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On singularity confinement for the pentagram map

Max Glick †

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

Abstract. The pentagram map, introduced by R. Schwartz, is a birational map on the configuration space of polygons in the projective plane. We study the singularities of the iterates of the pentagram map. We show that a "typical" singularity disappears after a finite number of iterations, a confinement phenomenon first discovered by Schwartz. We provide a method to bypass such a singular patch by directly constructing the first subsequent iterate that is well-defined on the singular locus under consideration. The key ingredient of this construction is the notion of a decorated (twisted) polygon, and the extension of the pentagram map to the corresponding decorated configuration space.

Résumé. L'application pentagramme de R. Schwartz est une application birationnelle sur l'espace des polygones dans le plan projectif. Nous étudions les singularités des itérations de l'application pentagramme. Nous montrons qu'une singularité "typique" disparaît après un nombre fini d'itérations, un phénomène découvert par Schwartz. Nous fournissons une méthode pour contourner une telle singularité en construisant la première itération qui est bien définie. L'ingrédient principal de cette construction est la notion d'un polygone décoré et l'extension de l'application pentagramme á l'espace de configuration décoré.

Keywords: pentagram map, singularity confinement, decorated polygon

1 Introduction

The pentagram map, introduced by R. Schwartz [9], is a geometric construction which produces one polygon from another. Successive applications of this operation (cf. Fig. 1) define a discrete dynamical system that has received considerable attention in recent years (see, e.g., $[2, 5, 6, 8, 10]$) due to its integrability properties and its connections to moduli spaces and cluster algebras. This paper is devoted to the study of singularity confinement for the pentagram map, a phenomenon first observed experimentally by Schwartz. Informally speaking, a singularity of a map at a point is said to be confined if some higher iterate of the map is well-defined at that point. We investigate singularities of the pentagram map and prove confinement in several cases.

The pentagram map is typically defined for objects called twisted polygons defined by Schwartz [8]. A *twisted polygon* is a sequence $A = (A_i)_{i\in\mathbb{Z}}$ of points in the projective plane that is periodic modulo some projective transformation ϕ, i.e., $A_{i+n} = \phi(A_i)$ for all $i \in \mathbb{Z}$. We will place the additional restriction that

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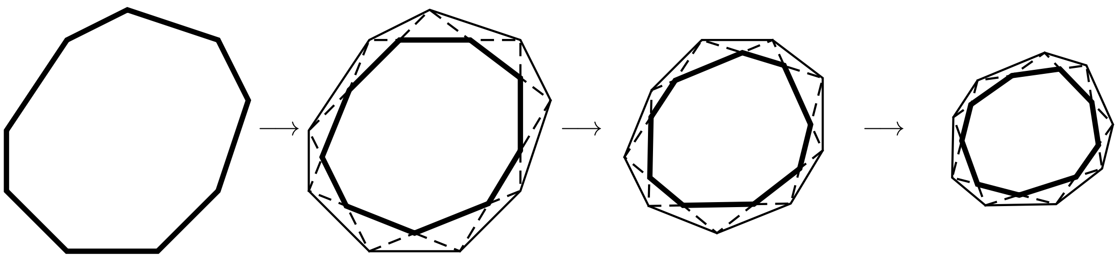

Fig. 1: Three iterations of the pentagram map in the space of 9-gons

every quadruple of consecutive points of A be in general position. Two twisted polygons A and B are said to be *projectively equivalent* if there exists a projective transformation ψ such that $\psi(A_i) = B_i$ for all i. Let $\mathcal{P}_n$ denote the space of twisted n-gons modulo projective equivalence.

It is convenient to also allow twisted polygons to be indexed by $\frac{1}{2}+\mathbb{Z}$ instead of $\mathbb{Z}$. Let $\mathcal{P}_n^*$ denote the space of twisted n-gons indexed by $\frac{1}{2}+\mathbb{Z}$, modulo projective equivalence.

The *pentagram map*, denoted T, inputs a twisted polygon A and constructs a new twisted polygon B defined by $B_i = \overleftrightarrow{A_{i-\frac{3}{2}}A_{i+\frac{1}{2}}} \cap \overleftrightarrow{A_{i-\frac{1}{2}}A_{i+\frac{3}{2}}}$. Note that if A is indexed by $\mathbb{Z}$ then B is indexed by $\frac{1}{2}+\mathbb{Z}$ and vice versa. The pentagram map preserves projective equivalence, so it induces maps

$$\alpha_1 : \mathcal{P}_n^* \to \mathcal{P}_n$$
$$\alpha_2 : \mathcal{P}_n \to \mathcal{P}_n^*$$

Schwartz [8] gives coordinates $x_1, \ldots, x_{2n}$ defined generically on $\mathcal{P}_n$ and on $\mathcal{P}_n^*$. These are naturally ordered cyclically, so let $x_{i+2n} = x_i$ for all $i \in \mathbb{Z}$. Expressed in these coordinates, the maps α_1 and α_2 take a simple form.

