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Chromatic roots as algebraic integers

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Abstract A chromatic root is a zero of the chromatic polynomial of a graph. At a Newton Institute workshop on Combinatorics and Statistical Mechanics in 2008, two conjectures were proposed on the subject of which algebraic integers can be chromatic roots, known as the "$\alpha + n$ conjecture" and the "$n\alpha$ conjecture". These say, respectively, that given any algebraic integer α there is a natural number n such that $\alpha + n$ is a chromatic root, and that any positive integer multiple of a chromatic root is also a chromatic root. By computing the chromatic polynomials of two large families of graphs, we prove the $\alpha + n$ conjecture for quadratic and cubic integers, and show that the set of chromatic roots satisfying the $n\alpha$ conjecture is dense in the complex plane.

Résumé. Une racine chromatique est un zéro du polynôme chromatique d'un graphe. A un atelier au Newton Institute sur la combinatoire et la mécanique statistique en 2008, deux conjectures ont été proposées dont le sujet des entiers algébriques peut être racines chromatiques, connus sous le nom "la conjecture $\alpha + n$" et "la conjecture $n\alpha$". Les conjectures veulent dire, respectivement, que pour chaque entier algébrique α il y a un nombre entier naturel n, tel que $\alpha + n$ est une racine chromatique, et que chaque multiple entier positif d'une racine chromatique est aussi une racine chromatique . En calculant les polynômes chromatiques de deux grandes familles de graphes, on prouve la conjecture $\alpha + n$ pour les entiers quadratiques et cubiques, et montre que l'ensemble des racines chromatiques qui confirme la conjecture $n\alpha$ est dense dans le plan complexe.

Keywords: chromatic polynomial, chromatic roots, algebraic integers

1 Introduction

A *proper k-colouring* of a graph G is a function from the vertices of G to a set of k colours, with the condition that no two adjacent vertices are assigned the same colour. The *chromatic polynomial* $P_G(x)$ of G counts the number of proper colourings of G, in that for some non-negative integer k, the evaluation of $P_G(x)$ at k is the number of proper k-colourings of G.

A *chromatic root* of a graph G is a zero of the chromatic polynomial of G. If we define the *chromatic number* $\chi(G)$ of G to be the least number of colours with which we can properly colour G, then it is not difficult to see that $0, 1, \ldots, \chi(G) - 1$ are chromatic roots of G; hence $P_G(x)$ always has a number of linear factors. However, the vast majority of chromatic polynomials are also divisible by at least one irreducible factor of higher degree, and it is the zeros of these non-linear factors that are of most interest in the study of chromatic roots.

Of particular interest has been the question of where in the real line and complex plane these non-integer chromatic roots are located. It follows from basic properties of the chromatic polynomial that there are no negative chromatic roots, and none in the interval $(0, 1)$. Much more unexpected is the result of Jackson

(1993) that the interval $(1, 32/27]$ also contains no chromatic roots, and the mysterious upper bound of this interval becomes even more significant in light of the subsequent result of Thomassen (1997) that chromatic roots are dense in $[32/27, \infty)$.

For some time it was thought that were no chromatic roots at all in the left half of the complex plane. However, this was shown to be false by a series of discoveries of chromatic polynomials having complex zeros with increasingly large negative real part, culminating with the proof by Sokal (2004) of the surprising result that chromatic roots are in fact dense in the whole complex plane.

As any chromatic polynomial is monic with integer coefficients, every chromatic root is an algebraic integer. However, very little is known about the converse question of which algebraic integers can be chromatic roots. Indeed, apart from those which lie in, or have a conjugate in one of the forbidden intervals of the real line, there are no algebraic integers that are known not to be chromatic roots.

In the absence of any techniques for proving such negative results, the following two conjectures – which appear in a paper by Cameron and Morgan – were proposed by the attendees of a workshop on Combinatorics and Statistical Mechanics in 2008 at the Newton Institute.

