for a Fano polytope $\mathcal{P}$:

- $\mathcal{P}$ is Gorenstein,
- $h(\mathcal{P})$ is symmetric, i.e., $h_j = h_{d-j}$ for $0 \leq j \leq d$,
- $i(\mathcal{P}, m) = (-1)^d i(\mathcal{P}, -m-1)$.

Gorenstein Fano polytopes are classified when $d \leq 4$ by [Kreuzer and Skarke(1997)] and [Kreuzer and Skarke(2000)]. The relevance of Gorenstein Fano polytopes to mirror symmetry is studied by [Batyrev(1994)]. A *smooth Fano polytope* is a Fano polytope such that the vertices of each facet form a $\mathbb{Z}$ basis of $\mathbb{Z}^d$. It is known that a smooth Fano polytope is Gorenstein. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and $\mathcal{A} \subset \mathbb{R}^N$ the affine subspace spanned by $\mathcal{P}$. One has an invertible affine transformation $\psi : \mathcal{A} \to \mathbb{R}^d$ with $\psi(\mathcal{A} \cap \mathbb{Z}^N) = \mathbb{Z}^d$. It follows that $\psi(\mathcal{P}) \subset \mathbb{R}^d$ is an integral convex polytope of dimension d. We say that $\psi(\mathcal{P})$ is a *standard form* of $\mathcal{P}$. In general, by abuse of terminology, we say that an integral convex polytope is a Gorenstein (resp. smooth) Fano polytope if one of its standard forms is a Gorenstein (resp. smooth) Fano polytope.

Let $d \geq 3$ be an integer and A_d, the $(d+1) \times (2d+1)$ matrix

$$A_d = \left(\begin{array}{c|cccc|cccc} 0 & 1 & & & -1 & -1 & & & 1 \\ 0 & -1 & \ddots & & & 1 & \ddots & & \\ \vdots & & \ddots & 1 & & & \ddots & -1 & \\ 0 & & & -1 & 1 & & & 1 & -1 \\ \hline 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \end{array} \right).$$

In this extended abstract, we study the convex hull $\mathrm{Conv}(A_d)$ of A_d. The matrix A_d is the *centrally symmetric configuration* [Ohsugi and Hibi(2011)] and $\mathrm{Conv}(A_d)$ is called the *symmetric edge polytope* of the cycle of length d. From the results in [Higashitani(2011), Matsui et al.(2011)], we have

Proposition 1.3 *The polytope* $\mathrm{Conv}(A_d)$ *is a Gorenstein Fano polytope (reflexive polytopes) of dimension* $d-1$. *In addition,* $\mathrm{Conv}(A_d)$ *is a smooth Fano polytope if and only if* d *is odd.*

Here, we first construct the reduced Gröbner basis $\mathcal{G}$ of I_{A_d}. Next, using $\mathcal{G}$, we compute the Ehrhart polynomial and the h-vector of $\mathrm{Conv}(A_d)$. Finally, we study the roots of the Ehrhart polynomial when d is odd. We show that the Ehrhart polynomial of $\mathrm{Conv}(A_{127})$ has a root whose real part is greater than $\dim(\mathrm{Conv}(A_{127}))$. This is a counterexample to the conjectures given in [Beck et al.(2005) , Matsui et al.(2011)].

Remark. This paper is an extended abstract of [Ohsugi and Shibata(2012)]. Proofs are given in the full version.

2 Gröbner bases of toric ideals

Let $\mathcal{R}_d = K[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}, s]$ be the Laurent polynomial ring over a field K and let $K[z, X, Y] = K[z, x_1, \dots, x_d, y_1, \dots, y_d]$ be the polynomial ring over K. We define the ring homomorphism $\pi : K[z, X, Y] \to \mathcal{R}_d$ by setting $\pi(x_i) = t_i t_{i+1}^{-1} s$, $\pi(y_i) = t_i^{-1} t_{i+1} s$ for $1 \le i \le d$ (here we set $t_{d+1} = t_1$) and $\pi(z) = s$. The *toric ideal* I_{A_d} is $\ker(\pi)$. Let $<$ be the reverse lexicographic order on $K[z, X, Y]$ with the ordering $z < y_d < x_d < \cdots < y_1 < x_1$. For $d \ge 3$, let $[d] = \{1, \dots, d\}$ and $k = \lceil \frac{d}{2} \rceil$.

Theorem 2.1 *The reduced Gröbner basis of* I_{A_d} *with respect to* $<$ *consists of*

$$\begin{array}{ll} x_i y_i - z^2 & (1 \le i \le d) \\ \displaystyle\prod_{l=1}^{k} x_{i_l} - z \prod_{l=1}^{k-1} y_{j_l} & ([d] = \{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{k-1}\}) \\ \displaystyle\prod_{l=1}^{k} y_{j_l} - z \prod_{l=1}^{k-1} x_{i_l} & ([d] = \{i_1, \dots, i_{k-1}\} \cup \{j_1, \dots, j_k\}) \end{array}$$