Proposition 1.1 ([8, (7)]) *Suppose that* $(x_1, \ldots, x_{2n})$ *are the* x*-coordinates of* A*. If* $A \in \mathcal{P}_n^*$ *then*

$$x_j(\alpha_1(A)) = \begin{cases} x_{j-1}\frac{1-x_{j-3}x_{j-2}}{1-x_{j+1}x_{j+2}}, & j \text{ even} \\ x_{j+1}\frac{1-x_{j+3}x_{j+2}}{1-x_{j-1}x_{j-2}}, & j \text{ odd} \end{cases} \tag{1.1}$$

Alternately, if $A \in \mathcal{P}_n$ *then*

$$x_j(\alpha_2(A)) = \begin{cases} x_{j+1}\frac{1-x_{j+3}x_{j+2}}{1-x_{j-1}x_{j-2}}, & j \text{ even} \\ x_{j-1}\frac{1-x_{j-3}x_{j-2}}{1-x_{j+1}x_{j+2}}, & j \text{ odd} \end{cases} \tag{1.2}$$

We will be interested in T^k, the kth iterate of the pentagram map. Defined on $\mathcal{P}_n$ it takes the form $T^k = \underbrace{\cdots \circ \alpha_2 \circ \alpha_1 \circ \alpha_2}_{k}$ and has image in either $\mathcal{P}_n$ or $\mathcal{P}_n^*$ depending on the parity of k. By (1.1) and (1.2), T^k is a rational map. The purpose of this paper is to better understand the singularities of the pentagram map and its iterates. For us, a singular point of a rational map is an input at which one of the components of the map has a vanishing denominator.

Let $A \in \mathcal{P}_n$ be a singular point of the pentagram map. Then typically A will be a singular point of T^k for all k less than some m, but not of T^m. This phenomenon is known as *singularity confinement* and was identified by Grammaticos, Ramani, and Papageorgiou [4] as a feature common to many discrete integrable systems. Now, the pentagram map is a discrete integrable system as proven by Ovsienko, Schwartz, and Tabachnikov [5,6] and Soloviev [10]. That singularity confinement holds in this setting has been observed experimentally by Schwartz. The current paper seeks to understand singularity confinement for the pentagram map from both an algebraic and geometric perspective.

Algebraically, (1.2) suggests that a polygon $A \in \mathcal{P}_n$ is a singular point of the pentagram map whenever $x_{2i}(A)x_{2i+1}(A) = 1$ for some $i \in \mathbb{Z}$. To check how many steps the singularity persists, one must determine for which k the rational expression for T^k has a vanishing denominator at the given point. We use generating function formulas for these denominators from [2] to better understand when this occurs.

What we discover is that the behavior of a singularity seems to depend on the set S of integers i for which $x_{2i}(A)x_{2i+1}(A) = 1$. We call S the *type* of the singularity and attempt to understand when singularity confinement holds for generic polygons of a given type. The simplest case is when S consists of a single element, in which event the singularity is confined to two iterations (i.e. A is a singular point of T and T^2 but not of T^3). More generally, suppose S is a finite arithmetic progression with common difference equal to 1 or 2. We prove that generic singularities of these types are confined to $l + 1$ steps where l is the size of the arithmetic progression.

We do not have as complete an understanding of the situation for other singularity types. If the number of sides n of the polygon is odd, we show that singularity confinement holds generically for every type except the worst case $S = \{1, \ldots, n\}$. In addition we have an upper bound for the number of iterations such singularities last. The case of n even seems to be more complicated and we only have a conjectural answer as to which types exhibit singularity confinement.

From a geometric perspective, the condition $x_{2i}(A)x_{2i+1}(A) = 1$ indicates that the triple of vertices A_{i-2}, A_i, and A_{i+2} are collinear. As such, a polygon A is a singular point of T if and only if this condition holds for some i. Although one can construct $B = T(A)$ in this case, the result will violate the condition that quadruples of consecutive vertices be in general position. In fact, $B_{i-\frac{3}{2}}$, $B_{i-\frac{1}{2}}$, $B_{i+\frac{1}{2}}$, and $B_{i+\frac{3}{2}}$ will be collinear making it impossible to carry the construction any further. The notion of singularity confinement also has a geometric interpretation. If A has a singularity which vanishes after m steps, then one can approximate A by nonsingular polygons, apply the construction T^m to them, and take a limit to find $T^m(A)$. Since A is a regular point of T^m, the result of this procedure does not depend on the approximations of A.

Our main result on the geometric side is a straightedge construction of the first defined iterate $T^m(A)$ of a polygon A of certain singularity types. The basic idea is to fix, up to the first order, a family of approximations of A by nonsingular polygons. The data needed to accomplish this is encoded by a collection of points and lines which we call a decoration of A. With this done, the iterates between A and $T^m(A)$ become well-defined. To determine $T^m(A)$, we iterate a procedure which constructs these intermediate polygons one by one.

This paper is organized as follows. Section 2 reviews previous work on the pentagram map, including a non-recursive formula for T^k as a rational map of the x-coordinates. This map factors into polynomials, some properties of which are given in Section 3. Section 4 identifies a hierarchy of singularity types of the pentagram map and establishes that generic polygons of these types exhibit singularity confinement. The remainder of the paper addresses the problem of moving past singularities by constructing $T^m(A)$ from A when A is a singular point of $T, T^2, \ldots, T^{m-1}$. An approach which works for the simplest singularity type is given in Section 5. Section 6 introduces decorated polygons which will serve as the underlying objects of the main construction. Section 7 states the main construction itself and discusses what is needed to prove its correctness for a given singularity type. Detailed proofs of all statements given in this paper can be found in the full version [3].