Conjecture 1.1 (The $\alpha + n$ conjecture) *For any algebraic integer α there is $n \in \mathbb{N}$ such that $\alpha + n$ is a chromatic root.*

Conjecture 1.2 (The $n\alpha$ conjecture) *If α is a chromatic root, then so too is $n\alpha$ for all $n \in \mathbb{N}$.*

There is some empirical evidence for these conjectures beyond that provided by Sokal's result. For example, let α be a chromatic root of a graph G, and let H be the join of G with the complete graph K_n on n vertices (that is, the graph formed by joining every vertex of G to all those of K_n). A simple combinatorial argument then gives us that:

$$P_H(x) = x(x-1)\cdots(x-n+1)P_G(x-n). \tag{1}$$

Thus if α is a chromatic root, then so too is $\alpha + n$ for any natural number n. Note that the $n\alpha$ conjecture is simply a multiplicative analogue of this result. Furthermore, this would seem to imply that a given algebraic integer has a higher likelihood of being a chromatic root if it has larger real part, lending some credibility to the $\alpha + n$ conjecture.

The purpose of this paper is to present partial proofs of both of these conjectures. In §2 we present two large families of graphs — bicliques and clique-theta graphs — and give a general formula for their chromatic polynomials. In §3 we show that, given any quadratic or cubic integer α, there is a natural number n such that $\alpha + n$ is a chromatic root of some biclique, thus proving the $\alpha + n$ conjecture for algebraic integers of degree 2 and 3. Finally, in §4, we use the family of clique-theta graphs and Sokal's result to show that the set of chromatic roots satisfying the $n\alpha$ conjecture is dense in the complex plane.

The main results presented in this abstract appear in Bohn (2011) and Bohn (2012).

2 Two families of chromatic polynomials

2.1 Bicliques

A *biclique* is simply the complement of a bipartite graph, consisting of two cliques joined by a number of edges. When we need to be more specific, we shall refer to a biclique in which the two cliques are of size j and k as a (j, k)*-biclique.* By convention, k will be greater than or equal to j, and the edges between the two cliques will be referred to as *bridging edges.* In this section we will give a construction of the chromatic polynomial of a general biclique G.

A *matching* of a graph is a set of edges of that graph, no two of which are incident to the same vertex. When we refer to the size of a matching, we refer to the number of edges in that matching; an i-*matching* is a matching of size i. Let m_G^i be the number of i-matchings of a graph G; then the *matching numbers* of G are the elements of the sequence $(m_G^0, m_G^1, m_G^2, \ldots)$. We will write that two graphs are *matching equivalent* if they have the same matching numbers.

Now, for some positive integers j and k, let G be a (j,k)-biclique, and let $\bar{G}$ be the complement of G (obtained by replacing edges of G with non-edges, and vice-versa). Then $\bar{G}$ is a subgraph of the complete bipartite graph $K_{j,k}$. We shall construct the chromatic polynomial of G by considering matchings of $\bar{G}$.

Given some matching of $\bar{G}$, partition the vertices of G such that two vertices are contained in the same part if and only if the corresponding vertices of $\bar{G}$ are joined by an element of the matching. Then, by assigning a different colour to each part of this partition, we obtain a proper colouring of G. Conversely, any proper colouring of G corresponds to a partition induced by some matching of $\bar{G}$. Thus we can compute the chromatic polynomial of G by counting x-colourings of partitions induced by matchings of $\bar{G}$, as follows.

Let $(x)_k$ denote the falling factorial $x(x-1)\cdots(x-k+1)$. If each part receives a different colour, then there are $(x)_{j+k-i}$ ways of assigning x colours to a partition induced by an i-matching of $\bar{G}$ (as any such partition consists of $j+k-i$ parts). Thus:

$$P_G(x) = \sum_M (x)_{j+k-|M|}, \tag{2}$$

where the sum is over all possible matchings M of $\bar{G}$. Note that this construction in fact gives us the unique decomposition of $P_G(x)$ into a sum of chromatic polynomials of complete graphs (the chromatic polynomial of the n-vertex complete graph K_n being $(x)_n$).

Now suppose that, for some $1 \le p \le j$, there are p vertices in the j-clique of G which are adjacent to every vertex of the k-clique. By the same argument as that leading to (1), we can then deduce that the chromatic polynomial of G will be of the form:

$$P_G(x) = (x)_p P_H(x-p),$$

where H is the $(j-p,k)$-biclique obtained from G by deleting each of the p vertices and all incident edges. A similar situation arises if some vertices of the k-clique are adjacent to every vertex of the j-clique. As we are concerned with algebraic properties of the chromatic polynomial, we shall discount these cases, and assume that no vertex of G is connected to every other vertex of the graph.

