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Hecke algebra characters and quantum chromatic symmetric functions

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Abstract. We evaluate induced sign characters of $H_n(q)$ at certain elements of $H_n(q)$ and conjecture an interpretation for the resulting polynomials as generating functions for P-tableaux by a certain statistic. Our conjecture relates the quantum chromatic symmetric functions of Shareshian and Wachs to $H_n(q)$ characters.

Résumé. Nous évaluons les caractères de signe induits de $H_n(q)$ à certains éléments de $H_n(q)$ et nous conjecturons une interprétation des polynômes résultants comme fonctions génératrices pour les tableaux-P par une certaine statistique. Cette conjecture établit un lien entre les fonctions chromatiques symmétriques de Shareshian et Wachs et les caractères de $H_n(q)$.

Keywords: Hecke algebra character, unit interval order, P-tableau, chromatic symmetric function, quantum analog.

1 Introduction

Let S_n be the symmetric group, let $s_i = (i, i+1)$, $i = 1, \ldots, n-1$ denote its standard generators, and for all $v \in S_n$, let $\ell(v)$ denote the length of any reduced (short as possible) expression $s_{i_1} \cdots s_{i_\ell}$ for v. (See [12] for more information on this material.) It is well known that each conjugacy class of S_n consists of the permutations whose cycle sizes in weakly decreasing order are equal to some fixed partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$. Define the *Young diagram* of λ to be the left-justified array of n boxes having λ_i boxes in row i. The $\mathbb{Z}$-module of S_n *class functions*, those functions $f : S_n \to \mathbb{Z}$ satisfying $f(w^{-1}vw) = f(v)$ for all $v, w \in S_n$, has several well-studied bases:

1. the irreducible characters $\{\chi^\lambda \,|\, \lambda \vdash n\}$,
2. the induced sign characters $\{\epsilon^\lambda = \mathrm{sgn}\!\uparrow_{S_\lambda}^{S_n} \,|\, \lambda \vdash n\}$,
3. the induced trivial characters $\{\eta^\lambda = \mathrm{triv}\!\uparrow_{S_\lambda}^{S_n} \,|\, \lambda \vdash n\}$,
4. the monomial class functions $\{\phi^\lambda \,|\, \lambda \vdash n\}$,

where S_λ is the Young subgroup of S_n of type λ. These bases are related to one another just as are the Schur, elementary, (complete) homogeneous, and monomial bases of the space of homogeneous degree n

symmetric functions. Specifically, we have

$$\begin{aligned} h_\lambda &= \sum_\mu K_{\mu,\lambda} s_\mu, \quad & e_\lambda &= \sum_\mu K_{\mu^\top,\lambda} s_\mu, \quad & s_\lambda &= \sum_\mu K_{\lambda,\mu} m_\mu, \\ \eta^\lambda &= \sum_\mu K_{\mu,\lambda} \chi^\mu, \quad & \epsilon^\lambda &= \sum_\mu K_{\mu^\top,\lambda} \chi^\mu, \quad & \chi^\lambda &= \sum_\mu K_{\lambda,\mu} \phi^\mu, \end{aligned} \tag{1}$$

where $K = (K_{\lambda,\mu})$ is the invertible matrix of *Kostka numbers*, and $\lambda^\top$ is the partition whose Young diagram is transpose (or *conjugate*) to that of λ. For each of these class functions f which is not an irreducible character, the integer $f(v)$ has a simple combinatorial formula; when $f = \chi^\lambda$, the number $\chi^\lambda(v)$ is computed by a cumbersome algorithm called the Murnaghan-Nakayama rule. This fact is somewhat unfortunate, since the irreducible characters are the most important of the above class functions.

The above class functions extend linearly to the group algebra $\mathbb{Z}[S_n]$. Work of Goulden and Jackson [4] and Greene [5] has led to the study of combinatorially interpreting $f(z)$ for f a class function and z in $\mathbb{Z}[S_n]$. In particular, these authors used S_n subgroups of the form $S_{[i,j]}$, which fix all letters outside of the interval $[i,j] = \{i, \dots, j\}$ of $[n] \underset{\text{def}}{=} \{1, \dots, n\}$, to construct elements

$$C'_{[i,j]} \underset{\text{def}}{=} \sum_{v \in S_{[i,j]}} v \tag{2}$$

of $\mathbb{Z}[S_n]$, and products of these. It is not difficult to show that we have

$$f(C'_{[i_1,j_1]} \cdots C'_{[i_m,j_m]}) \geq 0 \tag{3}$$

when f is η^λ or ϵ^λ, and to give a combinatorial interpretation of the resulting nonnegative integers in these cases. (See Theorems 2.2-2.3.) Stembridge showed furthermore [19, Cor. 3.3] that the inequality (3) holds for $f = \chi^\lambda$, and conjectured [20, Conj. 2.1] that it holds also for $f = \phi^\lambda$. In neither of these cases is there a proposed combinatorial interpretation, however. Stanley [15] initiated the study of the special case of (3) in which we have $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$, relating this special case of the equation to certain *chromatic symmetric functions*, and to posets and graphs which avoid certain substructures. In this special case Gasharov [2, Thm. 2] found an interpretation when $f = \chi^\lambda$, but when $f = \phi^\lambda$ we still have no proof of positivity and no conjectured combinatorial interpretation.