The following notation will be used throughout. If $a, b, k \in \mathbb{Z}$, $a \leq b$, $k \geq 1$ and $a \equiv b \pmod{k}$ then let $[a, b]_k$ denote the arithmetic progression

$$[a, b]_k = \{a, a + k, a + 2k, \ldots, b\}$$

Twisted polygons will be denoted by capital letters with individual vertices indexed by either $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$. The sides of a polygon (i.e. lines passing through two consecutive vertices) will be denoted by the corresponding lowercase letter and indexed using the opposite indexing scheme. For instance, if A is a twisted polygon indexed by $\mathbb{Z}$ then its vertices are denoted A_i for $i \in \mathbb{Z}$ and its sides are denoted $a_j = \overleftrightarrow{A_{j-\frac{1}{2}} A_{j+\frac{1}{2}}}$ for $j \in (\frac{1}{2} + \mathbb{Z})$.

2 Pentagram map background

The *cross ratio* of four real numbers a, b, c, d is defined to be

$$\chi(a, b, c, d) = \frac{(a - b)(c - d)}{(a - c)(b - d)}$$

This definition extends to the projective line, on which it gives a projective invariant of four points. We will be interested in taking the cross ratio of four collinear points in the projective plane, or dually, the cross ratio of four lines intersecting at a common point.

Let A be a twisted polygon. The x-coordinates of A are defined by Schwartz [8] as follows. For each index k of A, let

$$\begin{aligned} x_{2k}(A) &= \chi(A_{k-2}, A_{k-1}, B, D) \\ x_{2k+1}(A) &= \chi(A_{k+2}, A_{k+1}, C, D). \end{aligned}$$

where $B = \overleftrightarrow{A_{k-2}A_{k-1}} \cap \overleftrightarrow{A_k A_{k+1}}$, $C = \overleftrightarrow{A_{k-1}A_k} \cap \overleftrightarrow{A_{k+1}A_{k+2}}$, and $D = \overleftrightarrow{A_{k-2}A_{k-1}} \cap \overleftrightarrow{A_{k+1}A_{k+2}}$. Now $x_{j+2n} = x_j$ for all $j \in \mathbb{Z}$, and as mentioned in the introduction, $x_1, \ldots, x_{2n}$ give a set of coordinates on $\mathcal{P}_n$ and on $\mathcal{P}_n^*$.

In [2], we work with related quantities called the y-parameters and denoted y_j for $j \in \mathbb{Z}$. For each index k of A,

$$y_{2k}(A) = -\left(\chi(\overleftrightarrow{A_k A_{k-2}}, \overleftrightarrow{A_k A_{k-1}}, \overleftrightarrow{A_k A_{k+1}}, \overleftrightarrow{A_k A_{k+2}})\right)^{-1} \tag{2.1}$$

$$y_{2k+1}(A) = -\chi(B, A_k, A_{k+1}, E) \tag{2.2}$$

where $B = \overleftrightarrow{A_{k-2}A_{k-1}} \cap \overleftrightarrow{A_k A_{k+1}}$ and $E = \overleftrightarrow{A_k A_{k+1}} \cap \overleftrightarrow{A_{k+2}A_{k+3}}$ (see Fig. 2). As with the x_j, we have that $y_{j+2n} = y_j$ for all j. One can check that $y_{2k} = -(x_{2k}x_{2k+1})^{-1}$ and $y_{2k+1} = -x_{2k}x_{2k+1}$ for all k. Hence, $y_1, \ldots, y_{2n}$ do not give a set of coordinates as they satisfy the relation $y_1 y_2 \cdots y_{2n} = 1$.

The y-parameters transform under the pentagram map according to the Y-pattern dynamics of a certain cluster algebra. We used results of Fomin and Zelevinsky [7] to give formulas for the iterates of the

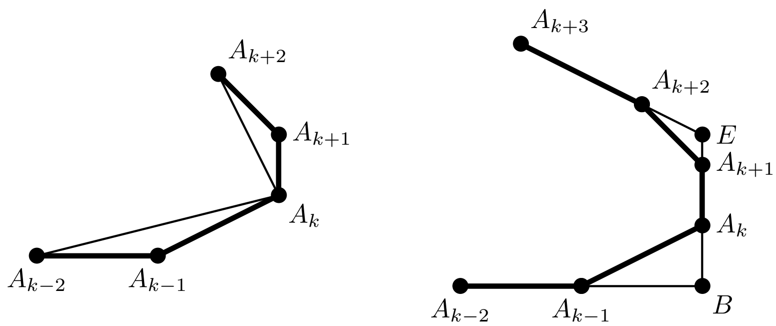

Fig. 2: The points and lines used in the definitions of $y_{2k}(A)$ (left) and $y_{2k+1}(A)$ (right)

pentagram map in terms of the F-polynomials $F_{j,k}$ of this cluster algebra. The $F_{j,k}$ can be defined recursively by $F_{j,-1} = F_{j,0} = 1$ and

$$F_{j,k+1} = \frac{F_{j-3,k}F_{j+3,k} + \left(\prod_{i=-k}^{k} y_{3i+j}\right) F_{j-1,k}F_{j+1,k}}{F_{j,k-1}} \tag{2.3}$$

for $k \geq 0$.

