FPSAC 2012, Nagoya, Japan DMTCS proc. **AR**, 2012, 491–502

Moment graphs and Kazhdan-Lusztig polynomials

Martina Lanini[1]

[1] *Department Mathematik Friedrich-Alexander-Universität Erlangen-Nürnberg*

Abstract. Motivated by a result of Fiebig (2007), we categorify some properties of Kazhdan-Lusztig polynomials via sheaves on Bruhat moment graphs. In order to do this, we develop new techniques and apply them to the combinatorial data encoded in these moment graphs.

Résumé. Motivés par un resultat de Fiebig (2007), nous categorifions certaines propriétés des polynômes de Kazhdan-Lusztig en utilisant faisceaux sur les graphes moment de Bruhat. Pour faire ça, nous développons de nouvelles techniques et les appliquons ensuite aux données combinatoires encodées dans ces graphes moment.

Keywords: Bruhat graphs, moment graphs, Kazhdan-Lusztig polynomials

1 Introduction

Moment graphs, as well as Kazhdan-Lusztig polynomials, straddle the intersection of algebraic combinatorics, representation theory and geometric representation theory. While the combinatorial core of the Kazhdan-Lusztig theory has been investigated for thirty years, after the seminal paper (Kazhdan and Lusztig, 1979) where these polynomials were defined, moment graphs have not yet been studied as combinatorial objects.

It is possible to develop a categorical and very general theory of moment graphs and sheaves on them, as we have done in (Lanini, 2011). In this extended abstract, however, we are going to focus our attention on a fundamental class of moment graphs: the regular and parabolic Bruhat (moment) graphs. In order to get a theory of moment graphs, we first had to choose if we were going to work with moment graphs on a vector space or on a lattice. The first possibility would enable us to associate a moment graph to any Coxeter system (cf. Fiebig, 2008b), while the second one has the advantage that a modular theory could be developed (cf. Fiebig, 2011). We decided to work with moment graphs on a lattice, because our results in characteristic zero categorify properties of Kazhdan-Lusztig polynomials, while in positive characteristic they give also information about the stalks of indecomposable parity sheaves (cf. Fiebig and Williamson, 2010). Thus, from now on we will speak of k-moment graphs, where k is any local ring with $2 \in k^{\times}$. However, our proofs can be adapted to moment graphs on a vector space, by slightly modifying some definitions.

As unlabelled oriented graphs, moment graphs were introduced by Dyer in 1991 (cf. Dyer, 1991) in order to study some properties of the Bruhat order on a Coxeter group; already in 1993, he considered them as edge–labelled oriented graphs. Actually, he was labelling the edges by reflections of the Coxeter

1365–8050 © 2012 Discrete Mathematics and Theoretical Computer Science (DMTCS), Nancy, France

group (cf. Dyer, 1993), instead of the corresponding positive coroots (see Def.2.2). Even if his definition seems equivalent to ours, the extra structure coming from the whole root lattice turns out to be fundamental when we are considering morphisms between two Bruhat (k-moment) graphs (see §2.2). Since in this work k-moment graphs will replace the role played by topological spaces in the usual sheaf theory, a good notion of morphisms between them is essential in order to define the pullback and pushforward functors (§3.3). The most important class of sheaves on a k-moment graph are the Braden-MacPherson—or canonical—sheaves. As their name suggests, they were introduced by Braden and MacPherson in (Braden and MacPherson, 2001) in order to give a combinatorial algorithm to compute the equivariant intersection cohomology of a "nice" projective algebraic variety X, equipped with an action of an algebraic torus and having a T-invariant Whitney stratification. Here, "nice" means that T acts equivariantly formally (cf. Goresky et al., 1998, §1.2) on X with finitely many 1-dimensional orbits and finitely many fixed points (all of which are isolated). In (Fiebig, 2008b), Fiebig defined the category, $\mathcal{H}$, whose objects are given by direct sums and direct summands of the spaces of global sections of Braden-MacPherson sheaves on a Bruhat k-moment graph and he proved it to be equivalent to the category of special Soergel bimodules, introduced by Soergel to categorify the Hecke algebra (cf. Soergel, 2007).

A famous conjecture of Soergel on this category may be now rephrased in terms of an explicit formula relating indecomposable Braden-MacPherson sheaves on a (possibly parabolic) Bruhat k-moment graph and (Deodhar's parabolic) Kazhdan-Lusztig polynomials (cf. Deodhar, 1987). In this form, the conjecture is due to Fiebig and we will briefly discuss it in §4.1. This conjecture motivates our work: we will try to interpret some properties of Kazhdan-Lusztig polynomials in the moment graph setting. In this extended abstract we explain to the reader some possible approaches to this problem that we developed in (Lanini, 2011). In particular, after recalling some basics definitions and properties of k-moment graphs and sheaves on them, we will present several different strategies to lift equalities concerning Kazhdan-Lusztig polynomials at the canonical sheaf level. We will see that, in some particularly favourable situations, it will be enough to find an isomorphism between the underlying Bruhat k-moment graphs (see Theor.4.1), while in other situations the solution will be rather difficult (see Theor.5.1).

2 The category of k-moment graphs

From now on, Y will denote a lattice of finite rank, k a local ring such that $2 \in k^\times$ and $Y_k := Y \otimes_{\mathbb{Z}} k$. We recall the definition of *k-moment graphs* on a lattice Y, following (Fiebig, 2010b). In (Goresky et al., 1998), Goresky, Kottwitz and MacPherson used moment graphs on a complex projective space, but the definition we will be working with will enable us to generalise all constructions to positive characteristic.