It is conceivable that quantum analogs of S_n characters may aid in the formulation of the unknown combinatorial interpretations mentioned above. In particular, the (type A) Hecke algebra $H_n(q)$ and certain functions χ_q^λ, ϵ_q^λ, η_q^λ, $\phi_q^\lambda : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$, known as $H_n(q)$ *traces*, serve as analogs of S_n and its class functions. Furthermore, there are well-studied quantum analogs $C'_{[i,j]}(q) \in H_n(q)$ of the elements (2) in $\mathbb{Z}[S_n]$, and Haiman [6, Lem. 1.1] has proved that we have

$$f_q(C'_{[i_1,j_1]}(q) \cdots C'_{[i_m,j_m]}(q)) \in \mathbb{N}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] \tag{4}$$

when f_q is η_q^λ, ϵ_q^λ, or χ_q^λ. He conjectured that the same holds for $f_q = \phi_q^\lambda$. Unfortunately, no simple combinatorial formulas are known or even conjectured for the evaluation of $H_n(q)$ traces at standard basis elements of $H_n(q)$ (in general), or at the special elements in (4). We will state and prove the first such formula for elements of the form (4) in the special case that $i_1 < \cdots < i_r$, $j_1 < \cdots < j_r$, and will relate this to the quantum chromatic symmetric functions introduced recently by Shareshian and Wachs [13].

In Section 2 we asssociate an element $\beta(P)$ of $\mathbb{Z}[S_n]$ to each labeled poset P. We evaluate induced sign characters at $\beta(P)$ and interpret the resulting nonnegative integers in terms of P-tableaux. In Section 3 we restrict our attention to special classes of posets P, especially unit interval orders. We evaluate other class functions at the associated elements $\beta(P)$, again interpreting the resulting nonnegative integers in terms of P-tableaux. We also discuss the appearance of these interpretations in Stanley's chromatic symmetric functions [15]. In Section 4 we present quantum analogs of much of the earlier material, including an element $\beta_q(P)$ of $H_n(q)$ which we associate to each labeled poset P. In Section 5 we present our main results. For each unit interval order P, we evaluate induced $H_n(q)$ sign characters at $\beta_q(P)$ and interpret the resulting elements of $\mathbb{N}[q]$ as generating functions for P-tableaux by a certain inversion statistic. Finally, we express the Shareshian-Wachs quantum chromatic symmetric functions [13] in terms of induced $H_n(q)$ sign characters.

2 P-tableaux and induced S_n sign characters

Given a poset P, define a *P-tableau of shape* λ to be a Young diagram of shape λ whose boxes contain the elements of P. A P-tableau is called *column-strict* if whenever elements $i_1, \ldots, i_r$ appear from top to bottom in a column, then we have that $i_1 <_P \cdots <_P i_r$ is a chain in P. A P-tableau is called *row-semistrict* if whenever i_1, i_2 appear consecutively (from left to right) in a row, then we have $i_1 <_P i_2$ or i_1 incomparable to i_2 in P. A P-tableau is called *semistandard* if it is both column-strict and row-semistrict. When P is simply the chain of integers $1 < \cdots < n$, we will refer to a P-tableau as a *Young tableau*.

It is often convenient to label the elements of a poset and to encode the labeled poset as a certain 0-1 matrix. If a labeled poset P has n elements, we will always assume the labels to be $\{1, \ldots, n\}$. We call P *naturally labeled* if whenever $i <_P j$, we have $i < j$ as integers. Every poset has at least one natural labeling. Given a labeled poset P, we define the *antiadjacency matrix* $A = A(P) = (a_{i,j})$ of P by

$$a_{i,j} = \begin{cases} 0 & \text{if } i <_P j, \\ 1 & \text{otherwise.} \end{cases} \tag{5}$$

Two different labelings of a poset give antiadjacency matrices which are conjugate to one another by a permutation matrix. If P is naturally labeled, then in its antiadjacency matrix, all entries on or below the diagonal are 1. We may use the antiadjacency matrix of P to count column-strict P-tableaux as follows. Call a sequence $(I_1, \ldots, I_r)$ of disjoint subsets of $[n]$ an *ordered set partition of type* λ if we have $|I_k| = \lambda_k$ for $k = 1, \ldots, r$ and $I_1 \cup \cdots \cup I_r = [n]$. For any $n \times n$ matrix A and any subset $I \subset [n]$, define $A_{I,I}$ to be the $|I| \times |I|$ submatrix $(a_{i,j})_{i,j \in I}$ of A.

