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Skew Pieri Rules for Hall-Littlewood Functions

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Abstract. We produce skew Pieri Rules for Hall–Littlewood functions in the spirit of Assaf and McNamara (FPSAC, 2010). The first two were conjectured by the first author (FPSAC, 2011). The key ingredients in the proofs are a q-binomial identity for skew partitions that are horizontal strips and a Hopf algebraic identity that expands products of skew elements in terms of the coproduct and antipode.

Résumé. Nous produisons quelques règles dissymétrique de Pieri pour les fonctions Hall–Littlewood au sens de Assaf et McNamara (FPSAC, 2010). Les premières deux règles ont été conjecturée par le premier auteur (FPSAC, 2011). Les principaux ingrédients dans les preuves sont une identité q-binomiale pour les partitions dissymétrique qui sont bandes horizontales et une identité de Hopf qui exprime les produits d'éléments dissymétrique en termes du coproduit et de l'antipode.

Keywords: Pieri Rules, Hall–Littlewood functions

Let $\Lambda[t]$ denote the ring of symmetric functions over $\mathbb{Q}(t)$, and let $\{s_\lambda\}$ and $\{P_\lambda(t)\}$ denote its bases of Schur functions and Hall–Littlewood functions, respectively, indexed by partitions λ. The Schur functions lead a rich life—making appearances in combinatorics, representation theory and Schubert calculus, among other places. See [Ful97, Mac95] for details. The Hall–Littlewood functions are nearly as ubiquitous (having as a salient feature that $P_\lambda(t) \to s_\lambda$ under the specialization $t \to 0$). See [LLT97] and the references therein for their place in the literature.

A classical problem is to determine cancellation-free formulas for multiplication in these bases,

$$s_\lambda\, s_\mu \;=\; \sum_\nu c^{\nu}_{\lambda,\mu}\, s_\nu \quad \text{and} \quad P_\lambda\, P_\mu \;=\; \sum_\nu f^{\nu}_{\lambda,\mu}(t)\, P_\nu .$$

The first problem was only given a complete solution in the latter half of the 20th century, while the second problem remains open. Special cases of the problem, known as *Pieri rules,* have been understood for quite a bit longer. The Pieri rules for Schur functions [Mac95, Ch. I, (5.16) and (5.17)] take the form

$$s_\lambda\, s_{1^r} = s_\lambda\, e_r = \sum_{\lambda^+} s_{\lambda^+}\ , \tag{1}$$

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with the sum over partitions λ^+ for which λ^+/λ is a vertical strip of size r, and

$$s_\lambda \, s_r = \sum_{\lambda^+} s_{\lambda^+} \,, \tag{2}$$

with the sum over partitions λ^+ for which λ^+/λ is a horizontal strip of size r. (See Section 1 for the definitions of vertical- and horizontal strip.) The Pieri rules for Hall–Littlewood functions [Mac95, Ch. III, (3.2) and (5.7)] state that

$$P_\lambda \, P_{1^r} = P_\lambda \, e_r = \sum_{|\lambda^+/\lambda|=r} \mathrm{vs}_{\lambda^+/\lambda}(t) P_{\lambda^+} \tag{3}$$

and

$$P_\lambda \, q_r = \sum_{|\lambda^+/\lambda|=r} \mathrm{hs}_{\lambda^+/\lambda}(t) P_{\lambda^+} \,, \tag{4}$$

with the sums again running over vertical strips and horizontal strips, respectively. Here q_r denotes $(1-t)P_r$ for $r > 0$ with $q_0 = P_0 = 1$, and $\mathrm{vs}_{\lambda/\mu}(t)$, $\mathrm{hs}_{\lambda/\mu}(t)$ are certain polynomials in t. (See Section 1 for their definitions, as well as those of $\mathrm{sk}_{\lambda/\mu}(t)$ and $\mathrm{br}_{\lambda/\mu}(t)$ appearing below.)

Our first result is a Pieri-type mixture of the two bases.

Theorem 1 *For a partition λ and $r \geq 0$, we have*

$$P_\lambda \, s_r = \sum_{\lambda^+} \mathrm{sk}_{\lambda^+/\lambda}(t) P_{\lambda^+} \,, \tag{5}$$

with the sum over partitions $\lambda^+ \supseteq \lambda$ for which $|\lambda^+/\lambda| = r$.

The main focus of this article is on the generalizations of Hall–Littlewood functions to skew shapes λ/μ. We introduce the question via the recent answer for skew Schur functions $s_{\lambda/\mu}$. In [AM11], Assaf and McNamara give a *skew Pieri rule* for Schur functions. They prove the following generalization of (2):

$$s_{\lambda/\mu} \, s_r = \sum_{\lambda^+,\, \mu^-} (-1)^{|\mu/\mu^-|} s_{\lambda^+/\mu^-} \,, \tag{6}$$

with the sum over pairs (λ^+, μ^-) of partitions such that λ^+/λ is a horizontal strip, μ/μ^- is a vertical strip and $|\lambda^+/\lambda| + |\mu/\mu^-| = r$. This elegant gluing-together of an s_r-type Pieri rule for the outer rim of λ/μ with an e_r-type Pieri rule for the inner rim of λ/μ demanded further exploration.