Definition 2.1 *(Fiebig, 2010b) Let Y be a lattice of finite rank. A* moment graph on the lattice Y *is given by $(\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$, where*

(MG1) $(\mathcal{V}, \mathcal{E})$ *is a directed graph without oriented cycles or multiple edges,*

(MG2) $\trianglelefteq$ *is a partial order on $\mathcal{V}$ such that if $x, y \in \mathcal{V}$ and $E : x \to y \in \mathcal{E}$, then $x \trianglelefteq y$,*

(MG3) $l : \mathcal{E} \to Y \setminus \{0\}$ *is a map called the* label function.

The fundamental example that we should always keep in mind is the following.

2.1 Moment graphs associated to a symmetrisable Kac-Moody Algebra

Let $\mathfrak{g}$ be a symmetrisable Kac-Moody algebra, with root datum $(\Pi, \mathfrak{h}, \Pi^\vee)$ and Weyl group $\mathcal{W}$. As subgroup of $GL(\mathfrak{h}^*)$, $\mathcal{W}$ is generated by the set of simple reflections $\mathcal{S} = \{s_\alpha | \alpha \in \Pi\}$ and it is known that

$(\mathcal{W}, \mathcal{S})$ is a Coxeter system (cf. Kac, 1983, §3.10). Let us take $J \subseteq \mathcal{S}$ and denote by $\mathcal{W}_J$ the subgroup of $\mathcal{W}$ generated by J and by $\mathcal{W}^J$ the set of minimal representatives for the equivalence classes $\mathcal{W}_J \setminus \mathcal{W}$. Finally, denote by Δ_+^{re} the set of positive real roots and recall that this is an indexing set for the set of all reflections $\mathcal{T}$ of $\mathcal{W}$.

Definition 2.2 *Let $\mathcal{W}$, $\mathcal{S}$ and J be as above. Then the* parabolic Bruhat (moment) graph $\mathcal{G}^J = \mathcal{G}(\mathcal{W}^J) = (\mathcal{V}, \mathcal{E}, \leq, l)$ *associated to $\mathcal{W}^J$ is a moment graph on $Q^\vee$ defined by*

(i) $\mathcal{V} = \mathcal{W}^J$

(ii) $\mathcal{E} = \left\{x \to y \,|\, x < y\,,\, \exists\, \alpha \in \Delta_+^{re},\, \exists w \in \mathcal{W}_J \,:\, y^{-1}wx = s_\alpha\right\}$

(iii) $l(x \to wxs_\alpha) := \alpha^\vee$*, the positive coroot corresponding to α.*

A priori, it is not clear that $\mathcal{G}^J$ does not have oriented cycles or multiple edges and that its label function is well–defined. However, it is not hard to verify these using certain properties of parabolic subgroups and quotients (cf. Bjorner and Brenti, 2005, §2.4) and the following Lemma.

Lemma 2.1 *(Lanini, 2011)Let $\mathcal{W}, \mathcal{S}, J$ be as before. Let $x, y, z \in \mathcal{W}$ and let $y^J = z^J \neq x^J$. If there exist $\alpha, \beta \in \Delta_+^{re}$ such that $x = ys_\alpha = zs_\beta$, then $\alpha = \beta$ and so $y = z$.*

We recall the following the definitions.

Definition 2.3 *(Lanini) Let $\mathcal{G}$ be a moment graph on the lattice Y. We say that $\mathcal{G}$ is a k-*moment graph *on Y if all labels are non–zero in Y_k*

Notice that a Bruhat moment graph is always a k-moment graph.

Definition 2.4 *(Fiebig, 2008a) The pair $(\mathcal{G}, k)$ is called a* GKM-pair *if all pairs of distinct edges E_1, E_2 containing a common vertex are such that $k\,l(E_1) \cap k\,l(E_2) = \{0\}$.*

2.2 Morphisms of moment graphs

In (Lanini) we defined the notion of morphism between two k-moment graphs, that we now recall. Since a moment graph is an ordered graph, whose edges are labelled by non–zero elements of Y, a morphism will be given by a morphism of oriented graphs together with a collection of automorphisms of the k-module Y_k, satisfying some technical requirements.

Definition 2.5 *A morphism between two k-moment graphs*

$$f : (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \to (\mathcal{V}', \mathcal{E}', \trianglelefteq', l') \tag{1}$$

is given by $(f_\mathcal{V}, \{f_{l,x}\}_{x \in \mathcal{V}})$, where

(MORPH1) $f_\mathcal{V} : \mathcal{V} \to \mathcal{V}'$ *is any map of posets such that, if $x \text{---} y \in \mathcal{E}$, then either $f_\mathcal{V}(x) \text{---} f_\mathcal{V}(y) \in \mathcal{E}'$, or $f_\mathcal{V}(x) = f_\mathcal{V}(y)$.*
For a vertex $E : x \text{---} y \in \mathcal{E}$ such that $f_\mathcal{V}(x) \neq f_\mathcal{V}(y)$, we define $f_\mathcal{E}(E) := f_\mathcal{V}(x) \text{---} f_\mathcal{V}(y)$.