Proposition 2.1 *Let P be a labeled poset with antiajacency matrix A, and let λ be a partition of $n = |P|$. Then the number of column-strict P-tableaux is given by*

$$\sum_{(I_1, \ldots, I_r)} \det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r}), \tag{6}$$

where the sum is over all ordered set partitions of $[n]$ of type λ.

Proof: Suppose that P is naturally labeled. It is straightforward to show that each minor in (6) satisfies

$$\det(A_{I_k,I_k}) = \begin{cases} 1 & \text{if } I_k \text{ is a chain in } P, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

and thus the claim is true in this case. (See, e.g., [16].) Furthermore, it is also straightforward to show that the polynomial function $\mathrm{Mat}_{n\times n}(\mathbb{Z}) \to \mathbb{Z}$ defined by the expression (6) is invariant under conjugation by a permutation matrix. $\square$

This relationship between the antiadjacency matrix and P-tableaux allows us to state a relationship between induced sign characters of S_n and P-tableaux. To any labeled n-element poset P we may associate an element $\beta(P)$ of $\mathbb{Z}[S_n]$ by

$$\beta(P) = \sum_v v, \tag{8}$$

where the sum is over all permutations v satisfying $i \not<_P v_i$ for all i. Equivalently, $\beta(P)$ is a sum of permutations which correspond to the placement of n nonattacking rooks on entries of $A(P)$ which are equal to 1 [18]. Now we have the following [1, Cor. 4.2]. (See also [18, Eqn. (5.1)].)

Theorem 2.2 *Let P be a labeled poset and let λ be a partition of $|P|$. Then $\epsilon^\lambda(\beta(P))$ equals the number of column-strict P-tableaux of shape $\lambda^\top$.*

Proof: Let $n = |P|$ and let $x = (x_{i,j})_{i,j\in[n]}$ be a matrix of n^2 variables. The Littlewood-Merris-Watkins identity for induced sign characters [10, Sec. 6.5], [11, Sec. 1] states that in $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ we have

$$\sum_{(I_1,\ldots,I_r)} \det(x_{I_1,I_1}) \cdots \det(x_{I_r,I_r}) = \sum_{v\in S_n} \epsilon^\lambda(v) x_{1,v_1} \cdots x_{n,v_n}. \tag{9}$$

Now let A be the antiadjacency matrix of P and evaluate (9) at $x = A$. Since $a_{1,v_1} \cdots a_{n,v_n} = 1$ precisely when $i \not<_P v_i$ for all i, we obtain

$$\epsilon^\lambda\Big(\sum_{v\in S_n} a_{1,v_1} \cdots a_{n,v_n} v\Big) = \epsilon^\lambda(\beta(P)),$$

and our result follows from Proposition 2.1. $\square$

A similar argument using permanents of submatrices applies to the induced trivial characters of S_n.

Theorem 2.3 *Let P be a labeled poset and let λ be a partition of $|P|$. Then $\eta^\lambda(\beta(P))$ equals the number of row-semistrict P-tableaux of shape λ.*

Proof: Omitted. $\square$

3 $(\mathbf{3}+\mathbf{1})$-free posets and chromatic symmetric functions

To state more results and conjectures concerning the interpretation of S_n class functions, we restrict our attention to special classes of posets. The following standard definitions concerning posets may be found in [17, Ch. 3] and [21, Sec. 9.2]. Given posets P, Q, let $P \oplus Q$ be the $(|P| + |Q|)$-element poset whose Hasse diagram is that for P written below that for Q, i.e., with every element of P being less than every element of Q. Letting $\mathbf{n}$ denote an n-element chain $x_1 <_P \cdots <_P x_n$, we have, e.g., $\mathbf{4} \cong \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$. Let $P + Q$ be the $(|P| + |Q|)$-element poset which is a disjoint union of P and Q, i.e., the poset whose Hasse diagrams is that for P written beside that for Q, with all elements of P being incomparable to all elements of Q. Thus $\mathbf{a}+\mathbf{b}$ is the disjoint union of an a-element chain and a b-element chain. We say that P is $(\mathbf{a}+\mathbf{b})$-free if no induced subposet of P is isomorphic to $\mathbf{a}+\mathbf{b}$. Call P an *interval order* if it is $(\mathbf{2}+\mathbf{2})$-free, and a *unit interval order* if it is $(\mathbf{2}+\mathbf{2})$-free and $(\mathbf{3}+\mathbf{1})$-free.