In [AM11], Lam, Sottile and the second author [LLS11] found a Hopf algebraic explanation for (6) that readily extended to many other settings. (For example, a skew Pieri rule for k-Schur functions was given.) Within the setting of Schur functions, it provided an easy extension of (6) to products of arbitrary skew Schur functions—a formula first conjectured by Assaf and McNamara in [AM11]. The results of this paper use the same Hopf machinery. We reprise most of the details and background in Section 2.

Around the same time, the first author [Kon] was motivated to give a skew Murnaghan–Nakayama rule in the spirit of [AM11]. Along the way, he gives a bijective proof of the conjugate form of (6) (only proven in [AM11] using the automorphism ω) and a *quantum* skew Murnaghan–Nakayama rule:

$$s_{\lambda/\mu} \, q_r = \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|} \, \mathrm{br}_{\lambda^+/\lambda}(t) \, \mathrm{br}_{(\mu/\mu^-)^c}(t) s_{\lambda^+/\mu^-} \,, \tag{7}$$

with the sum over pairs (λ^+, μ^-) of partitions such that λ^+/λ and μ/μ^- are broken ribbons and $|\lambda^+/\lambda| + |\mu/\mu^-| = r$. Note that since $P_r(0) = s_r$, we recover the skew Pieri rule for $t = 0$. Also, since $P_r(1) = p_r$, we recover a skew Murnaghan–Nakayama rule if we divide the formula by $1 - t$ and let $t \to 1$. This formula, like that in Theorem 1, may be viewed as a link between the two theories of Schur and Hall–Littlewood functions. One might ask for other examples of mixing, e.g., swapping the rolls of Schur and Hall–Littlewood functions in (7). Two such examples were found (conjecturally) in [Kon]. Their proofs, and a generalization of (6) to the Hall–Littlewood setting, are the main results of this paper.

Theorem 2 *For partitions* λ, μ, $\mu \subseteq \lambda$, *and* $r \geq 0$, *we have*

$$P_{\lambda/\mu}\, s_{1^r} = P_{\lambda/\mu}\, e_r = P_{\lambda/\mu}\, P_{1^r} = \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|}\, \mathrm{vs}_{\lambda^+/\lambda}(t)\, \mathrm{sk}_{\mu/\mu^-}(t)\, P_{\lambda^+/\mu^-}\ ,$$

where the sum on the right is over all $\lambda^+ \supseteq \lambda$, $\mu^- \subseteq \mu$ *such that* $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

Theorem 3 *For partitions* λ, μ, $\mu \subseteq \lambda$, *and* $r \geq 0$, *we have*

$$P_{\lambda/\mu}\, s_r = \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|}\, \mathrm{sk}_{\lambda^+/\lambda}(t)\, \mathrm{vs}_{\mu/\mu^-}(t)\, P_{\lambda^+/\mu^-}\ ,$$

where the sum on the right is over all $\lambda^+ \supseteq \lambda$, $\mu^- \subseteq \mu$ *such that* $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

Note that putting $\mu = \emptyset$ above recovers Theorem 1.

Theorem 4 *For partitions* λ, μ, $\mu \subseteq \lambda$, *and* $r \geq 0$, *we have*

$$P_{\lambda/\mu}\, q_r = \sum_{\lambda^+,\mu^-,\tau} (-1)^{|\mu/\mu^-|}(-t)^{|\tau/\mu^-|}\, \mathrm{hs}_{\lambda^+/\lambda}(t)\, \mathrm{vs}_{\mu/\tau}(t)\ \mathrm{sk}_{\tau/\mu^-}(t)\, P_{\lambda^+/\mu^-}\ ,$$

where the sum on the right is over all $\lambda^+ \supseteq \lambda$, $\mu^- \subseteq \tau \subseteq \mu$ *such that* $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

This paper is organized as follows. In Section 1, we prove some polynomial identities involving hs, vs and sk, prove Theorem 1, and find $\omega(q_r)$. In Section 2, we introduce our main tool, Hopf algebras. We conclude in Section 3 with the proofs of our main theorems.

1 Combinatorial Preliminaries

1.1 Notation and a key lemma

The conjugate partition of λ is denoted λ^c. We write $m_i(\lambda)$ for the number of parts of λ equal to i. For a partition λ, define $n(\lambda) = \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i^c}{2}$. The q-binomial coefficient is defined by

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(1-q^a)(1-q^{a-1})\cdots(1-q^{a-b+1})}{(1-q^b)(1-q^{b-1})\cdots(1-q)}$$

and is a polynomial in q that gives $\binom{a}{b}$ when $q = 1$.

We say that λ/μ is a *horizontal strip* (respectively *vertical strip*) if $[\lambda/\mu]$ contains no 2×1 (respectively 1×2) block, equivalently, if $\lambda_i^c \leq \mu_i^c + 1$ (respectively $\lambda_i \leq \mu_i + 1$) for all i. We say that λ/μ is a *ribbon* if $[\lambda/\mu]$ is connected and if it contains no 2×2 block, and that λ/μ is a *broken ribbon* if $[\lambda/\mu]$ contains no 2×2 block, equivalently, if $\lambda_i \leq \mu_{i-1}+1$ for $i \geq 2$. The Young diagram of a broken ribbon is a disjoint union of $\operatorname{rib}(\lambda/\mu)$ number of ribbons. The *height* $\operatorname{ht}(\lambda/\mu)$ (respectively *width* $\operatorname{wt}(\lambda/\mu)$) of a ribbon is the number of non-empty rows (respectively columns) of $[\lambda/\mu]$, minus 1. The height (respectively width) of a broken ribbon is the sum of heights (respectively widths) of the components.