(MORPH2) For all $x \in \mathcal{V}$ and $f_{l,x} : Y_k \to Y_k \in \mathrm{Aut}_k(Y_k)$ such that $E : x \text{---} y \in \mathcal{E}$ and $f_\mathcal{V}(x) \neq f_\mathcal{V}(y)$, the following two conditions are verified:

(MORPH2a) $f_{l,x}(l(E)) = h \cdot l'(f_\mathcal{E}(E))$*, for some $h \in k^*$*

(MORPH2b) $\pi \circ f_{l,x} = \pi \circ f_{l,y}$*, where π is the canonical quotient map $\pi : Y_k \to Y_k / l'(f_\mathcal{E}(E))Y_k$.*

If $f : \mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \to \mathcal{G}' = (\mathcal{V}', \mathcal{E}', \trianglelefteq', l')$ and $g : \mathcal{G}' \to \mathcal{G}'' = (\mathcal{V}'', \mathcal{E}'', \trianglelefteq'', l'')$ are two morphisms of k-moment graphs, then there is a natural way to define the composition. Namely, $g \circ f := (g_{\mathcal{V}'} \circ f_{\mathcal{V}}, \{g_{l', f_{\mathcal{V}}(x)} \circ f_{l,x}\}_{x \in \mathcal{V}})$.

Lemma 2.2 *(Lanini, 2011) The composition of two morphisms between k-moment graphs is again a morphism, and it is associative.*

For any k-moment graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$, we set $id_{\mathcal{G}} = (id_{\mathcal{V}}, \{id_{Y_k}\}_{x \in \mathcal{V}})$. Thus we may give the following definition.

Definition 2.6 *We denote by $\boldsymbol{MG}(Y_k)$ the category whose objects are the k-moment graphs on Y and whose morphisms are as in Def.2.5.*

It is possible to give an explicit characterisation of the isomorphisms in this category, namely,

Lemma 2.3 *(Lanini, 2011) Let $\mathcal{G}, \mathcal{G}' \in \boldsymbol{MG}(Y_k)$ and $f = (f_{\mathcal{V}}, \{f_{l,x}\}_{x \in \mathcal{V}}) \in Hom_{\boldsymbol{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$. Then f is an isomorphism if and only if the following two conditions hold.*

(ISO1) $f_{\mathcal{V}}$ is an isomorphism of posets

(ISO2) for all $u \to w \in \mathcal{E}'$, there exists exactly one $x \to y \in \mathcal{E}$ such that $f_{\mathcal{V}}(x) = u$ and $f_{\mathcal{V}}(y) = w$.

3 The category of sheaves on a k-moment graph

The notion of sheaf on a moment graph is due to Braden and MacPherson (cf. Braden and MacPherson, 2001) and it has been used by Fiebig in several papers (cf. Fiebig, 2008a,b, 2010a,b, 2011).

For any finite rank lattice Y and any local ring k (with $2 \in k^*$), we denote by $S = Sym(Y)$ its symmetric algebra and by $S_k := S \otimes_{\mathbb{Z}} k$ its extension. S_k is a polynomial ring and we provide it with the grading induced by setting $(S_k)_{\{2\}} = Y_k$. From now on, all the S_k-modules will be finitely generated and $\mathbb{Z}$–graded. Moreover, we will consider only degree zero morphisms between them. Finally, for $j \in \mathbb{Z}$ and M a graded S_k–module we denote by $M\{j\}$ the graded S_k-module obtained from M by shifting the grading by j, that is $M\{j\}_{\{i\}} = M_{\{j+i\}}$.

Definition 3.1 *(Braden and MacPherson, 2001) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$, then a* sheaf $\mathcal{F}$ on $\mathcal{G}$ *is given by the data, $(\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$, where*

(SH1) for all $x \in \mathcal{V}$, $\mathcal{F}^x$ is an S_k-module;

(SH2) for all $E \in \mathcal{E}$, $\mathcal{F}^E$ is an S_k-module such that $l(E) \cdot \mathcal{F}^E = \{0\}$;

(SH3) for $x \in \mathcal{V}$, $E \in \mathcal{E}$, $\rho_{x,E} : \mathcal{F}^x \to \mathcal{F}^E$ is a homomorphism of S_k-modules defined whenever x is in the border of the edge E.

Example 3.1 *(cf. Braden and MacPherson, 2001, §1) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$, then its structure sheaf $\mathcal{Z}$ is given by*

- *for all $x \in \mathcal{V}$, $\mathcal{Z}^x = S_k$*
- *for all $E \in \mathcal{E}$, $\mathcal{Z}^E = S_k / l(E) S_k$*
- *for all $x \in \mathcal{V}$ and $E \in \mathcal{E}$, such that x is in the border of the edge E, $\rho_{x,E} : S_k \to S_k / l(E)$ is the canonical quotient map*

Definition 3.2 *(Fiebig, 2009) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$ and let $\mathcal{F} = (\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$, $\mathcal{F}' = (\{\mathcal{F}'^x\}, \{\mathcal{F}'^E\}, \{\rho'_{x,E}\})$ be two sheaves on it. A* morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{F}'$ *is given by the data,*

(i) for all $x \in \mathcal{V}$, $\varphi^x : \mathcal{F}^x \to \mathcal{F}'^x$ is a homomorphism of S_k-modules

(ii) for all $E \in \mathcal{E}$, $\varphi^E : \mathcal{F}^E \to \mathcal{F}'^E$ is a homomorphism of S_k-modules such that, for any $x \in \mathcal{V}$ on the border of $E \in \mathcal{E}$, the following diagram commutes

$$\begin{tikzcd} \mathcal{F}^x \arrow[r, "\rho_{x,E}"] \arrow[d, "\varphi^x"] & \mathcal{F}^E \arrow[d, "\varphi^E"] \\ \mathcal{F}'^x \arrow[r, "\rho'_{x,E}"] & \mathcal{F}'^E \end{tikzcd} \tag{2}$$

Definition 3.3 *Let $\mathcal{G} \in \boldsymbol{MG}(Y_k)$. We denote by $\boldsymbol{Sh}_k(\mathcal{G})$ the category, whose objects are the sheaves on $\mathcal{G}$ and whose morphisms are as in Def.3.2.*

Even if $\mathbf{Sh}_k(\mathcal{G})$ is not a category of sheaves in the topological meaning, we may define, following (Fiebig, 2008a), the notion of sections.