With this condition on G, it is not difficult to see that $\bar{G}$ will always have a single 0-matching, along with at least one i-matching for all $0 < i \le j$, and that no larger matchings are possible. Hence we have:

$$P_G(x) = \sum_{i=0}^{j} m_{\bar{G}}^i (x)_{j+k-i},$$

where $m_{\bar{G}}^0 = 1$, and in general $m_{\bar{G}}^i$ is a positive integer. Thus the chromatic polynomial of G is a product of $(x)_k$ with a (usually irreducible) degree j factor $g(x)$ of the form:

$$g(x) = \sum_{i=0}^{j} m_{\bar{G}}^i (x-k)_{j-i}. \tag{3}$$

It is this factor which will be our main object of study, and we will henceforth refer to it as the "interesting factor" of $P_G(x)$.

2.2 Clique-theta graphs

In order to describe clique-theta graphs and present a formula for their chromatic polynomials, it is convenient to first discuss a special case. We define the *ring of cliques* $R(a_1, a_2, \ldots, a_n)$ to be an n-cycle in which, for each $1 \leq i \leq n$, the ith vertex has been blown up into an a_i-clique, and every vertex of this clique has been joined to each of those of its neighbouring a_{i-1}- and a_{i+1}-cliques. In Read (1981) the following general formula is given for the chromatic polynomial of a general ring of cliques:

$$P_{R(a_1,a_2,\ldots,a_n)}(x) = (x)_{a_1+a_2} \cdots (x)_{a_n+a_1} \sum_{k=0}^{n} (-1)^{nk} v_k(x) \left(\prod_{i=1}^{n} \frac{-(a_i)_k}{(x)_{a_i+k}} \right),$$

where $v_k(x) = \binom{x}{k} - \binom{x}{k-1}$.

Note that this is indeed a polynomial — the terms in the denominator of the summation are cancelled by some of the preceding linear factors. Interestingly, a permutation of the $\{a_i\}$ may change the linear factors, but does not affect the final more complicated factor. This is a desirable property for our purposes: as we are concerned with algebraic properties of chromatic polynomials, we will often disregard the linear terms of of a polynomial.

Now, suppose that $a_1 = 1$. The chromatic polynomial of $R(1, a_2, \ldots, a_n)$ reduces to the following considerably simpler expression:

$$x(x-1)_{a_{n-1}+a_n-1} \left(\prod_{i=2}^{n-2} (x - a_{i+1} - 1)_{a_i - 1} \right) r(1, a_1, \ldots, a_n), \tag{4}$$

where

$$r(1, a_1, \ldots, a_n) = \frac{1}{x} \left(\prod_{i=2}^{n} (x - a_i) - \prod_{i=2}^{n} (-a_i) \right).$$

This specialisation of the previous formula was discovered, but not published, by Read (a separate construction is given in Dong et al. (2002)). Now, we define the *generalised theta graph* $\Theta_{a_1,\ldots,a_k}$ to be the graph consisting of two endpoint vertices joined by k otherwise disjoint paths of lengths $a_1, \ldots, a_k$. Clique-theta graphs are obtained from generalised theta graphs in the same way that rings of cliques are obtained from cycle graphs, that is, by blowing up the vertices into cliques and replacing edges by all possible edges between neighbouring cliques. As such, a ring of cliques is a special case of a clique-theta graph in which the underlying structure has only two paths between endpoint vertices.

As might be expected, the general formula for the chromatic polynomials of these graphs is very complicated. However, we can simplify it considerably by fixing one of the endpoint vertices. Formally, let p be a positive integer, and let $S_1, \ldots, S_k$ be k non-empty ordered sets of positive integers with $S_i = (a_{i(1)}, a_{i(2)}, \ldots, a_{i(m_i)})$. We can construct the clique-theta graph $T(1, S_1, \ldots, S_k, p)$ by blowing one endpoint vertex of $\Theta_{a_1,\ldots,a_k}$ up into a clique of size p, and blowing up the jth vertex of the ith path into a clique of size $a_{i(j)}$.