Let us define some polynomials. For a horizontal strip λ/μ, define

$$\operatorname{hs}_{\lambda/\mu}(t) = \prod_{\substack{\lambda_j^c=\mu_j^c+1\\ \lambda_{j+1}^c=\mu_{j+1}^c}} (1-t^{m_j(\lambda)}).$$

If λ/μ is not a horizontal strip, define $\operatorname{hs}_{\lambda/\mu}(t)=0$. For a vertical strip λ/μ, define

$$\operatorname{vs}_{\lambda/\mu}(t) = \prod_{j\geq 1} \begin{bmatrix} \lambda_j^c-\lambda_{j+1}^c \\ \lambda_j^c-\mu_j^c \end{bmatrix}_t .$$

If λ/μ is not a vertical strip, define $\operatorname{vs}_{\lambda/\mu}(t)=0$. For a broken ribbon λ/μ, define

$$\operatorname{br}_{\lambda/\mu}(t) = (-t)^{\operatorname{ht}(\lambda/\mu)}(1-t)^{\operatorname{rib}(\lambda/\mu)}.$$

If λ/μ is not a broken ribbon, define $\operatorname{br}_{\lambda/\mu}(t)=0$. For any skew shape λ/μ, define

$$\operatorname{sk}_{\lambda/\mu}(t) = t^{\sum_j \binom{\lambda_j^c-\mu_j^c}{2}} \prod_{j\geq 1} \begin{bmatrix} \lambda_j^c-\mu_{j+1}^c \\ m_j(\mu) \end{bmatrix}_t .$$

Lemma 5 *For fixed λ,μ, $\mu\subseteq\lambda$, we have*

$$\sum_\nu (-t)^{|\lambda/\nu|} \operatorname{vs}_{\lambda/\nu}(t)\operatorname{sk}_{\nu/\mu}(t) = \operatorname{hs}_{\lambda/\mu}(t),$$

with the sum over all ν, $\mu\subseteq\nu\subseteq\lambda$, for which λ/ν is a vertical strip. □

Proof: Let $a_j=\lambda_j^c-\max(\mu_j^c,\lambda_{j+1}^c)\geq 0$. A partition ν, $\mu\subseteq\nu\subseteq\lambda$, for which λ/ν is a vertical strip is obtained by choosing k_j, $0\leq k_j\leq a_j$, and removing k_j bottom cells of column j in λ. See Figure 1 for the example $\lambda=98886666444$ and $\mu=77666633331$, where $a_4=3$, $a_6=2$, $a_8=3$, $a_9=1$ and $a_i=0$ for all other i.

We have $|\lambda/\nu|=\sum_j k_j$, $\nu_j^c=\lambda_j^c-k_j$. We make all such choices independently, which means that

$$\begin{aligned}\sum_\nu (-t)^{|\lambda/\nu|}\operatorname{sk}_{\nu/\mu}(t)\operatorname{vs}_{\lambda/\nu}(t) &= \sum_{k_1,k_2,\ldots} (-t)^{\sum_j k_j}\, t^{\sum_j \binom{\nu_j^c-\mu_j^c}{2}} \prod_j \begin{bmatrix} \nu_j^c-\mu_{j+1}^c \\ m_j(\mu)\end{bmatrix}_t \prod_j \begin{bmatrix} \lambda_j^c-\lambda_{j+1}^c \\ \lambda_j^c-\nu_j^c\end{bmatrix}_t \\ &= \prod_j \sum_{k_j=0}^{a_j} (-t)^{k_j}\, t^{\binom{\lambda_j^c-\mu_j^c-k_j}{2}} \begin{bmatrix} \lambda_j^c-k_j-\mu_{j+1}^c \\ m_j(\mu)\end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ k_j\end{bmatrix}_t . \qquad (8)\end{aligned}$$

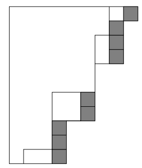


Fig. 1: A partition ν ($\mu \subseteq \nu \subseteq \lambda$) for which λ/ν is a vertical strip within λ/μ is built from λ by removing some number of the shaded cells of λ.

The rest of the proof consists of an involved case-by-case analysis which repeatedly uses the q-binomial theorem. Specifically, first we show that if λ/μ is a horizontal strip, then the term in the product (8) corresponding to j is $1 - t^{m_j(\lambda)}$ if $\lambda_j^c = \mu_j^c + 1, \lambda_{j+1}^c = \mu_{j+1}^c$, and 1 otherwise. Next, we show that if λ/μ is not a horizontal strip and j is the largest index for which $\lambda_j^c - \mu_j^c \geq 2$, then the term in the product (8) corresponding to j is 0. $\square$

1.2 Elementary Hall–Littlewood identities

We give two applications of Lemma 5, then prove some elementary properties of Hall–Littlewood functions that will be useful in Section 3. The first application is a formula for the product of a Hall–Littlewood polynomial with the Schur function s_r.