While no true analog of Theorem 2.2 and 2.3 is known for $\chi^\lambda(\beta(P))$ and $\phi^\lambda(\beta(P))$, some results exist for special cases of λ and for P belonging to one of the above classes of posets.

Proposition 3.1 *Let P be a labeled $(\mathbf{3}+\mathbf{1})$-free poset and let λ be a partition of $|P|$.*

(1) $\chi^\lambda(\beta(P))$ equals the number of standard P-tableaux of shape λ [2, Thm. 2].

(2) If P is a unit interval order and $\lambda_1 \leq 2$, then $\phi^\lambda(\beta(P))$ equals zero if there exists a column-strict P-tableaux of shape $\mu \prec \lambda$ in the dominance order; otherwise it equals the number of column-strict P-tableaux of shape λ [1, Thm. 5.7].

(3) If P is a unit interval order and $\lambda = (k, \ldots, k)$, then $\phi^\lambda(\beta(P))$ equals the number of ways to cover P with certain cycles [20, Thm. 2.8].

Furthermore, Stembridge's [20, Conj. 2.1] has the following special case.

Conjecture 3.2 *Let P be a labeled unit interval order and let λ be a partition of $|P|$. Then we have $\phi^\lambda(\beta(P)) \geq 0$.*

Even with this restriction on P, there is no conjectured combinatorial interpretation for $\phi^\lambda(\beta(P))$ which applies to all λ.

One property of unit interval orders P which facilitates combinatorial interpretation is the fact that certain labelings of P cause the antiadjacency matrix $A(P)$ and the $\mathbb{Z}[S_n]$ element $\beta(P)$ to have particularly nice forms. Define the *altitude* of a poset element i to be $\alpha(i) = \#\{x \,|\, x <_P i\} - \#\{x \,|\, x >_P i\}$. Call a labeling an *altitude respecting* or *ar-labeling* if $i < j$ whenever $\alpha(i) < \alpha(j)$. It is known that every *ar*-labeling of a $(\mathbf{3}+\mathbf{1})$-free poset is natural. It is also known that in the antiadjacency matrix corresponding to an *ar*-labeling of an n-element unit interval order, the zero entries form a right justified Young diagram which fits inside the right justified shape $(n-1, \ldots, 1)$. (See [21, Sec. 8.2].) In Proposition 4.1 we will prove that this special form of the antiadjacency matrix implies a nice expression for $\beta(P)$,

$$\beta(P) = \frac{C'_{[i_1,j_1]} \cdots C'_{[i_m,j_m]}}{k}, \qquad i_1 < \cdots < i_m, \quad j_1 < \cdots < j_m, \quad k \in \mathbb{N}. \tag{10}$$

S_n class functions may be used to define Stanley's *chromatic symmetric function* [15] for a poset P,

$$X_P = \sum_{\lambda \vdash n} c_\lambda m_\lambda, \tag{11}$$

where c_λ is the number of ways to partition P into a sequence of r chains of sizes $(\lambda_1, \ldots, \lambda_r)$ respectively, and to assign color k to the kth chain. Equivalently, we have $c_\lambda = \epsilon^\lambda(\beta(P))$. The transition matrices of Kostka numbers in (1) imply furthermore that we have

$$X_P = \sum_{\lambda \vdash n} \epsilon^\lambda(\beta(P)) m_\lambda = \sum_{\lambda \vdash n} \chi^{\lambda^\top}(\beta(P)) s_\lambda = \sum_{\lambda \vdash n} \phi^\lambda(\beta(P)) e_\lambda. \tag{12}$$

Monomial nonnegative by definition, the chromatic symmetric functions are sometimes Schur nonnegative and even elementary nonnegative as well. For example, the four posets $P = \mathbf{3}$, $\mathbf{2} + \mathbf{1}$, $(\mathbf{1} + \mathbf{1}) \oplus \mathbf{1}$, $\mathbf{1} + \mathbf{1} + \mathbf{1}$ and their chromatic symmetric functions X_P are

$$\begin{array}{llll} X_P = 6m_{111} + 3m_{21} + m_3 & X_P = 6m_{111} + m_{21} & X_P = 6m_{111} + 2m_{21} & X_P = 6m_{111} \\ \quad = s_{111} + 2s_{21} + s_3 & \quad = 4s_{111} + s_{21} & \quad = 2s_{111} + 2s_{21} & \quad = 6s_{111} \\ \quad = e_{111}, & \quad = 3e_3 + e_{21}, & \quad = 2e_{21}, & \quad = 6e_3. \end{array} \tag{13}$$

These nonnegativity phenomena are addressed in the following result of Gasharov [2, Thm. 2] and conjecture of Stanley and Stembridge [15, Conj. 5.1], [18, Conj. 5.5].