The proof of the following is too lengthy to include here, however it can be found in Bohn (2011):

Proposition 2.1 *The chromatic polynomial of $T(1, S_1, S_2, \ldots, S_k, p)$ is:*

$$\left[(x)_{a_{k(m_k)}+p}\left(\prod_{i=1}^{k-1}(x-p-1)_{a_{i(m_i)}-1}\right)\left(\prod_{i=1}^{k}\prod_{l=1}^{m_i-1}(x-a_{i(l+1)}-1)_{a_{i(l)}-1}\right)\right]$$
$$\times\left[\left(p(x-p)^{k-1}\prod_{i=1}^{k}r(1,a_{i(1)},\ldots,a_{i(m_i)})\right)+\left(\prod_{i=1}^{k}r(1,a_{i(1)},\ldots,a_{i(m_i)},p)\right)\right],$$

where $r(1, a_{i(1)}, \ldots, a_{i(m_i)})$ is the interesting factor from the chromatic polynomial of the ring of cliques $R(1, a_{i(1)}, \ldots, a_{i(m_i)})$.

3 The $\alpha + n$ Conjecture

We will show that, given any quadratic or cubic integer α, there is a natural number n such that $\alpha + n$ is a chromatic root of, respectively, a $(2, k)$-biclique or a $(3, k)$-biclique, thus proving the $\alpha + n$ conjecture for algebraic integers of degree 2 and 3. Note that we need only concern ourselves with the interesting factors of the chromatic polynomials.

For any given algebraic integer α of degree d, there is by definition some monic, irreducible polynomial $f(x) \in \mathbb{Z}[X]$ of degree d such that $f(\alpha) = 0$. The following observation gives us a more approachable way to state the $\alpha + n$ conjecture.

Let $f(x)$ be an irreducible polynomial of degree d. Then there is some integer m such that the coefficient of x^{d-1} in $f(x - m)$ lies between 0 and $d - 1$ (we will refer to a polynomial of this form as a *reduced polynomial.*) Let α be a root of $f(x) = 0$; then $\alpha + m$ is a zero of a unique reduced polynomial $f(x - m)$. Suppose now that we can show there is a factor $g(x)$ of a chromatic polynomial and integer k such that $f(x - m) = g(x + k)$. Then $f(x) = g(x + k + m)$, and letting $n = k + m$, we have that α is a zero of $g(x + n)$, which means that $\alpha + n$ is a zero of $g(x)$, and hence a chromatic root.

This means that, in order to prove the $\alpha + n$ conjecture for algebraic integers of degree d, we need only show that for every reduced polynomial $h(x)$ of degree d there is a factor $g(x)$ of a chromatic polynomial and integer n such that $g(x + n) = h(x)$.

3.1 Quadratic integers

Let G be a $(2, k)$-biclique. Label the vertices of the 2-clique v_1 and v_2, and let a and b represent, respectively, the number of neighbours of v_1 and v_2 in the k-clique. Then the interesting factor of $P_G(x)$ is, simply:

$$g(x) = (x - a)(x - b) - (x - a - b). \tag{5}$$

In order to prove our result it suffices to show that, given any reduced quadratic polynomial $h(x)$, there is an interesting factor $g(x)$ and natural number n such that $h(x) = g(x + n)$.

To begin with, suppose the x-coefficient of $h(x)$ is zero, so $h(x) = x^2 + a_0$ for some $a_0 \in \mathbb{Z}$. Let

$$a = -1/2 + n - (\sqrt{4n + 4a_0 - 3})/2$$
$$b = -1/2 + n + (\sqrt{4n + 4a_0 - 3})/2.$$

Then the polynomial obtained by substituting these values into 5 is $x^2 - 2nx + n^2 - a_0 = h(x - n)$, as desired. By choosing n high enough, and such that $-3 + 4n + 4a_0$ is a perfect square, we can always ensure that a and b are non-negative integers.

The second case we need to consider is where the x-coefficient of $h(x)$ is 1, that is, where: $h(x) = x^2 + x + a_0$ for some $a_0 \in \mathbb{Z}$. We can approach this in a similar way: this time let

$$\begin{aligned} a &= -1 + n - \sqrt{n - a_0 - 1} \\ b &= -1 + n + \sqrt{n - a_0 - 1}. \end{aligned}$$

Substituting these into 5 gives $x^2 + (1 - 2n)x + n^2 + a0 - n = h(x - n)$. Again, by choosing n high enough, and such that $n - a_0 - 1$ is a perfect square, we can ensure that a and b are non-negative integers.

So we have shown that, given any quadratic reduced polynomial $h(x)$, we can find non-negative integers a, b and n such that $g(x + n) = h(x)$, thus proving the quadratic case of the $\alpha + n$ conjecture.