Proof of Theorem 1: We induct on r. For $r = 0$, there is nothing to prove. For $r > 0$, we use the formula

$$q_r = \sum_{k=0}^{r} (-t)^k s_{r-k} e_k, \tag{9}$$

which is easy to prove using, say, [Sta99, Exercise 7.11]) and the conjugate Pieri rule. For $|\lambda^+/\lambda| = r$, the coefficient of P_{λ^+} in

$$P_\lambda\, s_r = P_\lambda \left(q_r - \sum_{k=1}^{r} (-t)^k s_{r-k} e_k \right)$$

is (by (3), (4) and induction) equal to

$$\mathrm{hs}_{\lambda^+/\lambda}(t) - \sum (-t)^{|\lambda^+/\nu|}\, \mathrm{sk}_{\nu/\lambda}(t)\, \mathrm{vs}_{\lambda^+/\nu}(t),$$

with the sum over all ν, $\lambda \subseteq \nu \subseteq \lambda^+$, for which λ^+/ν is a vertical strip of size at least 1. By Lemma 5, this is equal to $\mathrm{sk}_{\lambda^+/\lambda}(t)$. $\square$

Recall that $f^\lambda_{\sigma,\tau}(t)$ is the (polynomial) coefficient of P_λ in $P_\sigma P_\tau$.

Corollary 6 *The structure constants $f^\lambda_{\mu,\tau}(t)$ satisfy* $\displaystyle\sum_\tau t^{n(\tau)} f^\lambda_{\mu,\tau}(t) = \mathrm{sk}_{\lambda/\mu}(t)$.

Proof: This follows from $s_r = \sum_{\tau \vdash r} t^{n(\tau)} P_\tau$, which is (2) in [Mac95, page 219] and also Theorem 1 for $\lambda = \emptyset$. $\square$

The second application of Lemma 5 is a generalization of [Mac95, §III.3, Example 1].

Theorem 7 *For every λ, μ, we have*

$$\sum_{\nu} \mathrm{sk}_{\nu/\mu}(t)\,\mathrm{vs}_{\lambda/\nu}(t) y^{|\lambda/\nu|} = \sum_{\sigma} t^{n(\sigma)-\binom{\ell(\sigma)}{2}} f^{\lambda}_{\sigma\mu}(t) \prod_{j=1}^{\ell(\sigma)} (y + t^{j-1}). \tag{10}$$

Equivalently, for all m,

$$\sum_{\nu\colon |\lambda/\nu|=m} \mathrm{sk}_{\nu/\mu}(t)\,\mathrm{vs}_{\lambda/\nu}(t) = \sum_{\sigma} t^{n(\sigma)-\binom{m}{2}} f^{\lambda}_{\sigma\mu}(t) \begin{bmatrix} \ell(\sigma) \\ m \end{bmatrix}_{t^{-1}}. \tag{11}$$

Proof: Let us evaluate $P_\mu\, s_r\, (\sum_m e_m\, y^m)$ in two different ways. On the one hand,

$$P_\mu\, s_r \left(\sum_m e_m\, y^m \right) = \left(\sum_\nu \mathrm{sk}_{\nu/\mu}(t) P_\nu \right) \left(\sum_m e_m\, y^m \right) = \sum_{\nu,\lambda} \mathrm{sk}_{\nu/\mu}(t)\,\mathrm{vs}_{\lambda/\nu}(t) P_\lambda\, y^{|\lambda/\nu|}.$$

On the other hand, using Example 1 on page 218 of [Mac95],

$$P_\mu\, s_r \left(\sum_m e_m y^m \right) = P_\mu \sum_\sigma t^{n(\sigma)} P_\sigma \prod_{j=1}^{\ell(\sigma)} (1 + t^{1-j} y) = \sum_{\sigma,\lambda} t^{n(\sigma)-\binom{\ell(\sigma)}{2}} f^{\lambda}_{\sigma\mu}(t) P_\lambda \prod_{j=1}^{\ell(\sigma)} (y + t^{j-1}).$$

Now (10) follows by taking the coefficient of P_λ in both expressions. For (11), we use the q-binomial theorem and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{t^{-1}} = t^{\binom{k}{2}+\binom{n-k}{2}-\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_t .$$

□

Remark 8 The theorem is indeed a generalization of [Mac95, §III.3, Example 1]. For $\mu = \emptyset$, $\mathrm{sk}_{\nu/\mu}(t) = t^{n(\nu)}$, and the right-hand side of (11) is non-zero only for $\sigma = \lambda$, so the last equation on page 218 (*loc. cit.*) follows. It also generalizes Lemma 5: for $y = -t$, the right-hand side of (10) is non-zero if and only if $\ell(\sigma) = 1$, and is therefore equal to $\mathrm{hs}_{\lambda/\mu}(t)$.

We finish the section with two more lemmas.

Lemma 9 *Given $r > k \geq 0$, we have*

$$s_{r-k,1^k} = \sum_{\lambda\colon \ell(\lambda)\geq k+1} t^{\binom{\ell(\lambda)-k}{2}+\sum_{i=2}^{\lambda_1}\binom{\lambda_i^c}{2}} \begin{bmatrix} \ell(\lambda)-1 \\ k \end{bmatrix}_t P_\lambda.$$