Theorem 2.1 ([2, Theorem 4.2]) *Let $A \in \mathcal{P}_n$, $x_j = x_j(A)$, and $y_j = y_j(A)$. Then*

$$x_j(T^k(A)) = \begin{cases} x_{j-3k} \left(\prod_{i=-k}^{k-1} y_{j+1+3i} \right) \dfrac{F_{j+2,k-1}F_{j-3,k}}{F_{j-2,k-1}F_{j+1,k}}, & j+k \text{ even} \\ x_{j+3k} \left(\prod_{i=-k}^{k-1} y_{j+1+3i} \right) \dfrac{F_{j-3,k-1}F_{j+2,k}}{F_{j+1,k-1}F_{j-2,k}}, & j+k \text{ odd} \end{cases} \tag{2.4}$$

The $F_{j,k}$ are polynomials in the y_j (hence Laurent polynomials in the x_j) with positive coefficients. We found [2, Theorem 6.6] a simple combinatorial description of these polynomials as generating functions of order ideals of certain posets P_k first studied by Elkies, Kuperberg, Larsen, and Propp [1].

3 The F-polynomials

According to (1.1) and (1.2), the pentagram map has singularities for polygons with $x_j x_{j+1} = 1$, i.e., $y_j = -1$, for some j. According to (2.4), the iterate T^k has a singularity whenever $F_{j,k-1} = 0$ or $F_{j,k} = 0$ for some j. In this section we examine under which circumstances having $y_j = -1$ for certain j forces an F-polynomial to vanish. Results along these lines will indicate how many steps a given singularity persists.

For the purpose of this section, relax the assumptions $y_{i+2n} = y_i$ for all i and $y_1 y_2 \cdots y_{2n} = 1$. Instead consider the $F_{j,k}$ as polynomials in the countable collection of variables $\{y_i : i \in \mathbb{Z}\}$. By way of notation, if $S \subseteq \mathbb{Z}$ let $F_{j,k}|_S$ be the polynomial in $\{y_i : i \in \mathbb{Z} \setminus S\}$ obtained by substituting $y_i = -1$ for all $i \in S$ into $F_{j,k}$.

Proposition 3.1 *Fix $l \in [-(k-1), k-1]_2$. Let $S \subseteq \mathbb{Z}$ be either*

1. $[j+l-2(k-1), j+l+2(k-1)]_4$ *or*
2. $[j+2l-(k-1), j+2l+(k-1)]_2$.

Then $F_{j,k}|_S \equiv 0$.

Proposition 3.2 *Let $S \subseteq \mathbb{Z}$ be either $[a,b]_4$ or $[a,b]_2$ with $a \equiv j+k+1 \pmod 2$ and $|S| < k$. Then $F_{j,k}|_S \not\equiv 0$.*

Propositions 3.1 and 3.2 give a complete picture as to when $F_{j,k}|_S \equiv 0$ for S of the form $[a,b]_4$ or $[a,b]_2$. We will use these results to establish confinement for certain singularity types in the next section. There, we will assume that n is large relative to $|S|$ so that the relations among the y-variables do not enter into play. In contrast, the following proposition pertains to a more severe singularity type, so we will reintroduce those relations at this point.

Proposition 3.3 *Suppose that n is odd and $S = [2, 2n-2]_2$. Assume that $y_{i+2n} = y_i$ for all $i \in \mathbb{Z}$, $y_i = -1$ for all $i \in S$, and $y_1 \cdots y_{2n} = 1$. Let $j, k \in \mathbb{Z}$ with $j+k$ odd and $k \in \{n, n+1\}$. Then evaluated at this input, $F_{j,k} \neq 0$ provided $y_0 \neq -1$ and $y_i \neq 0$ for all i.*

4 Singularity patterns

For $i \in \mathbb{Z}$ let

$$X_i = \{A \in \mathcal{P}_n : y_{2i}(A) = -1\}$$

For $j \in (\frac{1}{2} + \mathbb{Z})$, let

$$Y_j = \{A \in \mathcal{P}_n : y_{2j}(A) = -1\}$$

The reason for the different notation is as follows.

Lemma 4.1 ([2, Lemma 7.2]) *Let $A \in \mathcal{P}_n$, $i \in \mathbb{Z}$, $j \in (\frac{1}{2} + \mathbb{Z})$.*

1. *$A \in X_i$ if and only if A_{i-2}, A_i, and A_{i+2} are collinear.*
2. *$A \in Y_j$ if and only if a_{j-2}, a_j, and a_{j+2} are concurrent.*

Define in the same way subvarieties $X_j \subseteq \mathcal{P}_n^*$ for $j \in (\frac{1}{2} + \mathbb{Z})$ and $Y_i \subseteq \mathcal{P}_n^*$ for $i \in \mathbb{Z}$.