Theorem 3.3 *If P is $(\mathbf{3} + \mathbf{1})$-free, then X_P is Schur nonnegative.*

Conjecture 3.4 *If P is $(\mathbf{3} + \mathbf{1})$-free, then X_P is elementary nonnegative.*

By (12), Theorem 3.3 is equivalent to Proposition 3.1 (1), Conjecture 3.4 is stronger than Conjecture 3.2, and two special cases of Conjecture 3.4 have been proved in Proposition 3.1 (2)-(3).

4 The Hecke algebra and quantum polynomial ring in n^2 variables

Let $H_n(q)$ be the (type A) *Hecke algebra*, the noncommutative $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra with multiplicative identity $T_e = 1$, generated by $T_{s_1}, \ldots, T_{s_{n-1}}$, subject to relations

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_e, && \text{for } i = 1, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j}, && \text{if } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i}, && \text{if } |i-j| \geq 2. \end{aligned} \tag{14}$$

If $s_{i_1} \cdots s_{i_\ell}$ is (any) reduced expression for $v \in S_n$ we define $T_v = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$. The $n!$ elements $\{T_v \,|\, v \in S_n\}$ are a basis of $H_n(q)$ as a $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module.

Let χ_q^λ be the irreducible $H_n(q)$ character indexed by λ, and let ϵ_q^λ and η_q^λ be the sign and trivial characters induced from Young subalgebras of type λ. All of these $H_n(q)$ characters are $H_n(q)$ *traces*, i.e., they satisfy $f_q(gh) = f_q(hg)$ for $g, h \in H_n(q)$. Like a class function, a trace f_q satisfies $f_q(g^{-1}hg) = f_q(h)$ whenever g is invertible. Define $\phi_q^\lambda = \sum_\mu K_{\lambda,\mu}^{-1}\chi_q^\mu$ to be the *monomial trace* indexed by λ. These sets of traces form four bases for the space of $H_n(q)$ traces and are related to one another by the same transition matrices of Kostka numbers (1) that relate the S_n class function bases used in Sections 1-3:

$$\eta_q^\lambda = \sum_{\mu \vdash n} K_{\mu,\lambda}\chi_q^\mu, \quad \epsilon_q^\lambda = \sum_{\mu \vdash n} K_{\mu^\top,\lambda}\chi_q^\mu, \quad \chi_q^\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu}\phi_q^\mu. \tag{15}$$

(See, e.g., [3].) For none of these traces f_q do we have a known combinatorial interpretation of $f_q(T_v)$ for all $v \in S_n$. Neither do we have a known combinatorial interpretation of

$$f_q(C'_{[i_1,j_1]}(q) \cdots C'_{[i_m,j_m]}(q)), \tag{16}$$

where

$$C'_{[i,j]}(q) \underset{\mathrm{def}}{=} q^{\frac{-\ell(w)}{2}} \sum_{v \in S_{[i,j]}} T_v,$$

and w is the longest element in $S_{[i,j]}$. We therefore consider special cases of the elements (16) which are quantum analogs of the elements $\beta(P)$ in $\mathbb{Z}[S_n]$. To any labeled n-element poset P we may associate an element $\beta_q(P)$ of $H_n(q)$ by

$$\beta_q(P) = \sum_{v \in S_n} T_v, \tag{17}$$

where the sum is over all permutations $v \in S_n$ satisfying $i \not<_P v_i$ for all i. The *ar*-labeling of a unit interval order P gives a nice expression for $\beta_q(P)$.

Proposition 4.1 *Let P be an* ar*-labeled n-element unit interval order. Then there exist subintervals $[i_1, j_1], \ldots, [i_m, j_m]$ of $[n]$ with $i_1 < \cdots < i_m$, $j_1 < \cdots < j_m$, a polynomial $p \in \mathbb{N}[q]$, a 3412-avoiding, 4231-avoiding permutation $w = w(P)$ in S_n, and a nonnegative integer c such that*

$$\beta_q(P) = \frac{q^{\frac{\ell(w)+c}{2}} C'_{[i_1,j_1]}(q) \cdots C'_{[i_m,j_m]}(q)}{p(q)} = \sum_{v \le w} T_v. \tag{18}$$

Proof: (Idea.) The n-element unit interval orders correspond bijectively to certain planar networks G having n *source* and n *sink* vertices, with each such network being composed of smaller networks indexed by intervals $[i_1, j_1], \ldots, [i_m, j_m]$. For each such planar network G, one defines a matrix $B = (b_{i,j})$ by setting $b_{i,j} = 1$ if there exists a path from source i to sink j, and $b_{i,j} = 0$ otherwise. One also defines an element $\beta_q(G) = \sum_v b_{1,v_1} \cdots b_{n,v_n} T_v$ of $H_n(q)$. If in the above bijection G corresponds to P, then one can show that the antiadjacency matrix A of P satsifies $a_{1,v_1} \cdots a_{n,v_n} = b_{1,v_1} \cdots b_{n,v_n}$ for all $v \in S_n$ (although A and B are not in general equal). Thus we have $\beta_q(P) = \beta_q(G)$.