Definition 3.4 *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$, $\mathcal{F} = (\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\}) \in \boldsymbol{Sh}_k(\mathcal{G})$ and $\mathcal{I} \subseteq \mathcal{V}$. Then the* set of sections of $\mathcal{F}$ over $\mathcal{I}$ *is denoted $\Gamma(\mathcal{I}, \mathcal{F})$ and defined as*

$$\Gamma(\mathcal{I}, \mathcal{F}) := \left\{ (m_x)_{x \in \mathcal{I}} \in \bigoplus_{x \in \mathcal{I}} \mathcal{F}^x \;\middle|\; \begin{array}{l} \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \\ \forall x, y \in \mathcal{I}, x \text{ --- } y \in \mathcal{E} \end{array} \right\} \tag{3}$$

We will denote $\Gamma(\mathcal{F}) := \Gamma(\mathcal{V}, \mathcal{F})$, that is the set of global sections of $\mathcal{F}$.

3.1 Flabby sheaves on a k-moment graph

Following (Braden and MacPherson, 2001), we define a topology on the set of vertices of a k-moment graph $\mathcal{G}$. We state a result about a very important class of flabby (with respect to this topology) sheaves: the *BMP*-sheaves. This notion, due to Fiebig and Williamson (cf. Fiebig and Williamson, 2010)), generalises the original construction of Braden and MacPherson.

Definition 3.5 *(Braden and MacPherson, 2001) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$, then the* Alexandrov topology *on $\mathcal{V}$ is the topology whose basis of open sets is given by the collection $\{\trianglerighteq x\} := \{y \in \mathcal{V} \mid y \trianglerighteq x\}$, for all $x \in \mathcal{V}$.*

A classical question in sheaf theory is to ask if a sheaf is flabby, that is whether or not any local section over an open set extends to a global one.

3.2 Braden-MacPherson sheaves

Definition 3.6 *(Fiebig and Williamson, 2010, Def. 6) Let $\mathcal{G} \in \boldsymbol{MG}(Y_k)$ and let $\mathcal{B} \in \boldsymbol{Sh}_k(\mathcal{G})$. We say that $\mathcal{B}$ is a* Braden-MacPherson sheaf *if it satisfies the following properties:*

(BMP1) for any $x \in \mathcal{V}$, $\mathcal{B}^x$ is a graded free S_k-module

(BMP2) for any $E : x \to y \in \mathcal{E}$, $\rho_{y,E} : \mathcal{B}^y \to \mathcal{B}^E$ is surjective with kernel $l(E)\mathcal{B}^y$

(BMP3) for any open set $\mathcal{I} \subseteq \mathcal{V}$, the map $\Gamma(\mathcal{B}) \to \Gamma(\mathcal{I}, \mathcal{B})$ is surjective

(BMP4) for any $x \in \mathcal{V}$, the map $\Gamma(\mathcal{B}) \to \mathcal{B}^x$ is surjective.

Hereafter, Braden-MacPherson sheaves will be referred to also as *BMP*-sheaves or *canonical* sheaves. An important theorem, characterising Braden-MacPherson sheaves, is the following one.

Theorem 3.1 *(Fiebig and Williamson, 2010, Theor. 6.3) Let $\mathcal{G} \in \boldsymbol{MG}(Y_k)$*

(i) For any $w \in \mathcal{V}$, there is, up to isomorphism, a unique Braden-MacPherson sheaf $\mathcal{B}(w) \in \boldsymbol{Sh}_k(\mathcal{G})$ with the following properties.

(BMP0) $\mathcal{B}(w)$ is indecomposable in $\boldsymbol{Sh}_k(\mathcal{G})$.
(BMP1a) $\mathcal{B}(w)^w \cong S_k$ and $\mathcal{B}(w)^x = 0$, unless $x \leq w$.

(ii) Let $\mathcal{B}$ be a Braden-MacPherson sheaf. Then there are $w_1, \ldots, w_r \in \mathcal{V}$ and $l_1 \ldots l_r \in \mathbb{Z}$ such that

$$\mathcal{B} \cong \mathcal{B}(w_1)[l_r] \oplus \ldots \oplus \mathcal{B}(w_r)[l_r] \tag{4}$$

3.3 Direct and inverse images

Let $f = (f_{\mathcal{V}}, \{f_{l,x}\}) : \mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \to \mathcal{G}' = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$ be a homomorphism of k-moment graphs. It is possible to define (cf. Lanini, 2011, §2.2), in analogy with classical sheaf theory, two functors

$$\begin{tikzcd} \mathbf{Sh}_k(\mathcal{G}) \arrow[rr, bend right, "f_*"'] & & \mathbf{Sh}_k(\mathcal{G}') \arrow[ll, bend right, "f^*"'] \end{tikzcd}.$$

We do not want to recall the definitions, but only the fact that if $\mathcal{F} \in \mathbf{Sh}_k(\mathcal{G}')$, then for any $v \in V$, $f^*\mathcal{F}^v \cong \mathcal{F}^{f_{\mathcal{V}}(v)}$ as graded S_k-modules and that the following fundamental property holds.