For $S \subseteq \mathbb{Z}$ or $S \subseteq (\frac{1}{2} + \mathbb{Z})$ let $X_S = \bigcap_{i \in S} X_i$ and $Y_S = \bigcap_{i \in S} Y_i$. For instance, $X_{\{3,5\}} = X_3 \cap X_5$ is the set of twisted polygons A for which A_1, A_3, A_5, A_7 are all collinear. On X_S, we have that $y_{2i} = -1$ for all $i \in S$. Therefore we can replace the $F_{j,k}$ in (2.4) with $F_{j,k}|_{2S}$ where $2S = \{2i : i \in S\}$. If all of these restricted polynomials are nonzero, then the corresponding iterate of the pentagram map is defined generically on X_S.

Theorem 4.2 *Let $i, m \in \mathbb{Z}$ with $1 \leq m < n/3 - 1$. Let*

$$S = [i-(m-1), i+(m-1)]_2$$
$$S' = \left[i - \frac{1}{2}(m-1), i + \frac{1}{2}(m-1)\right]_1$$

Then the map T^k is singular on X_S for $1 \leq k \leq m+1$, but T^{m+2} is nonsingular at generic $A \in X_S$. Moreover, $T^{m+2}(A) \in Y_{S'}$ for such A.

The roles of S and S' can be interchanged in Theorem 4.2. This is apparently an instance of projective duality.

Theorem 4.3 *Let S and S' be as in Theorem 4.2. Then the map T^k is singular on $X_{S'}$ for $1 \leq k \leq m+1$, but T^{m+2} is nonsingular at generic $A \in X_{S'}$. Moreover, $T^{m+2}(A) \in Y_S$ for such A.*

We now have singularity confinement on X_S for S an arithmetic sequence whose terms differ by 1 or 2. Generally, if S is a disjoint union of such sequences which are far apart from each other, then the corresponding singularities do not affect each other. Hence singularity confinement holds and the number of steps needed to get past the singularity is dictated by the length of the largest of the disjoint sequences.

Not all singularity types are of this form. However, it would seem that singularity confinement does hold for all types, outside of some exceptional cases. If n is odd, the only exceptional type is $S = [1, n]_1$. Moreover, for any other S and generic $A \in X_S$ the corresponding singularity lasts at most n steps. We establish this by considering the worst case where $|S| = n - 1$.

Proposition 4.4 *Suppose n is odd and let $S = [1, n]_1 \backslash \{i\}$ for some $i \in [1, n]_1$. Then T^{n+1} is nonsingular at generic $A \in X_S$.*

Corollary 4.5 *Suppose that n is odd and that $S \subsetneq [1, n]_1$. Then T^{n+1} is nonsingular for generic $A \in X_S$.*

Of course for general S, it will usually be the case that T^m is defined on X_S for some $m < n + 1$. The corollary only ensures that $n + 1$ steps will be sufficient. This appears to also be true for n even outside of some exceptional cases. We state this as a conjecture.

Conjecture 4.6 *Suppose that n is even.*

- *Singularity confinement holds generically on X_S unless $[1, n-1]_2 \subseteq S$ or $[2, n]_2 \subseteq S$.*
- *Whenever singularity confinement holds for a type, there exists an $m \leq n$ such that generic singularities of that type last m steps (i.e. T^m is singular but T^{m+1} is not).*

5 Straightedge constructions: a first attempt

Let $A \in \mathcal{P}_n$ be a singular point of T^k for $1 \leq k < m$ but not of T^m. The remainder of this paper focuses on the problem of constructing $B = T^m(A)$.

Suppose we choose a one-parameter family $A(t)$ of twisted polygons varying continuously with t such that

1. $A(0) = A$ and
2. $A(t)$ is a regular point of T^k for all $t \neq 0$ and $k \leq m$.

For small $t \neq 0$, we can obtain $B(t) = T^m(A(t))$ by iterating the geometric construction defining T. By continuity, B is given by $\lim_{t \to 0} B(t)$ which can be found numerically. Although this limit process gives a correct description of B, more satisfying would be a finite construction. Away from singularities, the pentagram map can be carried out with a straightedge alone, so it seems likely that there is a straightedge construction producing B from A in the present setting.

In this section we introduce an iterative approach to finding such a straightedge construction, which works in simple situations. The idea is to attempt to make sense of the polygon $T^k(A)$ for $k < m$ despite the presence of the singularity. Let $A(t)$ be as above, and fixing $k < m$, let $C(t) = T^k(A(t))$. For each appropriate index i, let

$$C_i = \lim_{t \to 0}(C_i(t))$$

We say that C_i is well-defined if this limit always exists and is independent of the choice of the curve $A(t)$ through A. We can define sides c_j of $T^k(A)$ in the same way. In fact it is possible that each of the C_i and c_j are well-defined, despite the singularity. This would simply indicate that the resulting polygon C fails to satisfy the property that quadruples of consecutive vertices be in general position, which is needed for all the x-coordinates to be defined.