By [14, Thm. 4.3], there exist a certain nonnegative integer c and a 3412-, 4231-avoiding permutation $w \in S_n$ associated to G such that $q^{\frac{-\ell(w)}{2}} \beta_q(G)$ is equal to $q^{\frac{c}{2}} C'_{[i_1,j_1]}(q) \cdots C'_{[i_m,j_m]}(q)$ divided by a polynomial in $\mathbb{N}[q]$. By [14, Lem. 5.3], w also satisfies

$$b_{1,v_1} \cdots b_{n,v_n} = \begin{cases} 1 & \text{if } v \le w \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Thus we have $\beta_q(P) = \beta_q(G) = \sum_{v \le w} T_v$. □

For P an *ar*-labeled unit interval order, let $w(P)$ denote the permutation appearing in the above proposition. We remark that by results in [9], $\beta_q(P)$ is equal to $q^{\frac{\ell(w)}{2}}$ times the Kazhdan-Lusztig basis element $C'_{w(P)}(q)$ of $H_n(q)$ [7], and the factors $C'_{[i,j]}(q)$ also are Kazhdan-Lusztig basis elements.

Even for the special case (18) of (16), there is no published combinatorial interpretation for $f_q(\beta_q(P))$ when $f_q \in \{\chi_q^\lambda, \eta_q^\lambda, \epsilon_q^\lambda, \phi_q^\lambda\}$. On the other hand, Haiman's [6, Lem. 1.1, Conj. 2.1] have the following special cases which suggest that combinatorial interpretations for some of these expressions should exist.

Proposition 4.2 *Let P be an* ar*-labeled unit interval order and let λ be a partition of $|P|$. Then we have $\chi_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$, and therefore $\epsilon_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$ and $\eta_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$.*

Conjecture 4.3 *Let P be an* ar*-labeled unit interval order and let λ be a partition of $|P|$. Then we have $\phi_q^{\lambda}(\beta_q(P)) \in \mathbb{N}[q]$.*

In Section 5 we give a combinatorial interpretation for the polynomial $\epsilon_q^{\lambda}(q_{e,w}\beta_q(P))$ for P an *ar*-labeled unit interval order. To do this, we employ a quantum analog of the identity (9) which holds in the *quantum polynomial ring* $\mathcal{A}_n(q)$. Let $\mathcal{A}_n(q)$ be the noncommutative $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra generated by n^2 variables $x = (x_{1,1}, \ldots, x_{n,n})$, subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell}, \qquad & x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k}, \qquad & x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \tag{20}$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$.

$\mathcal{A}_n(q)$ is a multigraded $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module whose components $\mathcal{A}_{L,M}(q)$ are spanned by monomials having a fixed multiset L of row indices and a fixed multiset M of column indices. In particular, the component $\mathcal{A}_{[n],[n]}(q)$ contains monomials having total degree n, with all row indices and all column indices appearing exactly once. Defining $x^{u,v} = x_{u_1,v_1} \cdots x_{u_n,v_n}$ for $u, v \in S_n$, we thus have

$$\mathcal{A}_{[n],[n]}(q) = \mathrm{span}\{x^{u,v} \mid u, v \in S_n\}.$$

By the relations (20), each multigraded component has a $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-basis of monomials in which variables appear in lexicographic order. Thus the component $\mathcal{A}_{[n],[n]}(q)$ has the natural basis $\{x^{e,v} \mid v \in S_n\}$, and contains the *quantum determinant*, defined by

$$\det_q(x) \underset{\mathrm{def}}{=} \sum_{v \in S_n} (-q^{\frac{-1}{2}})^{\ell(v)} x^{e,v}.$$

The quantized Littlewood-Merris-Watkins identity, due to Konvalinka and the second author [8, Thm. 5.4], gives generating functions for the induced sign characters of $H_n(q)$.

Theorem 4.4 *Fix any partition $\lambda \vdash n$. Then in $\mathcal{A}_{[n],[n]}(q)$ we have the identity*

$$\sum_{(I_1,\ldots,I_r)} \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}) = \sum_{v \in S_n} \epsilon_q^{\lambda}(q^{\frac{-\ell(v)}{2}} T_v) x^{e,v}, \tag{21}$$

where the first sum is over all ordered set partitions of $[n]$ of type λ.