Lemma 3.1 *(Lanini) Let $\mathcal{G}, \mathcal{G}' \in \boldsymbol{MG}(Y_k)$ and let $f : \mathcal{G} \longrightarrow \mathcal{G}'$ be an isomorphism. Let $w \in \mathcal{V}$ and $w' = f_{\mathcal{V}}(w)$, then, if $\mathcal{B}(w)$ and $\mathcal{B}'(w')$ are the corresponding indecomposable* BMP-*sheaves, then $\mathcal{B}(w) \cong f^*\mathcal{B}'(w')$ in $\boldsymbol{Sh}_k(\mathcal{G})$.*

This result will provide us with an important technique to compare indecomposable canonical sheaves on different k-moment graphs, that we will use in what follows.

4 KL-properties of canonical sheaves

In this section, we recall a conjecture, due to Fiebig, that motivates our work, and we try to explain some techniques we partially developed in (Lanini) to categorify some equalities concerning Kazhdan-Lusztig polynomials.

4.1 The multiplicity conjecture

In 1979 Kazhdan and Lusztig (cf. Kazhdan and Lusztig, 1979) introduced a family of polynomials $\{P_{y,w}\}$ indexed by pairs of elements in a Coxeter group $\mathcal{W}$ with $\mathcal{S}$, the set of simple reflections. Some years later, Deodhar generalised this notion to the parabolic setting, defining two families of polynomials $\{P^{J,-1}_{x,y}\}$ and $\{P^{J,q}_{x,y}\}$, where y and w are now varying in $\mathcal{W}^J$, for $J \subseteq \mathcal{S}$ (see §2.1). If $\mathcal{W}$ was a Weyl group, these polynomials were related to the intersection cohomology of the corresponding Schubert variety in partial flag variety (cf. Kazhdan and Lusztig, 1979, 1980) and to the representation theory of the complex Lie algebras (cf. Kazhdan and Lusztig, 1979), resp. of the semisimple, simply connected, reductive algebraic groups over a field of positive characteristic (cf. Lusztig, 1980b), whose Weyl group is $\mathcal{W}$.

The following conjecture motivates our work.

Conjecture 4.1 *(Fiebig, 2010b, Conj. 4.4) Let $y, w \in \mathcal{W}^J$ and let k be such that $(\mathcal{G}^J_{|_{[y,w]}}, k)$ is a GKM-pair. Then $\underline{rk}\,(\mathcal{B}(w)^J)^y = P^{J,-1}_{y,w}$.*

Here, $\mathcal{B}(w)^J$ denotes the indecomposable Braden–MacPherson sheaf on the Bruhat moment graph $\mathcal{G}^J$, corresponding to the vertex $w \in \mathcal{W}^J$, $\underline{rk}$ is the graded rank of a graded (finitely generated) S_k-module and $P^{J,-1}_{y,w}$ Deodhar's parabolic analogue of Kazhdan-Lusztig polynomials, corresponding to the pair y, w and to the parameter $u = -1$ (cf. Deodhar, 1987).

This conjecture is proved in characteristic zero and in this case it is equivalent to Kazhdan-Lusztig's conjecture (cf. Fiebig, 2008a). In characteristic p it is proved for p bigger than a large (but explicit) bound and it implies Lusztig's conjecture (cf. Fiebig, 2011, 2010b). Our goal is now to interpret combinatorial properties of Kazhdan-Lusztig polynomials in term of Braden-MacPherson sheaves.

4.2 Three techniques

From now on, $\mathcal{G}$ will denote the regular Bruhat moment graph, that is $\mathcal{G}^\emptyset$. Moreover, if $w \in \mathcal{W}$, we will write $\mathcal{B}(w)$ instead of $\mathcal{B}(w)^\emptyset$.

We mainly use three strategies to prove our claims.

a) *Pullback of canonical sheaves.* We look for isomorphisms of k-moment graphs and then, via the pullback functor (see Lemma 3.1), we get the desired equality between stalks of Braden-MacPherson sheaves.

b) *Looking at the set of invariants.* For any $s \in \mathcal{S}$ we define an involution σ_s of the set of local sections of a canonical sheaf on an s–invariant interval of $\mathcal{G}$. In this case, the study of the space of the invariants gives us the property we wanted to show.

c) *Flabbiness of the structure sheaf.* It is known (cf. Fiebig, 2010a) that the so–called *structure sheaf* (see Example 3.1) is isomorphic to an indecomposable Braden-MacPherson sheaf if and only if it is flabby and this is the case if and only if the corresponding Kazhdan–Lusztig polynomials evaluated in 1 are all 1. We prove in a particularly explicit way that the structure sheaf is flabby in order to categorify the fact that the associated polynomials evaluated in 1 are 1.

4.3 Some applications

From now on, $\mathcal{G}$ will denote the regular Bruhat moment graph, that is $\mathcal{G}^\emptyset$. Moreover, if $w \in \mathcal{W}$, we will write $\mathcal{B}(w)$ instead of $\mathcal{B}(w)^\emptyset$. Using the techniques we listed above, it is possible to prove the following results.

Theorem 4.1 *Lanini Let $y, w \in \mathcal{W}$, then*

(i) $\mathcal{B}(w)^y \cong \mathcal{B}(w^{-1})^{y^{-1}}$.
Let $s \in \mathcal{S}$ be such that $ws < w$, then

(ii) if $y \not\leq ws$, $\mathcal{B}(w)^y \cong \mathcal{B}(sw)^{sy}$
All isomorphisms are isomorphisms of S_k-modules, for any k.