As before, suppose $A \in \mathcal{P}_n$ is a singular point of T^k for $1 \le k < m$ but not of T^m. In addition, assume that all of the vertices and sides of $T^k(A)$ for $1 \le k < m$ are well-defined. Then it should be possible to construct the components of these intermediate polygons successively. Ideally, each individual side or vertex can be constructed by a simple procedure depending only on nearby objects.

Consider the simplest case $m = 1$ of Theorem 4.2, and without loss of generality let $i = 3$. If $A \in X_3$ (meaning A_1, A_3, and A_5 are collinear), then the theorem implies that $D = T^3(A) \in Y_3$ (i.e. d_1, d_3, and d_5 are concurrent). For such A, let $B = T(A)$ and $C = T^2(A)$.

Proposition 5.1 *All of the vertices and sides of B, C, and D are well-defined and each of them can be constructed from A using a straightedge.*

The construction alluded to in Proposition 5.1 is pictured in Fig. 3. Two parts of the construction require explanation, namely how to find the vertex C_3 and the side d_3, the latter being needed to construct $D_{2.5}$ and $D_{3.5}$. We use a generalization of the cross ratio called the triple ratio defined as follows. The *triple ratio* of six collinear points $P_1, \ldots, P_6$ is

$$[P_1, P_2, P_3, P_4, P_5, P_6] = \frac{P_1P_2}{P_2P_3}\frac{P_3P_4}{P_4P_5}\frac{P_5P_6}{P_6P_1}$$

where P_iP_j denotes the signed distance from P_i to P_j. Define using projective duality the triple ratio of six concurrent lines.

Now the vertex C_3 is the unique point on the line common to A_1, A_3, and A_5 such that

$$[B_{1.5}, B_{2.5}, A_3, B_{4.5}, B_{3.5}, C_3] = -1$$

Similarly, the side d_3 of D can be constructed as the unique line passing through C_2 satisfying

$$[c_{1.5}, c_{3.5}, b_3, c_{4.5}, c_{2.5}, d_3] = -1$$

The next simplest case, $m = 2$, of Theorem 4.2 concerns a singularity which disappears after four steps. Specifically, taking $i = 4$ there is a map $T^4 : X_{\{3,5\}} \to Y_{\{3.5,4.5\}}$. Suppose $A \in X_{\{3,5\}}$ which means that A_1, A_3, A_5, and A_7 are collinear. Then $E = T^4(A) \in Y_{\{3.5,4.5\}}$, i.e., $e_{1.5}, e_{3.5}, e_{5.5}$ are concurrent and $e_{2.5}, e_{4.5}, e_{6.5}$ are also concurrent. Here, not all of the intermediate polygons are completely well-defined.

Proposition 5.2 *Let $A \in X_{\{3,5\}}$ and let $A(t)$ be a curve in $\mathcal{P}_n$ with $A(0) = A$. Let $C = \lim_{t\to 0} T^2(A(t))$. Then the vertex C_4 of C depends not only on A but also on the choice of the curve $A(t)$.*

The fact that an intermediate vertex is not well-defined causes great difficulty in the current approach to devising straightedge constructions. In the following sections, we demonstrate how enriching the input A with first-order data counteracts this difficulty and leads to a general algorithm.

