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An injection from standard fillings to parking functions

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Abstract. The Hilbert series of the Garsia-Haiman module can be written as a generating function of standard fillings of Ferrers diagrams. It is conjectured by Haglund and Loehr that the Hilbert series of the diagonal harmonics can be written as a generating function of parking functions. In this paper we present a weight-preserving injection from standard fillings to parking functions for certain cases.

Résumé. La série Hilbert du module Garsia-Haiman peut être écrite comme fonction génératrice de tableaux des diagrammes Ferrers. Haglund et Loehr conjecturent que la série Hilbert de l'harmonic diagonale peut être écrite comme fonction génératrice des fonctions parking. Dans cet essai nous présentons une injection des tableaux vers les fonctions parking pour certains cas.

Keywords: Macdonald polynomials, parking functions, diagonal harmonics

1 Introduction

Over the past twenty years, Macdonald polynomials have become a central object of study in the theory of symmetric functions. The classical Macdonald polynomials, $\{P_\mu\}$, where μ ranges over all partitions of n, provide a basis for the vector space of symmetric functions, Λ_n [Mac88, Mac95]. In addition, the P_μ specialize to many of the other common bases of Λ_n such as the elementary, monomial, and Schur polynomials [Mac88, Mac95]. In [Mac88], Macdonald also introduces the dual basis, Q_μ, and the integral basis, J_μ, which can both be computed from P_μ by multiplication by a suitable rational expression that depends on μ.

The modified Macdonald polynomials, $\tilde{H}_\mu(X;q,t)$, are obtained algebraically from the integral basis J_μ by a suitable plethystic transformation [Mac88, Mac95]. The $\tilde{H}_\mu(X;q,t)$ have connections to the representation theory of the symmetric group. In fact, Haiman showed [Hai01] that $\tilde{H}_\mu(X;q,t)$ is the Frobenius series of the Garsia-Haiman module [GH93], $M_\mu \subset \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. We show the construction of M_μ for $\mu = (2,2,1)$. First, we label each cell of the Ferrers diagram of μ with the Cartesian coordinates of that cell, as shown in Fig. 1. We then use the labels as exponents for $x_i y_i$ and

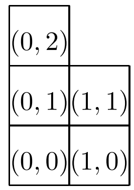


Fig. 1: Labeled Ferrers diagram for $\mu = (2, 2, 1)$

construct the determinant Δ_μ in (1).

$$\Delta_\mu = \begin{vmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 & x_5^0y_5^0 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 & x_5^1y_5^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 & x_5^0y_5^1 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 & x_5^1y_5^1 \\ x_1^0y_1^2 & x_2^0y_2^2 & x_3^0y_3^2 & x_4^0y_4^2 & x_5^0y_5^2 \end{vmatrix} \tag{1}$$

After calculating the determinant, M_μ is obtained by taking linear combinations of all possible partial derivatives of Δ_μ. This module can be written as a doubly graded module based on the degree of the x and y coordinates [GH93]. Thus

$$M_\mu = \bigoplus_{a,b\geq 0} M_\mu^{(a,b)} \tag{2}$$

where each $f \in M_\mu^{(a,b)}$ is homogeneous of degree a in the x_i and homogeneous of degree b in the y_j. Then the *Hilbert series* of M_μ is

$$\mathrm{Hilb}(M_\mu) = \sum_{a,b\geq 0} \dim(M_\mu^{(a,b)}) q^a t^b. \tag{3}$$

Haglund conjectured [Hag04] and Haglund, Haiman, and Loehr proved [HHL05a] a combinatorial definition for the Hilbert series of M_μ as a generating function of certain fillings of the Ferrers diagram of μ, which we describe in §2.