5 Main results

Recall that for P a unit interval order with antiadjacency matrix A, the combinatorial interpretation of $\beta(P)$ in Theorem 2.2 follows immediately from the evaluation of both sides of the classical Littlewood-Merris-Watkins identity (9) at $x = A$. However, the quantum analog (21) of this identity does not allow us to compute $\epsilon^{\lambda}(\beta_q(P))$ as the evaluation of the left-hand side at $x = A$, since the noncommuting variables $(x_{i,j})$ satisfy nontrivial relations. If we are to use the left-hand side of (21) to understand $\epsilon^{\lambda}(\beta_q(P))$, we must first express it in terms of the natural basis $\{x^{e,v} \mid v \in S_n\}$. We therefore define a family of maps as follows. For each matrix $A \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$, let $\sigma_A : \mathcal{A}_{[n],[n]}(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-linear map defined by

$$\sigma_A(x^{e,v}) = q^{\frac{\ell(v)}{2}} a_{1,v_1} \cdots a_{n,v_n}.$$

This definition has the following immediate consequence.

Proposition 5.1 *Let P be an* ar*-labeled poset with antiadjacency matrix A, let $w = w(P)$, and let λ be a partition of $n = |P|$. Then we have*

$$\sum_{(I_1,\ldots,I_r)} \sigma_A(\det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r})) = \epsilon_q^\lambda(\beta_q(P)), \tag{22}$$

where the sum is over all ordered set partitions of $[n]$ of type λ.

Proof: The left-hand side of (22) equals

$$\sigma_A\Big(\sum_{v \in S_n} \epsilon_q^\lambda(q^{\frac{-\ell(v)}{2}} T_v) x^{e,v} \Big) = \sum_{v \in S_n} \epsilon_q^\lambda(T_v) q^{\frac{-\ell(v)}{2}} \sigma_A(x^{e,v}) = \epsilon_q^\lambda \Big(\sum_{v \leq w} T_v \Big) = \epsilon_q^\lambda(\beta_q(P)),$$

since the proof of Proposition 4.1 implies that $a_{1,v_1} \cdots a_{n,v_n}$ is 1 if $v \leq w$ and is 0 otherwise. □

To combinatorially interpret each term on the left-hand side of (22), we introduce the following statistic on P-tableaux. Given a P-tableau U, define an *inversion* in U to be a pair (i,j) of incomparable elements in P satisfying $i < j$, with i appearing in a column of U to the right of the column containing j. Let $\textsc{inv}(U)$ denote the number of inversions in U.

Lemma 5.2 *Let P be a labeled unit interval order with antiadjacency matrix A, let λ be a partition of $n = |P|$, let $(I_1, \ldots, I_r)$ be an ordered set partition of $[n]$ of type λ, and let U be the P-tableau whose jth column contains the poset elements labeled by I_j, with labels increasing upward in that column. If U is a column-strict P-tableau, then we have*

$$\sigma_A(\det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r})) = q^{\textsc{inv}(U)}. \tag{23}$$

Proof: Omitted. □

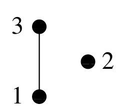
The authors believe that if we keep the hypotheses of Lemma 5.2, but assume that the P-tableau U is *not* column-strict, then $\sigma_A(\det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}))$ vanishes. In light of Proposition 5.1, we thus have the following conjecture.

Conjecture 5.3 *Let P be an* ar*-labeled unit interval order, and let λ be a partition of $|P|$. Then we have*

$$\epsilon_q^\lambda(\beta_q(P)) = \sum_U q^{\textsc{inv}(U)}, \tag{24}$$

where the sum is over all column-strict P-tableaux of shape $\lambda^\top$.

For example, let P be the poset $\mathbf{2} + \mathbf{1}$ with its unique *ar*-labeling



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and its two pairs of incomparable elements $(1,2)$ and $(2,3)$. Then there are six column-strict P-tableaux of shape $111^\top = 3$, one of shape $21^\top = 21$, and none of shape $3^\top = 111$. Enumerating these and indicating inversions by arrows, we have

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&3&2\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 2&1&3\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 2&3&1\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 3&1&2\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 3&2&1\\\hline\end{array} \quad \begin{array}{|c|c}\cline{1-1} 3 & \\\hline 1&2\\\hline\end{array}\;. \tag{25}$$

Comparing these tableaux with the known evaluations

$$\epsilon_q^{111}(\beta_q(P)) = 1 + 4q + q^2, \qquad \epsilon_q^{21}(\beta_q(P)) = q, \qquad \epsilon_q^{3}(\beta_q(P)) = 0, \tag{26}$$

we see that Conjecture 5.3 holds in this case.