if d *is odd and*

$$\begin{array}{ll} x_i y_i - z^2 & (1 \le i \le d) \\ \displaystyle\prod_{l=1}^{k} x_{i_l} - y_d \prod_{l=1}^{k-1} y_{j_l} & ([d-1] = \{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{k-1}\}) \\ \displaystyle\prod_{l=1}^{k} y_{j_l} - x_d \prod_{l=1}^{k-1} x_{i_l} & ([d-1] = \{i_1, \dots, i_{k-1}\} \cup \{j_1, \dots, j_k\}) \end{array}$$

if d *is even. The initial monomial of each binomial is the first monomial.*

3 Ehrhart polynomials and roots

For $0 \leq i \leq d$, let $r_d(i)$ denote the number of squarefree monomials in $K[X,Y]$ of degree i that do not belong to the initial ideal $\mathrm{in}_<(I_{A_d})$ and let $s_d(i+1)$ denote the number of squarefree monomials in $K[z,X,Y]$ of degree $(i+1)$ that are divided by z and do not belong to $\mathrm{in}_<(I_{A_d})$. For example, $r_d(0) = s_d(1) = 1$ and $r_d(d) = 0$.

Lemma 3.1 *For $0 \leq i \leq d-1$, we have*

$$r_d(i) = \binom{d}{i} \sum_{\ell=1}^{d-i} \binom{i}{k-\ell}$$

and $s_d(i+1) = r_d(i)$. In particular, $r_d(i) = \binom{d}{i} 2^i$ for $0 \leq i \leq k-1$.

It is known [Sturmfels(1996), Chapter 8] that $\mathrm{in}_<(I_{A_d}) = \sqrt{\mathrm{in}_<(I_{A_d})}$ is the Stanley–Reisner ideal of a regular unimodular triangulation Δ of $\mathrm{Conv}(A_d)$. Thus, $r_d(i) + r_d(i+1)$ is the number of i-dimensional faces of Δ. From Lemma 3.1 and [Stanley(1996), Theorem 1.4], we have the following:

Theorem 3.2 *The Ehrhart polynomial of $\mathrm{Conv}(A_d)$ is*

$$\sum_{i=0}^{d-1} r_d(i) \binom{m}{i}.$$

Moreover, the normalized volume of $\mathrm{Conv}(A_d)$ equals $k\binom{d}{k}$.

Let $(h_0^{(d)}, h_1^{(d)}, \ldots, h_{d-1}^{(d)})$ be the h-vector of $\mathrm{Conv}(A_d)$. Note that $h_0^{(d)} = 1$. Since $\mathrm{Conv}(A_d)$ is Gorenstein, we have $h_j^{(d)} = h_{d-1-j}^{(d)}$ for each $0 \leq j \leq d-1$. Thus, it is enough to study $h_j^{(d)}$ for $1 \leq j \leq k-1$.

Theorem 3.3 *For $1 \leq j \leq k-1$, we have*

$$h_j^{(d)} = (-1)^j \sum_{i=0}^{j} (-2)^i \binom{d}{i} \binom{d-i-1}{j-i} = \begin{cases} 2^{d-1} & j = k-1 \text{ and } d \text{ is odd,} \\ h_j^{(d-1)} + h_{j-1}^{(d-1)} & \text{otherwise.} \end{cases}$$

Finally, we study the roots of the Ehrhart polynomial when d is odd. In this case, $\mathrm{Conv}(A_d)$ is a smooth Fano polytope of dimension $d-1$. Since $\mathrm{Conv}(A_d)$ is a Gorenstein Fano polytope, the roots of the Ehrhart polynomial are symmetrically distributed in the complex plane with respect to the line $\mathrm{Re}(z) = -1/2$. Here, $\mathrm{Re}(z)$ is the real part of $z \in \mathbb{C}$. The following conjectures are given in [Beck et al.(2005) , Matsui et al.(2011)]:

Conjecture 3.4 ([Beck et al.(2005)]) All roots α of Ehrhart polynomials of D dimensional lattice polytopes satisfy $-D \leq \mathrm{Re}(\alpha) \leq D-1$.

Conjecture 3.5 ([Matsui et al.(2011)]) All roots α of Ehrhart polynomials of D dimensional Gorenstein Fano polytopes satisfy $-D/2 \leq \mathrm{Re}(\alpha) \leq D/2 - 1$.

Using the software packages `Maple`, `Mathematica`, and `Maxima`, we computed the largest real part of roots of the Ehrhart polynomial of $\mathrm{Conv}(A_d)$:

d	$\dim(\mathrm{Conv}(A_d))$	the largest real part	
35	34	16.35734046	a counterexample to Conjecture 3.5
125	124	123.5298262	a counterexample to Conjecture 3.4
127	126	126.5725840	greater than its dimension

Remark 3.6 Recently, a simplex (not a Fano polytope) that does not satisfy the condition "$\mathrm{Re}(\alpha) \leq D - 1$" in Conjecture 3.4 was presented in [Higashitani(2012)]. Our polytope $\mathrm{Conv}(A_{125})$ is the first example satisfying neither "$-D \leq \mathrm{Re}(\alpha)$" nor "$\mathrm{Re}(\alpha) \leq D - 1$" in Conjecture 3.4.

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