For μ a partition of n, the Garsia-Haiman module, M_μ, is a submodule of the module of diagonal harmonics, DH_n [Hag08, Hai94]. The *diagonal harmonics* of order n are defined as

$$DH_n = \left\{ f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] : \sum_{i=1}^{n} \frac{\partial^h}{\partial x_1^h} \frac{\partial^k}{\partial y_i^k} f = 0 \text{ for } 1 \leq h + k \leq n \right\}. \tag{4}$$

We can also write DH_n as a doubly graded module by setting $DH_n^{(a,b)}$ to be the set of $f \in DH_n$ that are homogeneous of degree a in the x_i and homogeneous of degree b in the y_j. Then

$$DH_n = \bigoplus_{a,b\geq 0} DH_n^{(a,b)}. \tag{5}$$

The Hilbert series of DH_n is

$$\mathrm{Hilb}(DH_n) = \sum_{a,b\geq 0} \dim(DH_n^{(a,b)}) q^a t^b. \tag{6}$$

Haglund and Loehr have conjectured [HL05] a combinatorial formula for (6) as a weighted sum over parking functions on n cars, which is described in §3. Assuming Haglund and Loehr's conjecture holds, then, since M_μ is a submodule of DH_n, there exists a weight-preserving injection from standard fillings to parking functions. We want to find an explicit combinatorial definifition of the injection to provide more evidence for the conjecture in [HL05]. Such a definition would also help us to better understand the relation between the algebraic structures and the combinatorial structures. In this paper we first review the necessary combinatorial definitions (§2,3) and then establish such an injection for certain cases (§4).

2 Haglund's combinatorial definition of $\mathrm{Hilb}(M_\mu)$

In this section we define the Hilbert series of the Garsia-Haiman module, M_μ, using Haglund's combinatorial definition [Hag04, HHL05a]. Recall that a *partition* μ of the positive integer n is a sequence $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$ with $\mu_1 + \mu_2 + \cdots + \mu_k = n$. The *diagram* of μ is

$$\mathrm{dg}(\mu) = \{(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ : 1 \leq i \leq k, 1 \leq j \leq \mu_i\}.$$

We visualize $\mathrm{dg}(\mu)$ by using a *Ferrers diagram*, a collection of left justified boxes (following the French convention) such that the bottom row has μ_1 boxes, the next has μ_2, etc. A *standard filling* of μ is a bijection $T : \mathrm{dg}(\mu) \to \{1, \ldots, n\}$ which we visualize by placing $T(c)$ in cell c of the Ferrers diagram of μ as in Fig. 2. Let $\mathcal{F}_\mu = \{\text{standard fillings of shape } \mu\}$. Given $T \in \mathcal{F}_\mu$, construct μ_1 column words

$$T = \begin{array}{|c|c|c|} \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & 1 & \multicolumn{1}{c}{} \\ \hline 8 & 2 & 7 \\ \hline 5 & 9 & 6 \\ \hline \end{array}$$

Fig. 2: T is a standard filling of $\mu = (3,3,2,1)$

by reading down each column individually. Thus, for T in Fig. 2, the column words are 3485, 129, and 76. In a word $w = w_1 w_2 \cdots w_k$, a *descent* occurs when $w_i > w_{i+1}$. The *descent set* of w is defined $\mathrm{Des}(w) = \{i : w_i > w_{i+1}\}$. The *major index* of w is $\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i$. We now can define the μ-major index of T as

$$\mathrm{maj}_\mu(T) = \sum_{\substack{w \text{ a column} \\ \text{word of } T}} \mathrm{maj}(w).$$

Thus, for the filling in Fig. 2,

$$\begin{aligned} \mathrm{maj}_\mu(T) &= \sum_{\substack{w \text{ a column} \\ \text{word of } T}} \mathrm{maj}(w) \\ &= \mathrm{maj}(3485) + \mathrm{maj}(129) + \mathrm{maj}(76) \\ &= 3 + 0 + 1 \\ &= 4. \end{aligned}$$

A cell (i,j) in the diagram of μ *attacks* cells in the set $\{(a,b) : a = i \text{ and } b > j\} \cup \{(a,b) : a = i-1 \text{ and } b < j\}$. That is, (i,j) attacks cells in the same row and to the right, or in the row below and to

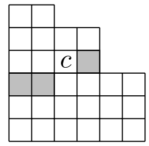


Fig. 3: Cells attacked by c

the left, as seen in Fig. 3. An *attack inversion* of T occurs when cell c attacks cell d and $T(c) > T(d)$. We define

$$\mathrm{Inv}(T) = \{(T(c), T(d)) : c \text{ attacks } d \text{ and } T(c) > T(d)\}.$$