3.2 Cubic integers

Let G be a $(3, k)$-biclique. As established previously, $P_G(x)$ is a product of $(x)_k$ with a cubic "interesting factor" $g(x)$. Observe that $(x)_k$ is the number of ways to properly x-colour the k-clique of G; thus we can view $g(x)$ as an expression for the number of proper x-colourings of the 3-clique. We can construct this expression independently of the rest of the polynomial using the Principle of Inclusion-Exclusion, as follows.

Label the three vertices of the 3-clique v_1, v_2 and v_3; let a, b and c represent, respectively, the number of neighbours of v_1, v_2 and v_3 in the k-clique; and let d, e and f represent the number of vertices in the k-clique joined to both v_2 and v_3, both v_1 and v_3, and both v_1 and v_2 respectively. Then, if we remove all edges between any of the $\{v_i\}$, the number of ways to properly x-colour these three vertices is:

$$(x - a - e - f)(x - b - d - f)(x - c - d - e). \tag{6}$$

From these we must subtract those colourings in which two vertices receive the same colour. There are, for example:

$$(x - a - b - d - e - f)(x - c - d - e)$$

ways in which to properly x-colour the 3 vertices such that v_1 and v_2 receive the same colour. The other two pairs provide similar expressions, giving us three such expressions to subtract from (6). Finally we consider those colourings in which all three vertices receive the same colour. The number of these is, simply:

$$(x - a - b - c - d - e - f).$$

As we have effectively discounted these three times in the previous step, we must add this expression twice. Our final interesting factor is:

$$\begin{aligned} &(x - a - e - f)(x - b - d - f)(x - c - d - e) \\ &-(x - a - b - d - e - f)(x - c - d - e) \\ &-(x - a - c - d - e - f)(x - b - d - f) \\ &-(x - b - c - d - e - f)(x - a - e - f) + 2(x - a - b - c - d - e - f). \end{aligned} \tag{7}$$

We can now prove the $\alpha + n$ conjecture for cubic integers by employing a similar method to that used in the previous section. That is, we will show that given any reduced cubic polynomial $h(x)$ there is an interesting factor $g(x)$ of the above form and natural number n such that $h(x) = g(x+n)$.

We will proceed with each of the three types of reduced polynomial in turn, showing that for each type, and for every choice of the $x-$coefficient and constant term, the parameters $a, \ldots, f$ can be chosen in such a way as to produce the desired chromatic polynomial. There are no doubt many possible ways in which to correctly choose the parameters; in each case we will mention just one.

Case 1: $a_2 = -1$

Let $h(x) = x^3 - x^2 + a_1 x + a_0$, and let i represent any number. Assign the below values to the parameters a, b, c, d, e, f:

$$\begin{aligned}
a &= (2n + a_0)^2 - 11a_0 + 35 + a_1 - (8a_0 - 45)i - (16i + 24)n + 16i^2 \\
b &= -2i + n - 3 \\
c &= (2n + a_0)^2 - 13a_0 + 46 + a_1 - (8a_0 - 53)i - (16i + 28)n + 16i^2 \\
d &= i + 1 \\
e &= -(2n + a_0)^2 + 12a_0 - 41 - a_1 + (8a_0 - 50)i + (16i + 27)n - 16i^2 \\
f &= i
\end{aligned}$$

Let $g(x)$ be the polynomial obtained by substituting these values into (7). Then we have

$$g(x) = x^3 + (-3n - 1)x^2 + (3n^2 + 2n + a_1)x - n^3 - n^2 - a_1 n + a_0 = h(x - n),$$

as desired. It remains to show that, for any a_0 and a_1, appropriate values for i and n can be found such that each of the above parameters are non-negative integers. From the expressions for b, d and f we have that i must be non-negative, and n must satisfy $n \geq 2i + 3$. We introduce a new variable t by making the substitution

$$n = -a_0/2 + 2i + t,$$

giving us new expressions for a, c and e:

$$\begin{aligned}
a &= a_0 + 35 + a_1 - 3i - 24t + 4t^2 \\
c &= a_0 + 46 + a_1 - 3i - 28t + 4t^2 \\
e &= -3a_0/2 - 41 - a_1 + 4i + 27t - 4t^2.
\end{aligned}$$