Proof: The lemma follows from the formula due to Lascoux and Schützenberger, see, e.g., [Mac95, Ch. III, (6.5)]. In that terminology, we have to evaluate $K_{(r-k,1^k),\lambda}(t)$. We choose a semistandard Young tableau T of shape $(r-k, 1^k)$ and type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. Clearly, such tableaux are in one-to-one correspondence with k-subsets of the set $\{2, \ldots, \ell\}$. For such a subset S, write s for the word with the elements of S in increasing order, and write $\overline{s}$ for the word with the elements of $\{2, \ldots, \ell\} \backslash S$ in decreasing order. The reading word of the tableau corresponding to S is $\ell^{\lambda_\ell - 1} \cdots 3^{\lambda_3-1} 2^{\lambda_2-1} 1^{\lambda_1} s$. The subwords $w_2, w_3, \ldots$ are all strictly decreasing, and $w_1 = \overline{s}1s$. The charges of $w_2, w_3, \ldots$ are $\binom{\lambda_2^c}{2}, \binom{\lambda_3^c}{2}, \ldots$, while the charge of w_1 is $\sum_{i \notin S}(\ell - i + 1)$. The formula

$$\sum_{S \subseteq \{2,\ldots,\ell+1\}, |S|=k} t^{\sum_{i\notin S}(\ell-i+1)} = t^{\binom{\ell-k}{2}} \begin{bmatrix} \ell - 1 \\ k \end{bmatrix}_t$$

can be proved by induction. □

Lemma 10 *Let ω be the fundamental involution on $\Lambda[t]$ defined by $\omega(s_\lambda) = s_{\lambda^c}$. We have*

$$\omega(q_r) = (-1)^r \sum_{\lambda \vdash r} c_\lambda(t) P_\lambda,$$

where

$$c_\lambda(t) = t^{\sum_{i=2}^{\lambda_1} \binom{\lambda_i^c + 1}{2}} \prod_{i=1}^{\ell(\lambda)} (-1 + t^i).$$

Proof: We have

$$\begin{aligned}
\omega(P_r) &= \omega\left(\sum_{k=0}^{r-1} (-t)^{r-k-1} s_{k+1,1^{r-k-1}}\right) = \sum_{k=0}^{r-1} (-t)^{r-k-1} s_{r-k,1^k} = \\
&= \sum_{k=0}^{r-1} (-t)^{r-k-1} \left(\sum_{\ell(\lambda) \geq k+1} t^{\binom{\ell(\lambda)-k}{2} + \sum_{i=2}^{\lambda_1} \binom{\lambda_i^c}{2}} \begin{bmatrix} \ell(\lambda) - 1 \\ k \end{bmatrix}_t P_\lambda \right) = \\
&= \sum_{\lambda \vdash r} \left(\sum_{k=0}^{\ell(\lambda)-1} (-t)^{r-k-1} \, t^{\binom{\ell(\lambda)-k}{2} + \sum_{i=2}^{\lambda_1} \binom{\lambda_i^c}{2}} \begin{bmatrix} \ell(\lambda) - 1 \\ k \end{bmatrix}_t \right) P_\lambda.
\end{aligned}$$

Now by the q-binomial theorem,

$$\prod_{i=2}^{\ell(\lambda)} (-1 + t^i) = t^{2(\ell(\lambda)-1)} \prod_{i=0}^{\ell(\lambda)-2} (-1/t^2 + t^i) = t^{2(\ell(\lambda)-1)} \sum_{k=0}^{\ell(\lambda)-1} t^{\binom{\ell(\lambda)-1-k}{2}} \begin{bmatrix} \ell(\lambda) - 1 \\ k \end{bmatrix}_t \left(-\frac{1}{t^2}\right)^k,$$

and simple calculations show that the coefficient of P_λ in $\omega(q_r) = (1-t)\omega(P_r)$ is indeed $(-1)^r c_\lambda(t)$. □

2 Hopf Perspective on Skew Elements

Recall that $\Lambda[t]$ has another important basis $\{Q_\lambda\}$, defined by $Q_\lambda = b_\lambda(t) P_\lambda$, where $b_\lambda(t) = \prod_{i \geq 1} (1 - t)(1-t^2) \cdots (1 - t^{m_i(\lambda)})$. The (extended) Hall scalar product on $\Lambda[t]$ is defined by either of the equivalent conditions

$$\langle P_\lambda, Q_\mu \rangle = \delta_{\lambda\mu} \quad \text{or} \quad \langle p_\lambda, p_\mu \rangle = z_\mu(t)\, \delta_{\lambda\mu}\,,$$

where $z_\mu(t) := z_\mu \cdot \prod_{j=1}^r (1-t^{\mu_j})^{-1} = \prod_{i=1}^k (i^{a_i} a_i!) \prod_{j=1}^r (1-t^{\mu_j})^{-1}$ for $\mu = (\mu_1, \mu_2, \ldots, \mu_r) = \langle 1^{a_1}, 2^{a_2}, \cdots, k^{a_k} \rangle$. See [Mac95, §III.4]. The skew Hall–Littlewood function $P_{\lambda/\mu}$ is defined [Mac95, Ch. III, (5.1′)] as the unique function satisfying

$$\left\langle P_{\lambda/\mu}, Q_\nu \right\rangle = \langle P_\lambda, Q_\nu \, Q_\mu \rangle \tag{12}$$

for all $Q_\nu \in \Lambda[t]$. (Likewise for $Q_{\lambda/\mu}$.) If we choose to read $P_{\lambda/\mu}$ as, "Q_μ skews P_λ," then we allow ourselves access to the machinery of Hopf algebra actions on their duals. We introduce the basics in Subsection 2.1 and return to $\Lambda[t]$ and Hall–Littlewood functions in Subsection 2.2.