Proof: (Idea) In both cases, we use the strategy a). It is indeed possible to define isomorphisms between the corresponding k-moment graphs. This is done only using the combinatorics of $\mathcal{G}$. □

The theorem above categorifies two elementary properties of regular Kazhdan-Lusztig polynomials. Namely, for any pair $y, w \in \mathcal{W}$ it holds $P_{y,w} = P_{y^{-1},w^{-1}}$ and, if $s \in \mathcal{S}$ is such that $sw < w$ but $y \not\leq sw$, $P_{y,w} = P_{sy,sw}$. We want to stress the fact that Theorem 4.1 holds for any local ring k with $2 \in k^{\times}$.

Remark 4.1 *Observe that we are not simply looking for isomorphisms of the underlying (intervals of) Bruhat graphs and that it is not trivial at all to lift isomorphisms of graphs to isomorphisms of moment graphs. If it were possible, indeed, to prove that any isomorphism between the corresponding intervals of Bruhat graphs induces an isomorphism between k-moment graphs, at least for $k = \mathbb{Q}$, the Lusztig-Dyer combinatorial invariance conjecture would follow. See (Brenti et al., 2006) for partial results on this conjecture.*

Clearly not all equalities concerning Kazhdan-Lusztig polynomials come from k-isomorphisms of the underlying Bruhat graphs. Inspired by a theorem of Deodhar (cf. Deodhar, 1987), we prove a relation between indecomposable canonical sheaves on a regular Bruhat graph $\mathcal{G}$ and the ones on the corresponding parabolic Bruhat graphs $\mathcal{G}^J$, for J such that the subgroup $\mathcal{W}_J = \langle J \rangle$ is finite. Consider the epimorphism $\pi^J : \mathcal{G} \to \mathcal{G}^J$ induced by the action of $\mathcal{W}_J$ on $\mathcal{W}$.

Theorem 4.2 *(Lanini) Let $w \in \mathcal{W}^J$ and let w_J be the longest element of $\mathcal{W}_J$. If $(\mathcal{G}^J_{|_{\leq w}}, k)$ is a GKM-pair, then $\pi^J(\mathcal{B}^J(w)) \cong \mathcal{B}(w_J w)$.*

Proof: (Idea) Here we use the second technique we quoted. In particular, we are able to define an action of $\mathcal{W}_J$ on the set of global sections of $\mathcal{B}(w)$ and to prove that the data we need to build the indecomposable canonical sheaf are contained in the invariants with respect to this action. □

Using the third technique it is possible to prove the following result.

Theorem 4.3 *(Lanini, 2011) Let $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ and $J = \{\alpha\}$ the (unique) positive simple root of A_1. If $v_j \leq v_i$ and $(\mathcal{G}^J_{|_{[v_j, v_i]}}, k)$ is a GKM-pair, then $(\mathcal{B}^J(v_i))^{v_j} \cong S_k$.*

Proof: (Idea) In this case the corresponding k-moment graph has an explicit description. It is a complete graph, whose set of vertices is totally ordered and indexed by $\mathbb{Z}$. Moreover, the edge joining n and m is labelled by the positive coroot corresponding to $\alpha + (n+m)\delta$. It is then possible to show that the structure sheaf is flabby via direct calculations. □

5 The stabilisation phenomenon

Let $\mathfrak{g}$ be an affine Kac-Moody algebra, with Weyl group $\mathcal{W}$. Let J be the corresponding set of finite simple reflections. We will write $\mathcal{G}^{par}$, resp. $\mathcal{B}^{par}$, resp. $P^{par}_{y,w}$, instead of $\mathcal{G}^J$, resp. $\mathcal{B}^J$, resp. $P^{J,-1}_{y,w}$. In this case, we can identify the $\mathcal{W}$ with its set of alcoves $\mathcal{A}$ and $\mathcal{W}^J$ with the set of alcoves $\mathcal{A}^+$ lying in the fundamental chamber $\mathcal{C}^+$.

The aim of this section is to interpret in the moment graph setting a result in (Lusztig, 1980a). In particular, Lusztig proved that, for any pair of alcoves $A, B \in \mathcal{A}^+$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \in \mathbb{N}$, with $n, m > n_0$

$$P^{par}_{A+n\rho,B+n\rho} = P^{par}_{A+m\rho,B+m\rho} = Q_{A,B}. \tag{5}$$

where ρ is half the sum of the positive finite coroots and $Q_{A,B}$ is called the *generic polynomial* of the pair A, B. The $Q_{A,B}$'s turn out to have a realisation very similar to the one of the regular Kazhdan-Lusztig polynomials. Indeed, Lusztig in (Lusztig, 1980a) associated to every affine Weyl group $\mathcal{W}$ its periodic module **M**, that is, the free $\mathcal{L} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$–module with set of generators—or standard basis—indexed by the set of all alcoves $\mathcal{A}$. It is possible to define an involution and to prove that there exists a self–dual basis of **M**: the *canonical basis*. In this setting, the generic polynomials are the coefficients of the change basis matrix. Our interest in the periodic module was motivated by the fact that **M** governs the representation theory of the affine Kac–Moody algebra, whose Weyl group is $\mathcal{W}$, at the critical level (Arakawa and Fiebig).

The aim of this section is to study the behaviour of indecomposable Braden-MacPherson sheaves on finite intervals of the parabolic Bruhat graph far enough in $\mathcal{C}^+$. In particular, we claim the following.

Theorem 5.1 *Lanini (2011) Let k be a filed of characteristic 0. Let $\mathcal{I} = [B, A]$ be an interval far enough in the fundamental chamber and $(\mathcal{G}^{par}_{|[A,B]}, k)$ be a GKM-pair. Then, for all $\mu \in \mathcal{C}^+$, $\mathcal{B}(A)^B \cong \mathcal{B}(A+\mu)^{B+\mu}$*

Our first hope was that we could define an isomorphism of the corresponding k-moment graphs and then conclude by applying the technique of the pullback. Actually, via a very explicit description of finite intervals far enough in the fundamental chamber, we discovered that such an isomorphism does not exist.