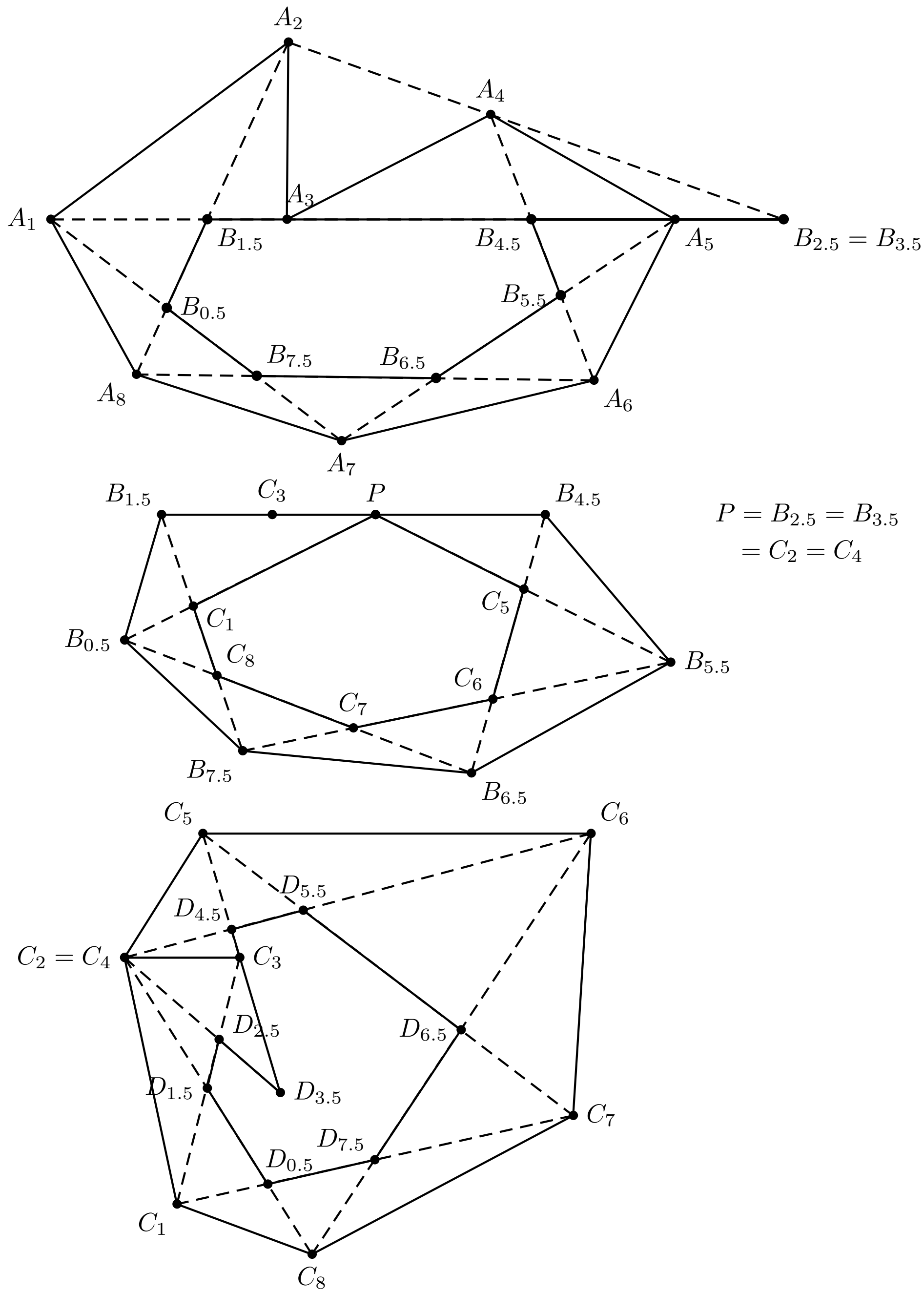

Fig. 3: Illustrations of the three steps of the construction of $D = T^3(A)$ for $A \in X_3$

6 Decorated polygons

Let A be a twisted polygon which is a singular point of T^k. As explained in the previous section, we can attempt to define $T^k(A)$ as a limit of $T^k(A(t))$ where $A(t)$ is a curve in the space of polygons passing through $A = A(0)$. As we saw, the result sometimes depends on the choice of the curve. This suggests a different approach to constructing the first nonsingular iterate $T^m(A)$. Start by fixing arbitrarily the one-parameter family $A(t)$. With respect to this choice the intermediate polygons $T^k(A)$ are well-defined. Constructing them in turn we eventually get $T^m(A)$. Since A is not a singular point of T^m, the final result will not depend on the choice of $A(t)$.

Working with curves themselves would be difficult. However, all that will actually matter will be the first order behavior of the curve near $t = 0$. This information can be encoded using geometric data which we call decorations.

Let A be a point in the projective plane, and let γ be a smooth curve with $\gamma(0) = A$. Define the associated *decoration* of A, denoted A^*, to be the tangent line of γ at A:

$$A^* = \lim_{t\to 0} \overleftrightarrow{A\gamma(t)}$$

When defined, A^* is a line passing through A.

By the same token, if a is a line in the projective plane then a can be thought of as a point in the dual plane. Given a curve γ through that point we can define the *decoration* a^* as

$$a^* = \lim_{t\to 0} a \cap \gamma(t)$$

When defined, a^* is a point lying on a.

Finally, let A be a twisted polygon and γ a curve in the space of twisted polygons with $\gamma(0) = A$. Then γ determines a curve in the plane through each vertex of A and a curve in the dual plane through each side of A. By the above, we can define decorations on each of these individual objects.

Definition 6.1 *A* decorated polygon *is a twisted polygon A together with the decorations of each of its vertices and sides induced by some curve γ in the space of twisted polygons with $\gamma(0) = A$.*

Decorated polygons will be denoted by the appropriate script letter. For instance if the underlying polygon is A then the decorated polygon will be called $\mathcal{A}$.

Proposition 6.2 *Let $A(t)$ be a curve in the space of polygons and let $B(t) = T(A(t))$ for all t. Let $\mathcal{A}$ and $\mathcal{B}$ be the corresponding decorations of $A = A(0)$ and $B = B(0)$ respectively. Then $\mathcal{B}$ is uniquely determined by $\mathcal{A}$. Moreover, $\mathcal{B}$ can be constructed from $\mathcal{A}$ using only a straightedge.*

The map taking $\mathcal{A}$ to $\mathcal{B}$ above should be though of as a lift of the pentagram map to the space of decorated polygons. To distinguish this operation from the original map, write $\mathcal{B} = \tilde{T}(\mathcal{A})$.

Remark 6.3 *We will only be using decorated polygons and the map $\tilde{T}$ as tools in our straightedge constructions. However, these are likely interesting objects to study in their own right. Some immediate questions come to mind such as*

- *What would be a good set of coordinates on the space of decorated polygons?*
- *In such coordinates, does the map $\tilde{T}$ take a nice form?*

- *Does $\tilde{T}$ define a discrete integrable system?*

A *degenerate polygon* is a collection of vertices and sides which can be realized as a limit of twisted polygons. Nearby vertices can be collinear, and even equal to each other. If A is a degenerate polygon and $A(t)$ is a curve with $A(0) = A$ and $A(t)$ generic for $t \neq 0$, we can define decorations on A as above. The result is a degenerate decorated polygon $\mathcal{A}$.