2.1 Hopf preliminaries

Let $H = \bigoplus_n H_n$ be a graded algebra over a field $\Bbbk$. Recall that H is a Hopf algebra if there are algebra maps $\Delta\colon H \to H \otimes H$ and $\varepsilon\colon H \to \Bbbk$, and an algebra antimorphism $S\colon H \to H$, called the *coproduct*, *counit* and *antipode*, respectively, satisfying some additional compatibility conditions. See [Mon93]. We use Sweedler's notation for the coproduct, denoting $\Delta(h)$ by $\sum_{(h)} h' \otimes h''$ for $h \in H$.

Let $H^* = \bigoplus_n H_n^*$ denote the graded dual of H. If each H_n is finite dimensional, then the pairing $\langle \cdot , \cdot \rangle : H \otimes H^* \to \Bbbk$ defined by $\langle h, a \rangle = a(h)$ is nondegenerate. This pairing naturally endows H^* with a Hopf algebra structure, with product and coproduct uniquely determined by the formulas:

$$\langle h, a \cdot b \rangle := \langle \Delta(h), a \otimes b \rangle \quad \text{and} \quad \langle g \otimes h, \Delta(a) \rangle := \langle g \cdot h, a \rangle$$

for all homogeneous $g, h \in H$ and $a, b \in H^*$. (Extend to all of H^* by linearity, insisting that $\langle H_n, H_m^* \rangle = 0$ for $n \neq m$.)

We recall some standard actions ("$\rightharpoonup$") of H and H^* on each other. Given $h \in H$ and $a \in H^*$, put

$$a \rightharpoonup h := \sum_{(h)} \langle h'', a \rangle \, h' \quad \text{and} \quad h \rightharpoonup a := \sum_{(a)} \langle h, a'' \rangle \, a'. \tag{13}$$

Equivalently, $\langle g, h \rightharpoonup a \rangle = \langle g \cdot h, a \rangle$ and $\langle a \rightharpoonup h, b \rangle = \langle h, b \cdot a \rangle$. We call these *skew elements* (in H and H^*, respectively) to keep the nomenclature consistent with that in symmetric function theory.

Our skew Pieri rules (Theorems 2, 3 and 4) come from an elementary formula relating products of elements h and skew elements $a \rightharpoonup g$ in a Hopf algebra H:

$$(a \rightharpoonup g) \cdot h = \sum \big(S(h'') \rightharpoonup a \big) \rightharpoonup (g \cdot h'). \tag{14}$$

See $(*)$ in the proof of [Mon93, Lemma 2.1.4] or [LLS11, Lemma 1]. Before turning to the proofs of these theorems, we first recall the Hopf structure of $\Lambda[t]$.

2.2 The Hall–Littlewood setting

The ring $\Lambda[t]$ is generated by the one-part power sum symmetric functions p_r ($r > 0$), so the definitions

$$\Delta(p_r) := 1 \otimes p_r + p_r \otimes 1, \quad \varepsilon(p_r) := 0, \quad \text{and} \quad S(p_r) := -p_r \tag{15}$$

completely determine the Hopf structure of $\Lambda[t]$.

Proposition 11 *For $r > 0$ and c_λ given by Lemma 10,*

$$\Delta(e_r) = \sum_{k=0}^{r} e_k \otimes e_{r-k} \qquad \Delta(s_r) = \sum_{k=0}^{r} s_k \otimes s_{r-k} \qquad \Delta(q_r) = \sum_{k=0}^{r} q_k \otimes q_{r-k}$$
$$S(e_r) = (-1)^r s_r \qquad S(s_r) = (-1)^r e_r \qquad S(q_r) = \sum_{\lambda \vdash r} c_\lambda P_\lambda .$$

Proof: Equalities for e_r and s_r are elementary consequences of (15) and may be found in [Mac95, §I.5, Example 25]. The coproduct formula for q_r is (2) in [Mac95, §III.5, Example 8]. The antipode formula for q_r is identical to Lemma 10, as the fundamental morphism ω and the antipode S are related by $S(h) = (-1)^r \omega(h)$ on homogeneous elements h of degree r. □

It happens that $\Lambda[t]$ is a self-dual Hopf algebra. This may be deduced from [Mac95, §III.5, Example 8] and is easy to see on the power sum basis. (Write p^*_λ for $z_\lambda(t)^{-1} p_\lambda$ and use (15) to check that $\langle p_\lambda, p^*_\mu \cdot p^*_\nu \rangle = \langle \Delta(p_\lambda), p^*_\mu \otimes p^*_\nu \rangle$ and $\langle p_\mu \otimes p_\nu, \Delta(p^*_\lambda) \rangle = \langle p_\mu \cdot p_\nu, p^*_\lambda \rangle$ for all partitions λ, μ and ν.)

After (12), (13) and self-duality, we see that $P_{\lambda/\mu} = Q_\mu \rightharpoonup P_\lambda$, and similarly, $Q_{\lambda/\mu} = P_\mu \rightharpoonup Q_\lambda$.

3 Proofs of the main theorems

We specialize (14) to Hall–Littlewood polynomials, putting $a \rightharpoonup g = P_{\lambda/\mu}$.