Given a cell c, the *arm* of c, $\mathrm{ARM}(c)$ is the set of cells in the diagram of μ in the same row and to the right of c. We define the μ-inversions of T to be

$$\mathrm{inv}_\mu(T) = |\,\mathrm{Inv}(T)| - \sum_{c \in \mathrm{Des}(T)} |\,\mathrm{ARM}(c)|.$$

For T in Fig. 2, $\mathrm{Inv}(T) = \{(9,6),(8,2),(8,7),(7,5),(4,1)\}$ and $\sum_{c\in \mathrm{Des}(T)} |\,\mathrm{ARM}(c)| = 2$. Thus $\mathrm{inv}_\mu(T) = |\,\mathrm{Inv}(T)| - |\,\mathrm{ARM}(\mathrm{Des}(T))| = 5 - 2 = 3$.

Definition 1 ([HHL05a, HHL05b]) *The Hilbert series for the Garsia-Haiman module, M_μ, is*

$$\mathrm{Hilb}(M_\mu) = \tilde{F}_\mu(q,t) = \sum_{T \in \mathcal{F}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{maj}_\mu(T)}.$$

3 Parking functions and $\mathrm{Hilb}(DH_n)$

A combinatorial formula for $\mathrm{Hilb}(DH_n)$, as a weighted sum of *parking functions*, is conjectured in [HL05]. A function $f : \{1,\dots,n\} \to \{1,\dots,n\}$ is a parking function of order n if and only if

$$|\{x : f(x) \le i\}| \ge i \text{ for } 1 \le i \le n.$$

Let $\mathcal{P}_n$ denote the set of all parking functions of order n. It is known that $|\mathcal{P}_n| = (n+1)^{n-1}$ [Sta99].

A *Dyck path* of order n is a path from $(0,0)$ to (n,n) consisting of n north and n east steps such that no step goes strictly below the diagonal $x = y$. A *labeled Dyck path* is a Dyck path with vertical steps labeled $1,\dots,n$ such that labels of consecutive vertical steps increase from bottom to top, as seen in Fig. 4. There is a bijection between parking functions and labeled Dyck paths [Loe05, Loe11]. Let f be a parking function of order n. Place all x such that $f(x) = y$ in column y. Thus the parking function defined by $f(1) = 2, f(2) = 1, f(3) = 4, f(4) = 1, f(5) = 4, f(6) = 4$ maps to the labeled Dyck path in Fig. 4. Similarly, when given a labeled Dyck path P, for all entries x in row y, set $f(x) = y$. The resulting function is a parking function.

The *area* of a labeled Dyck path P is the number of complete lattice squares between the path and the line $x = y$. We can construct the *area vector*, $g(P)$, by setting g_i to be the number of complete squares between the path and $x = y$ in row i, where row 1 is the bottom row of the path. Thus, for the path in Fig. 4, $g(P) = (0,1,1,0,1,2)$. Then

$$\mathrm{area}(P) = \sum_{i=1}^{n} g_i \tag{7}$$

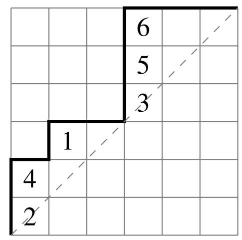


Fig. 4: A labeled Dyck path

and $\mathrm{area}(P) = 0 + 1 + 1 + 0 + 1 + 2 = 5$ for the path in Fig. 4.

We can also construct a *content vector*, $p(P)$, by setting p_i to be the entry in row i of P. For the path in Fig. 4, $p(P) = (2, 4, 1, 3, 5, 6)$. A Dyck path can be constructed from a pair of vectors (g, p) if and only if [Loe05]

1. g and p have length n.
2. $g_1 = 0$.
3. $g_i \geq 0$ for $1 \leq i \leq n$.
4. $g_{i+1} \leq g_i + 1$ for $1 \leq i \leq n - 1$.
5. p is a permutation of $\{1, \ldots, n\}$.
6. $g_{i+1} = g_i + 1$ implies $p_i < p_{i+1}$.