Requiring that all these be non-negative then gives us the three inequalities:

$$3i \leq a_0 + 35 + a_1 - 24t + 4t^2 \tag{8}$$

$$3i \leq a_0 + 46 + a_1 - 28t + 4t^2 \tag{9}$$

$$4i \geq 3a_0/2 + 41 + a_1 - 27t + 4t^2. \tag{10}$$

Let t be an integer that is greater than 3, greater than $a_0/2 + 3$, and otherwise large enough to satisfy:

$$\frac{a_0 + 46 + a_1 - 28t + 4t^2}{3} \geq \frac{3a_0/2 + 41 + a_1 - 27t + 4t^2}{4} + 1.$$

There is at least one integer between the expression on the left and that on the right. Choose i to be such an integer; then the chosen values for i and t satisfy (9) and (10). Because $t \geq 3$, (9) implies (8). Finally set $n = \lceil -a_0/2 \rceil + 2i + t$. Because $t > a_0/2 + 3$, we then have that n satisfies the condition $n \geq 2i + 3$.

The remaining two cases are similar, and so will be more briefly described.

Case 2: $a_2 = 0$

Let $h(x) = x^3 + a_1 x + a_0 x$, and again let i be any number. This time set:

$$
\begin{aligned}
a &= (n + a_0)^2 + a_1 + 14 + 19i + 9i^2 - (6i + 8)n - (6i + 6)a_0 \\
b &= -2i + n - 3 \\
c &= (n + a_0)^2 + a_1 + 20 + 25i + 9i^2 - (6i + 10)n - (6i + 8)a0 \\
d &= i + 1 \\
e &= -(n + a_0)^2 - a_1 - 18 - 23i - 9i^2 + (6i + 10)n + (6i + 7)a_0 \\
f &= i
\end{aligned}
$$

Let $g(x)$ be the polynomial obtained by substituting these values into (7). Then

$$g(x) = x^3 - 3nx^2 - (3n^2 - a_1 + 3n^2)x - n^3 - a_1 n + a_0 = h(x - n).$$

Now make the substitution

$$n = -a_0 + 3i + t.$$

This gives us the following expressions for a, c and e:

$$
\begin{aligned}
a &= t^2 + a_1 + 14 - 5i + 2a_0 - 8t \\
c &= t^2 + a_1 + 20 - 5i + 2a_0 - 10t \\
e &= -t^2 - a_1 - 18 + 7i - 3a_0 + 10t,
\end{aligned}
$$

leading to the inequalities:

$$
\begin{aligned}
5i &\leq t^2 + a_1 + 14 + 2a_0 - 8t \\
5i &\leq t^2 + a_1 + 20 + 2a_0 - 10t \\
7i &\geq t^2 + a_1 + 18 + 3a_0 + 10t.
\end{aligned}
$$

Again, by choosing t to be very large, a positive value for i can be found to satisfy these for any a_0, a_1.

Case 3: $a_2 = 1$

Let $h(x) = x^3 + x^2 + a_1 x + a_0 x$, and set:

$$
\begin{aligned}
a &= a_0^2 + 5 - a_0 + a1 + (3 - 4a_0)i - 2n + 4i^2 \\
b &= -2i + n - 3 \\
c &= a_0^2 + 6 - 3a_0 + a_1 + (7 - 4a_0)i - 2n + 4i^2 \\
d &= i + 1 \\
e &= -a_0^2 - 7 + 2a_0 - a_1 - (6 - 4a_0)i + 3n - 4i^2 \\
f &= i
\end{aligned}
$$

Substituting into (7) we get

$$g(x) = x^3 + (1 - 3n)x^2 + (3n^2 - 2n + a1)x - n^3 + n^2 - a1n + a0 = h(x - n).$$

We now express i in terms of a new parameter t, setting:

$$i = a_0/2 - t.$$

This gives us

$$\begin{aligned} a &= 5 + a_0/2 + a_1 - 3t - 2n + 4t^2 \\ c &= 6 + a_0/2 + a_1 - 7t - 2n + 4t^2 \\ e &= -7 - a_0 - a_1 + 6t + 3n - 4t^2, \end{aligned}$$

and so we must satisfy

$$\begin{aligned} 2n &\leq 5 + a_0/2 + a_1 - 3t + 4t^2 \\ 2n &\leq 6 + a_0/2 + a_1 - 7t + 4t^2 \\ 3n &\geq 7 + a_0 + a_1 - 6t + 4t^2. \end{aligned}$$

This time we need to choose a large negative value for t. If it is large enough then d and f will be non-negative, and we can easily find a positive n to satisfy the three inequalities, as well as the requirement $n \geq 2i + 3$.