Proof of Theorem 2: Taking $h = e_r$ in (14), we get

$$P_{\lambda/\mu} \cdot e_r = (Q_\mu \rightharpoonup P_\lambda) \cdot e_r = \sum_{(e_r)} (S(e_r{}'') \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot e_r{}'\big) \tag{16}$$

$$= \sum_{k=0}^{r} (S(e_k) \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot e_{r-k}\big) \tag{17}$$

$$= \sum_{k=0}^{r} (-1)^k \,(s_k \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot e_{r-k}\big) \tag{18}$$

$$= \sum_{k=0}^{r} (-1)^k \left(\sum_\tau t^{n(\tau)} Q_{\mu/\tau} \right) \rightharpoonup \big(P_\lambda \cdot e_{r-k}\big) \tag{19}$$

$$= \sum_{k=0}^{r} (-1)^k \left(\sum_{|\mu/\mu^-|=k} \left(\sum_\tau t^{n(\tau)} f^{\,\mu}_{\mu^-,\tau}(t) \right) Q_{\mu^-} \right) \rightharpoonup \left(\sum_{|\lambda^+/\lambda|=r-k} \mathrm{vs}_{\lambda^+/\lambda}(t) P_{\lambda^+} \right) \tag{20}$$

$$= \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|}\, \mathrm{sk}_{\mu/\mu^-}(t)\, \mathrm{vs}_{\lambda^+/\lambda}(t) P_{\lambda^+/\mu^-} \,. \tag{21}$$

For (17) and (18), we used Proposition 11. For (19), we used $s_k \rightharpoonup Q_\mu = \big(\sum_{\tau \vdash k} t^{n(\tau)} P_\tau\big) \rightharpoonup Q_\mu = \sum_{\tau \vdash k} t^{n(\tau)} Q_{\mu/\tau}$. We use (3) and (12) to pass from (19) to (20). Explicitly, the coefficient of Q_{μ^-} in the expansion of $Q_{\mu/\tau}$ is equal to the coefficient of P_μ in $P_{\mu^-} P_\tau$. Finally, (21) follows from Corollary 6. □

Proof of Theorem 3: Taking $h = s_r$ in (14), we get

$$P_{\lambda/\mu} \cdot s_r = (Q_\mu \rightharpoonup P_\lambda) \cdot s_r = \sum_{(s_r)} (S(s_r{}'') \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot s_r{}'\big) \tag{22}$$

$$= \sum_{k=0}^{r} (S(s_k) \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot s_{r-k}\big) \tag{23}$$

$$= \sum_{k=0}^{r} (-1)^k \, (e_k \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot s_{r-k}\big) \tag{24}$$

$$= \sum_{k=0}^{r} (-1)^k Q_{\mu/1^k} \rightharpoonup \big(P_\lambda \cdot s_{r-k}\big) \tag{25}$$

$$= \sum_{k=0}^{r} (-1)^k \left(\sum_{|\mu/\mu^-|=k} \mathrm{vs}_{\mu/\mu^-}(t) Q_{\mu^-} \right) \rightharpoonup \left(\sum_{|\lambda^+/\lambda|=r-k} \mathrm{sk}_{\lambda^+/\lambda}(t) P_{\lambda^+} \right) \tag{26}$$

$$= \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|} \, \mathrm{vs}_{\mu/\mu^-}(t) \, \mathrm{sk}_{\lambda^+/\lambda}(t) P_{\lambda^+/\mu^-} \, . \tag{27}$$

For (23) and (24), the proof is the same as above. For (25), we used $e_k = P_{1^k}$, while for (26), we used (3) and (5). Equation (27) is obvious. □

Proof of Theorem 4: Our first proof is along the lines of the preceding proofs of Theorems 2 and 3.

$$P_{\lambda/\mu} \cdot q_r = (Q_\mu \rightharpoonup P_\lambda) \cdot q_r = \sum_{(q_r)} (S(q_r{}'') \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot q_r{}'\big) \tag{28}$$

$$= \sum_{k=0}^{r} (S(q_k) \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot q_{r-k}\big) \tag{29}$$

$$= \sum_{k=0}^{r} \left(\sum_{\tau \vdash k} c_\tau(t) P_\tau \rightharpoonup Q_\mu \right) \rightharpoonup \big(P_\lambda \cdot q_{r-k}\big) \tag{30}$$

$$= \sum_{k=0}^{r} \left(\sum_{\tau \vdash k} c_\tau(t) Q_{\mu/\tau} \right) \rightharpoonup \big(P_\lambda \cdot q_{r-k}\big) \tag{31}$$

$$= \sum_{k=0}^{r} \left(\sum_{|\mu/\mu^-|=k} \left(\sum_{\tau} c_\tau(t) f^{\mu}_{\mu^-,\tau}(t) \right) Q_{\mu^-} \right) \rightharpoonup \left(\sum_{|\lambda^+/\lambda|=r-k} \mathrm{hs}_{\lambda^+/\lambda}(t) P_{\lambda^+} \right) \tag{32}$$

$$= \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|} (-t)^{|\tau/\mu^-|} \, \mathrm{vs}_{\mu/\tau}(t) \, \mathrm{sk}_{\tau/\mu^-} \, \mathrm{hs}_{\lambda^+/\lambda}(t) P_{\lambda^+/\mu^-} \, . \tag{33}$$

The only line that needs a comment is (33).