5.1 Parabolic intervals far enough in the fundamental chamber

The following result tells us that finite intervals of $\mathcal{G}^{par}$ far enough in $\mathcal{C}^+$ are very regular.

Lemma 5.1 *Lanini (2011) Let $A, B \in \mathcal{A}^+$, then there exists an integer $n_0 = n_0(A, B)$ such that for any $\lambda \in X \cap n\rho + \mathcal{C}^+$, with $n \geq n_0$, for any pair $C, D \in [A + \lambda, B + \lambda]$ there is an edge C —— D if and only if there exists $\alpha \in \Delta^{re}_+$ such that*

(i) either $D = Cs_\alpha$

(ii) or $D = C + a\alpha^\vee$ for some $a \in \mathbb{Z} \setminus \{0\}$

Proof: (idea) The fundamental step in the proof is to show that there exists a $K > 0$, depending only on the root system, such that if $\lambda \in \mathcal{C}^+$ and d_λ is the minimum of the distances of λ from the borders of $\mathcal{C}^+$, then if $\mu \in \mathcal{C}^+$ such that $\lambda = w(\mu + x\alpha^\vee)$, for $x \in \mathbb{R}$, α a positive fine root and w in a the finite Weyl group, and $|\lambda - \mu| \leq K \cdot d_\lambda$, then $\lambda = \mu + t\beta^\vee$, for some $t \in \mathbb{R}$ and β finite positive root. □

We immediately get a corollary.

Corollary 5.1 *Lanini (2011) For any pair $A, B \in \mathcal{A}^+$, $B \leq A$ and for any pair $\lambda_1 = n_1\rho$, $\lambda_2 = n_2\rho \in X \cap n\rho + \mathcal{C}^+$ $(n_1, n_2 \geq n_0(A, B))$ then $\mathcal{G}^{par}_{|_{[A+\lambda_1, B+\lambda_1]}}$ and $\mathcal{G}^{par}_{|_{[A+\lambda_2, B+\lambda_2]}}$ are isomorphic as oriented graphs.*

□

We say that the edges of type (i), that is given by reflections, are *stable*, while the ones of type (ii), that is given by translations, are *non–stable*. We denote the corresponding sets $\mathcal{E}_S$, resp. $\mathcal{E}_{NS}$.

Remark 5.1 *We want to stress the fact that in Lemma 5.1 we are* not *proving the existence of an isomorphism of moment graphs, but only between the underlying oriented graphs, that is, we are not considering labels. We are going to see that this isomorphism does not induce an isomorphism of moment graphs.*

Using a combinatorial property of affine Weyl groups it is possible to prove the following result.

Lemma 5.2 *Lanini (2011) Let $A \in \mathcal{A}^+$, $t \in \mathcal{T}$, $\alpha \in \Delta^{re}_+$ and $n \in \mathbb{Z}$ be such that $A, At, A + n\alpha^\vee, At + n\alpha^\vee \in \mathcal{A}^+$. Then, $l(A \text{ —— } At) = l(A + n\alpha^\vee \text{ —— } At + n\alpha^\vee) = {\alpha_t}^\vee$.*

Moreover, let $\beta \in \Delta^{re}_+$ be such that $A, A + b\beta^\vee, A + n\alpha^\vee, A + n\alpha^\vee, A + b\beta^\vee + n\alpha^\vee \in \mathcal{A}^+$. Then, $l(A \text{ —— } A + b\beta^\vee) \neq l(A + n\alpha^\vee \text{ —— } A + b\beta^\vee + n\alpha^\vee)$.

5.2 The stable moment graph

Now it is clear that we cannot use technique a), but it is easy to see that there is a subgraph of $\mathcal{G}^{par}$ such that any finite interval is invariant by root translation (as a moment graph). Here we define the *stable moment graph* $\mathcal{G}^{stab}$ as follows. This is the moment graph having as set of vertices the alcoves in the fundamental chamber (that we identify with the corresponding elements of the Weyl group), equipped with the induced Bruhat order: we connect two vertices A and B if and only if there exists an affine reflection t such that $A = Bt$, and in this case we set $l(A \text{ —— } Bt) := {\alpha_t}^\vee$.

Then we have.

Lemma 5.3 *(Lanini, 2011) For any Bruhat interval $[A, B] \subset \mathcal{A}^+$ and for any $\mu \in Q^\vee$ there exists an isomorphism of k-moment graphs $\mathcal{G}^{stab}_{|_{[A,B]}} \longrightarrow \mathcal{G}^{stab}_{|_{[A+\mu,B+\mu]}}$ for all k.*

We have just showed that the two moment graphs $\mathcal{G}^{par}_{|_{[A,B]}}$ and $\mathcal{G}^{par}_{|_{[A+\mu,B+\mu]}}$ are in general not isomorphic, while there is always an isomorphism of moment graphs between $\mathcal{G}^{stab}_{|_{[A,B]}}$ and $\mathcal{G}^{stab}_{|_{[A+\mu,B+\mu]}}$. Since the stable moment graph is a subgraph of $\mathcal{G}^{par}$, there is a monomorphism $\mathcal{G}^{stab} \hookrightarrow \mathcal{G}^{par}$. The following diagram summarises this situation.