Thus we have given an explicit means by which to construct a $(3, k)$-biclique with a chromatic root $\alpha + n$ for any cubic integer α, thereby proving the cubic case of the $\alpha + n$ conjecture.

Remark

Given the roughly exponential increase in the number of chromatic polynomials of (j, k)-bicliques as j increases (when constructed in the same manner as was used to obtain 7, the interesting factor of the chromatic polynomial of a (j, k)-biclique has $2^j - 2$ parameters), it seems entirely plausible that they might satisfy the general conjecture. Unfortunately the increase in parameters leads to difficulties in finding correct specialisations in the manner of the two cases proved so far, and it seems likely that a different method from that used here would need to be found for higher degree algebraic numbers.

4 The $n\alpha$ Conjecture

The following theorem implies that chromatic roots of clique-theta graphs satisfy the assertions of the $n\alpha$ conjecture.

Theorem 4.1 *If $S_i = (a_{i(1)}, \ldots, a_{i(m_i)})$, then let $nS_i = (na_{i(1)}, \ldots, na_{i(m_i)})$. Suppose α is a non-integer chromatic root of $T(1, S_1, S_2, \ldots, S_k, p)$; then $n\alpha$ is a chromatic root of $T(1, nS_1, nS_2, \ldots, nS_k, np)$.*

Proof: Again we need only consider the interesting factors of the relevant chromatic polynomials. For $T(1, S_1, S_2, \ldots, S_k, p)$ this is, by Proposition 2.1:

$$\left[p(x-p)^{k-1} \prod_{i=1}^{k} r(1, a_{i(1)}, \ldots, a_{i(m_i)}) \right] + \left[\prod_{i=1}^{k} r(1, a_{i(1)}, a_{i(2)}, \ldots, a_{i(m_i)}, p) \right]. \qquad (11)$$

Expanding the interesting factors of the rings of cliques, this becomes:

$$\left[p(x-p)^{k-1}\prod_{i=1}^{k}\frac{1}{x}\left(\prod_{l=1}^{m_i}(x-a_{i(l)})-\prod_{l=1}^{m_i}(-a_{i(l)})\right)\right] \\ +\left[\prod_{i=1}^{k}\frac{1}{x}\left((x-p)\prod_{l=1}^{m_i}(x-a_{i(l)})+p\prod_{l=1}^{m_i}(-a_{i(l)})\right)\right]. \tag{12}$$

For $T(1, nS_1, nS_2, \ldots, nS_k, np)$, we have:

$$\left[np(x-np)^{k-1}\prod_{i=1}^{k}\frac{1}{x}\left(\prod_{l=1}^{m_i}(x-na_{i(l)})-\prod_{l=1}^{m_i}(-na_{i(l)})\right)\right] \\ +\left[\prod_{i=1}^{k}\frac{1}{x}\left((x-np)\prod_{l=1}^{m_i}(x-na_{i(l)})+np\prod_{l=1}^{m_i}(-na_{i(l)})\right)\right]. \tag{13}$$

Let $s=\sum_{i=1}^{k}(m_i+1)$. Then dividing (13) by n^s gives:

$$\left[p(x/n-p)^{k-1}\prod_{i=1}^{k}\frac{1}{x}\left(\prod_{l=1}^{m_i}(x/n-a_{i(l)})-\prod_{l=1}^{m_i}(-a_{i(l)})\right)\right] \\ +\left[\prod_{i=1}^{k}\frac{1}{x}\left((x/n-p)\prod_{l=1}^{m_i}(x/n-a_{i(l)})+p\prod_{l=1}^{m_i}(-a_{i(l)})\right)\right]. \tag{14}$$

If α is a zero of (12), then $n\alpha$ is a zero of (14).

□

Corollary 4.2 *The set of chromatic roots satisfying the $n\alpha$ conjecture is dense in the complex plane.*

Proof: By Sokal (2004), the chromatic roots of generalised theta graphs [(i)] are dense in the complex plane (if we include integer shifts of roots — see Corollary 1.3 in Sokal's paper). As a generalised theta graph is simply a clique-theta graph with all cliques of size one, the result follows from Theorem 4.1. □

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[(i)] A number of different definitions for generalised theta graphs are used in the literature. Note that in Sokal's definition, the disjoint paths between the endpoints are assumed to all have same length.

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