A degenerate decorated polygon does not always have a well-defined image under the pentagram map. However, given two such polygons which occur as consecutive iterates of the pentagram map, it is usually possible to determine the iterate immediately following the second one.

Proposition 6.4 *Let $A(t)$ be a curve in the space of twisted polygons that is generic away from $t = 0$. Let $B(t) = T(A(t))$ and $C(t) = T(B(t))$ for $t \neq 0$. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be the decorated polygons associated to these curves. Suppose that*

$$\forall j, (B_{j-1} = B_{j+1} \Longrightarrow A_{j-\frac{1}{2}} = A_{j+\frac{1}{2}} = B_{j-1}) \tag{6.1}$$

$$\forall i, (c_{i-\frac{1}{2}} = c_{i+\frac{1}{2}} \Longrightarrow b_{i-1} = b_{i+1} = c_{i-\frac{1}{2}}) \tag{6.2}$$

where j and i run over the vertex indices of B and C respectively. Then $\mathcal{C}$ is uniquely determined by $\mathcal{A}$ and $\mathcal{B}$. Moreover, $\mathcal{C}$ can be constructed using only a straightedge.

More specifically, it is always possible to determine the sides of $\mathcal{C}$, but if (6.1) fails for some j then the decoration of c_j is not well-defined. If we choose these unknown decorations arbitrarily, then we can determine the vertices of $\mathcal{C}$. However, if (6.2) fails for some i then C_i^* is not well-defined.

In the context of Proposition 6.4, write $\mathcal{C} = \tilde{T}_2(\mathcal{A}, \mathcal{B})$. Extend the construction $\tilde{T}_2$ to pairs $\mathcal{A}, \mathcal{B}$ that fail to satisfy (6.1) and/or (6.2) by choosing random decorations when necessary.

7 The main algorithm

The goal of our main algorithm is to construct $B = T^m(A)$ from A when the usual construction fails, i.e. when A is a singular point of various T^k for $k < m$. According to the previous section, it is typically possible to construct and decorate $T^2(A)$ from A, $T(A)$, and the corresponding decorations, even when singularities arise. The main construction, given in Algorithm 1, simply iterates this procedure.

Algorithm 1 main(A,m)

```
𝒜 := DecorateRandomly(A)
Iterates[0] := 𝒜
Iterates[1] := T̃(A)
for k := 2 to m do
    Iterates[k] := T̃₂(Iterates[k − 2], Iterates[k − 1])
end for
ℬ := Iterates[m]
return B
```

Given $S \subseteq \{1, 2, \ldots, n\}$ such that singularity confinement holds on X_S, let m be the smallest positive integer such that T^m is generically defined on X_S. We want to say that for generic $A \in X_S$, the main algorithm given A and m as input produces $T^m(A)$. For the simplest singularity types, $S = \{i\}$, this result can be proven using Propositions 6.2 and 6.4.

For more complicated S, a difficulty arises because the assumptions (6.1) and (6.2) in Proposition 6.4 will not hold at every step. Hence, some applications of $\tilde{T}_2$ in the main algorithm will produce random decorations. To prove correctness of the algorithm for such S, it is necessary to determine at which steps this occurs and to demonstrate that the outcome is independent of the random choices. Using this method, we have verified that the algorithm works for the singularity types of Theorem 4.2 when $|S|$ is small. In general, experimental evidence suggests that the algorithm behaves correctly for many, but not all, types.

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References

[1] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, Alternating-sign matrices and domino tilings, *J. Algebraic Combin.* **1** (1992), 111–132.

[2] M. Glick, The pentagram map and Y-patterns, *Adv. Math.* **227** (2011), 1019–1045.

[3] M. Glick, On singularity confinement for the pentagram map, arXiv:1110.0868.

[4] B. Grammaticos, A. Ramani, and V. Papageorgiou, Do integrable mappings have the Painlevé property?, *Phys. Rev. Lett.* **67** (1991), 1825–1828.

[5] V. Ovsienko, R. Schwartz, and S. Tabachnikov, The pentagram map: a discrete integrable system, *Comm. Math. Phys.* **299** (2010), 409-446.

[6] V. Ovsienko, R. Schwartz, and S. Tabachnikov, Liouville-Arnold integrability of the pentagram map on closed polygons, arXiv:1107.3633.

[7] S. Fomin and A. Zelevinsky, Cluster algebras IV: Coefficients, *Compos. Math.* **143** (2007), 112–164.

[8] R. Schwartz, Discrete monodromy, pentagrams, and the method of condensation, *J. Fixed Point Theory Appl.* **3** (2008), 379–409.

[9] R. Schwartz, The pentagram map, *Experiment. Math.* **1** (1992), 71–81.

[10] F. Soloviev, Integrability of the Pentagram Map, arXiv:1106.3950.