$$
\begin{tikzcd}
\mathcal{G}^{par}_{|_{[A,B]}} & \mathcal{G}^{par}_{|_{[A+\mu,B+\mu]}} \\
\mathcal{G}^{stab}_{|_{[A,B]}} \arrow[u, hook, "i"] \arrow[r] & \mathcal{G}^{stab}_{|_{[A+\mu,B+\mu]}} \arrow[u, hook, "i_\mu"']
\end{tikzcd}
$$

We then get a functor $i^* = \cdot^{stab} : \mathbf{Sh}_{\mathcal{G}^{par}_{|_{[A,B]}}} \to \mathbf{Sh}_{\mathcal{G}^{stab}_{|_{[A,B]}}}$. The following theorem is our main result

Theorem 5.2 *(Lanini, 2011) Let k be a field of characteristic 0. The functor $\cdot^{stab} : \boldsymbol{Sh}_k(\mathcal{G}^{par}_{|_{[A,B]}}) \to \boldsymbol{Sh}_k(\mathcal{G}^{stab}_{|_{[A,B]}})$ preserves indecomposable Braden-MacPherson sheaves.*

Proof: (idea) In the case of $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$, we are able to prove the claim via the third technique mentioned above, that is, for any finite interval of $\mathcal{G}^{stab}$, we show that in characteristic zero its structure sheaf is flabby, so it is invariant by weight translation for all integral weights $\mu \in \mathcal{C}^+$. On the other hand, we know already that the structure sheaf for $\mathcal{G}^{par}(\widehat{\mathfrak{sl}_2})$ is flabby (see Theorem 4.3) and this concludes the $\mathfrak{sl}_2$-case.

For the general case, we apply a localisation technique due to Fiebig , that enables us to use the $\mathfrak{sl}_2$-case, together with other results of (Fiebig, 2011). □

The stabilisation property follows by applying the technique of the pullback to Theorem 5.2.

Acknowledgements

I would like to thank my advisor Peter Fiebig, for helpful and fruitful discussions, and Rocco Chirivì for useful advice. Many thanks go to Rollo Jenkins for help with the language. I also owe my thanks to the Frauenbeauftragte der Universität Erlangen-Nürnberg, which provided me with the funding to stay a year longer in Erlangen, where this extended abstract was written.

References

T. Arakawa and P. Fiebig. The linkage principle for restricted critical level representations of affine Kac-Moody algebras. *Compos. Math*. to appear.

A. Bjorner and F. Brenti. *Combinatorics of Coxeter Groups*, volume 231 of *Graduate Texts in Mathematics*. New York, 2005.

T. Braden and R. MacPherson. From moment graphs to intersection cohomology. *Math. Ann.*, 321: 533–551, 2001.

F. Brenti, F. Caselli, and M. Marietti. Special matchings and Kazhdan-Lusztig polynomials. *Adv. Math.*, 202:555–601, 2006.

V. Deodhar. On some geometric aspects of Bruhat orderings ii. The parabolic analogue of Kazhdan-Lusztig polynomials. *J. of Algebra*, 111:483–506, 1987.

M. Dyer. On the "Bruhat graph" of a Coxeter system. *Compos. Math*, 78:185–191, 1991.

M. Dyer. Hecke algebras and shellings of Bruhat intervals. *Compos. Math*, 89:91–115, 1993.

P. Fiebig. Sheaves on moment graphs and a localization of Verma flags. *Adv. Math.*, 217:683–712, 2008a.

P. Fiebig. The combinatorics of Coxeter categories. *Trans. Amer. Math. Soc.*, 360:4211–4233, 2008b.

P. Fiebig. *Moment graphs in representation theory and geometry*, 2009. Notes.

P. Fiebig. The multiplicity one case of Lusztig's conjecture. *Duke Math. J.*, 153:551–571, 2010a.

P. Fiebig. Lusztig's conjecture as a moment graph problem. *Bull. London Math. Soc.*, 42(6):957–972, 2010b.

P. Fiebig. Sheaves on affine Schubert varieties, modular representations and Lusztig's conjecture. *J. Amer. Math. Soc.*, 24:133–181, 2011.

P. Fiebig and G. Williamson. Parity sheaves, moment graphs and the p-smooth locus of Schubert varieties. *preprint, arXiv:1008.0719*, 2010.

M. Goresky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality and the localization theorem. *Invent. Math.*, 131(1):23–83, 1998.

V. Kac. *Infinite dimensional Lie algebras*, volume 44 of *Progress in Mathematics*. Birkhaeuser Boston Inc., Boston, MA, 1983.

D. Kazhdan and G. Lusztig. Representation of Coxeter groups and Hecke algebras. *Invent. Math.*, 53: 165–184, 1979.

D. Kazhdan and G. Lusztig. Schubert varieties and Poincaré duality. *Geometry of the Laplace Operator. Proc. Symp. Pure Math (Amer. Math. Soc.)*, 36:185–203, 1980.

M. Lanini. Kazhdan-Lusztig combinatorics in the moment graph setting. *J. of Algebra*. to appear.

M. Lanini. Kazhdan -Lusztig combinatorics in the moment graph setting. PhD thesis, FAU Erlangen-Nuernberg and Universita degli Studi Roma Tre, 2011.

G. Lusztig. Hecke algebras and Jantzen's generic decomposition patterns. *Adv. Math.*, 37:121–164, 1980a.

G. Lusztig. Some problems in the representation theory of finite Chevalley groups. In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Criz, Calif., 1979) Adv. Proc. Symp. Pure Math.*, number 44, pages 313–317. Amer. Math Soc, Boston, MA, 1980b.

W. Soergel. Kazhdan-Lusztig Polynome und unzerlegbare Bimoduln ueber Polynomringen,. *J. Inst. Math. Jussieu*, 6(3):501–525, 2007.