Shareshian and Wachs [13] defined a quantum analog of Stanley's chromatic symmetric function. Given a labeled poset P, their *quantum chromatic quasisymmetric function* for P is the sum

$$X_{P,q} = \sum_{\kappa} q^{\#\{(i,j)\,|\, i \not\leq_P j,\; j \not\leq_P i,\; i<j,\; \kappa(i)>\kappa(j)\}} \prod_{k \in \mathbb{P}} x_k^{\#\{i \in P\,|\,\kappa(i)=k\}} \tag{27}$$

over all colorings $\kappa : P \to \mathbb{N}$ of P, where a function κ is termed a *coloring* if it satisfies $\kappa(i) \neq \kappa(j)$ whenever i and j are incomparable in P. By [13, Prop. 4.4], when P is an n-element *ar*-labeled unit interval order then $X_{P,q}$ is in fact symmetric,

$$X_{P,q} = \sum_{\lambda \vdash n} c_{\lambda,q} m_\lambda, \tag{28}$$

and it follows that $c_{\lambda,q} = \sum_U q^{\mathrm{INV}(U)}$, where the sum is over all column-strict P-tableaux of shape $\lambda^\top$. For example, when the four unit interval orders in (13) are *ar*-labeled, their quantum chromatic symmetric functions are

$$\begin{aligned} X_{\mathbf{3},q} &= 6m_{111} + 3m_{21} + m_3 \\ &= s_{111} + 2s_{21} + s_3 \\ &= e_{111}, \end{aligned} \qquad \begin{aligned} X_{\mathbf{2}+\mathbf{1},q} &= (1+4q+q^2)m_{111} + qm_{21} \\ &= (1+2q+q^2)s_{111} + qs_{21} \\ &= (1+q+q^2)e_3 + qe_{21}, \end{aligned}$$

$$\begin{aligned} X_{(\mathbf{1}+\mathbf{1})\oplus\mathbf{1},q} &= (3+3q)m_{111} + (1+q)m_{21} \\ &= (1+q)s_{111} + (1+q)s_{21} \\ &= (1+q)e_{21}, \end{aligned} \qquad \begin{aligned} X_{\mathbf{1}+\mathbf{1}+\mathbf{1},q} &= (1+2q+2q^2+q^3)m_{111} \\ &= (1+2q+2q^2+q^3)s_{111} \\ &= (1+2q+2q^2+q^3)e_3. \end{aligned}$$

Now Conjecture 5.3 and Equation (15) imply the following.

Conjecture 5.4 *If P is an n-element* ar*-labeled unit interval order then the Shareshian-Wachs quantum chromatic symmetric function satisfies*

$$X_{P,q} = \sum_{\lambda \vdash n} \epsilon_q^\lambda(\beta_q(P)) m_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top}(\beta_q(P)) s_\lambda = \sum_{\lambda \vdash n} \phi_q^\lambda(\beta_q(P)) e_\lambda.$$

Shareshian and Wachs have formulated the following quantum analog [13, Conj. 4.8] of Stanley and Stembridge's [15, Conj. 5.1] ([18, Conj. 5.5]).

Conjecture 5.5 *If P is an* ar*-labeled unit interval order, then the coefficients arising in the elementary expansion of $X_{P,q}$ belong to $\mathbb{N}[q]$.*

By Conjecture 5.4, the authors believe this to be equivalent to the special case of Haiman's [6, Conj. 2.1] which we have stated in Conjecture 4.3. Furthermore, Shareshian and Wachs have proved the following weaker result [13, Thm. 4.11], which is a quantum analog of the special case of Gasharov's [2, Thm. 2] stated in Theorem 3.3.

Theorem 5.6 *If P is an* ar-*labeled unit interval order, then the coefficient of s_λ in the Schur expansion of $X_{P,q}$ is $\sum_U q^{\mathrm{INV}(U)}$, where the sum is over all semistandard P-tableaux of shape $\lambda^\top$.*

This coincides with the authors' conjectured interpretation of the irreducible $H_n(q)$ characters.

Conjecture 5.7 *If P is an* ar-*labeled unit interval order and $w = w(P)$, then $\chi_q^\lambda(\beta_q(P)) = \sum_U q^{\mathrm{INV}(U)}$, where the sum is over all semistandard P-tableaux of shape $\lambda^\top$.*

Since Theorem 3.3 and Conjecture 3.4 apply to $(\mathbf{3}+\mathbf{1})$-free posets, it would be interesting to see if the symmetry of $X_{P,q}$ holds more generally for *ar*-labeled $(\mathbf{3}+\mathbf{1})$-free posets, and to interpret $\epsilon_q^\lambda(\beta_q(P))$, $\eta_q^\lambda(\beta_q(P))$, and $\chi_q^\lambda(\beta_q(P))$ in these cases.

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