Substitute $y = -1/t$, $\lambda = \mu$, $\mu = \mu^-$ and $\nu = \tau$ into Theorem 7. We get

$$\sum_{\tau} \mathrm{sk}_{\tau/\mu^-}(t)\,\mathrm{vs}_{\mu/\tau}(t)(-1/t)^{|\mu/\tau|} = \sum_{\sigma} t^{n(\sigma)-\binom{\ell(\sigma)}{2}} f^{\mu}_{\tau,\mu^-}(t) \prod_{j=1}^{\ell(\sigma)} (-1/t + t^{j-1}),$$

which after a little arithmetic yields

$$\sum_{\sigma} c_\sigma f^{\mu}_{\sigma,\mu^-}(t) = \sum_{\tau} (-1)^{|\mu/\tau|}\, t^{|\tau/\mu^-|}\, \mathrm{sk}_{\tau/\mu^-}(t)\,\mathrm{vs}_{\mu/\tau}(t).$$

Our second proof uses Theorems 1, 2 and 3. Recall from (9) that $q_r = \sum_{k=0}^{r} (-t)^k s_{r-k} e_k$. We have

$$\begin{aligned}
P_{\lambda/\mu} \cdot q_r &= P_{\lambda/\mu} \cdot \left(\sum_{k=0}^{r} (-t)^k s_{r-k} e_k \right) = \sum_{k=0}^{r} (-t)^k (P_{\lambda/\mu} s_{r-k}) e_k \\
&= \sum_{k=0}^{r} (-t)^k \sum_{\sigma,\tau} (-1)^{|\mu/\tau|}\, \mathrm{vs}_{\mu/\tau}(t)\, \mathrm{sk}_{\sigma/\lambda}(t) P_{\sigma/\tau} e_k \\
&= \sum_{\sigma,\tau,\mu^-,\lambda^+} (-t)^{|\tau/\mu^-|+|\lambda^+/\sigma|} (-1)^{|\mu/\tau|+|\tau/\mu^-|}\, \mathrm{vs}_{\mu/\tau}(t)\, \mathrm{sk}_{\sigma/\lambda}(t)\, \mathrm{sk}_{\tau/\mu^-}(t)\, \mathrm{vs}_{\lambda^+/\sigma}(t) P_{\lambda^+/\mu^-} \\
&= \sum_{\tau,\mu^-,\lambda^+} (-1)^{|\mu/\mu^-|} (-t)^{|\tau/\mu^-|}\, \mathrm{vs}_{\mu/\tau}(t)\, \mathrm{sk}_{\tau/\mu^-}(t) \left(\sum_{\sigma} (-t)^{|\lambda^+/\sigma|}\, \mathrm{vs}_{\lambda^+/\sigma}(t)\, \mathrm{sk}_{\sigma/\lambda}(t) \right) P_{\lambda^+/\mu^-} \\
&= \sum_{\tau,\mu^-,\lambda^+} (-1)^{|\mu/\mu^-|} (-t)^{|\tau/\mu^-|}\, \mathrm{vs}_{\mu/\tau}(t)\, \mathrm{sk}_{\tau/\mu^-}(t)\, \mathrm{hs}_{\lambda^+/\lambda}(t) P_{\lambda^+/\mu^-},
\end{aligned}$$

where we used Lemma 5 in the final step. □

Remark 12 It would be preferable to have a simpler expression for the polynomial

$$d_{\lambda/\mu}(t) = \sum_{\nu} (-t)^{|\nu/\mu|}\, \mathrm{vs}_{\lambda/\nu}(t)\, \mathrm{sk}_{\nu/\mu}(t), \tag{34}$$

i.e., one involving only the boxes of λ/μ in the spirit of $\mathrm{hs}_{\lambda/\mu}(t)$, so that we could write

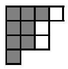
$$P_{\lambda/\mu} \cdot q_r = \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|}\, \mathrm{hs}_{\lambda^+/\lambda}(t) d_{\mu/\mu^-}(t)\, P_{\lambda^+/\mu^-}\ ,$$

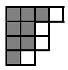
where the sum is over all $\lambda^+ \supseteq \lambda$, $\mu^- \subseteq \mu$ such that $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

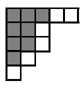
Toward this goal, we point out a hidden symmetry in the polynomials $d_{\lambda/\mu}$(t). Writing q_r as the sum $\sum_{k=0}^{r} (-t)^k e_k s_{r-k}$ before running through the second proof of Theorem 4 (i.e., applying Theorems 2 and 3 in the reverse order) reveals

$$d_{\lambda/\mu}(t) = \sum_{\nu} (-t)^{|\lambda/\nu|}\, \mathrm{sk}_{\lambda/\nu}(t)\ \mathrm{vs}_{\nu/\mu}(t)\,. \tag{35}$$

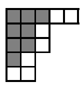
Basic computations suggest some hint of a polynomial-product description for $d_{\lambda/\mu}(t)$,

 $: \ -(t-1)^2(t+1)\left(t^3+t^2+t-1\right)$

 $: \ (t-1)^2(t+1)\left(t^3+t^2+t-1\right)^2$

 $: \ t(t-1)^2(t+1)\left(t^2+t-1\right)\left(t^3+t^2+t-1\right)^2,$

but others suggest that such a description will not be tidy,

 $: \ -t^2(t-1)^2(t+1)^2\left(t^3+t^2+t-1\right)\left(t^7+t^6+2t^5-t^3-2t^2-t+1\right).$

We leave a concise description of the $d_{\lambda/\mu}(t)$ as an open problem.

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