For a parking function $f \in \mathcal{P}_n$ corresponding to the labeled Dyck path P, set $\mathrm{area}(f) = \mathrm{area}(P)$, $g(f) = g(P)$, and $p(f) = p(P)$. We define a second weight on f by

$$\begin{aligned}\mathrm{dinv}(f) = \sum_{i<j} & \left[\chi\left(g_i(f) = g_j(f) \text{ and } p_i(f) < p_j(f)\right)\right. \\ & \left.+\chi\left(g_i(f) = g_j(f) + 1 \text{ and } p_i(f) > p_j(f)\right)\right] \end{aligned} \tag{8}$$

where, for a logical statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false.

Conjecture 2 ([HL05]) *The Hilbert series of DH_n is given by the polynomial*

$$CH_n(q, t) = \sum_{f \in \mathcal{P}_n} q^{\mathrm{dinv}(f)} t^{\mathrm{area}(f)}. \tag{9}$$

4 Weight-preserving injection

For two classes of fillings, namely those with a μ-major index of zero and those with shape $\mu = (1^{n-k}k)$, we will define an injection from standard fillings to parking functions. Given a filling $T \in \mathcal{F}_\mu$, the *reading word* of T, denoted $\mathrm{RW}(T)$, is obtained by reading the entries of T from left to right, top to bottom. To find the reading word of a labeled Dyck path, denoted $\mathrm{RW}(P)$, read down the diagonals from right to left, top to bottom. The *reverse* of a word $w = w_1 \dots w_k$ is $w^R = w_k w_{k-1} \dots w_1$. The *inverse descent set* of a filling (resp. labeled Dyck path), denoted $\mathrm{IDes}(T)$ (resp. $\mathrm{IDes}(P)$), is the descent set of the inverse of $\mathrm{RW}(T)$ (resp. $\mathrm{RW}(P)$) as a permutation. Thus $\mathrm{IDes}(T) = \{i : i+1 \text{ appears earlier than } i \text{ in } \mathrm{RW}(T)\}$. Ultimately, we want to define an injection $f : \mathcal{F}_\mu \to \mathcal{P}_n$ with the following properties:

1. $\mathrm{inv}_\mu(T) = \mathrm{dinv}(f(T))$,
2. $\mathrm{maj}_\mu(T) = \mathrm{area}(f(T))$, and
3. $\mathrm{IDes}(T) = \mathrm{IDes}(f(T))$.

4.1 μ-major index of zero

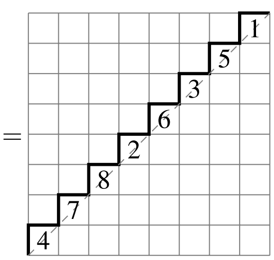
We consider fillings T of shape μ with $\mathrm{maj}_\mu(T) = 0$. Note that if $\mathrm{maj}_\mu(T) = 0$ then each column must increase from top to bottom and $\mathrm{Des}(T) = \emptyset$. Then $\mathrm{inv}_\mu(T) = |\operatorname{Inv}(T)|$, the number of attack inversions in T, since $\sum_{c \in \mathrm{Des}(T)} |\operatorname{ARM}(c)| = 0$. These fillings must be sent to a parking function $P = (g, p)$ with $\mathrm{area}(P) = 0$. Setting $p = \mathrm{RW}(T)^R$ can introduce many new inversions that will be counted by $\mathrm{dinv}(P)$ as seen in Fig. 5, so the injection must rearrange $\mathrm{RW}(T)$ to compensate for these new inversions.

$$T = \begin{array}{|c|c|c|} \cline{1-2} 1 & 5 & \multicolumn{1}{c}{} \\ \hline 3 & 6 & 2 \\ \hline 8 & 7 & 4 \\ \hline \end{array}$$

$\mathrm{inv}_\mu(T) = 6$

$\mathrm{RW}(T) = 15362874$ and

$\mathrm{RW}(T)^R = 47826351$

$P =$ 

$\mathrm{dinv}(P) = 8$

Fig. 5: A map taking T to P that does **not** preserve inversions

In order to address the additional inversions present in the reading word of T we define a new type of inversion that may be present in T but is not counted by $\mathrm{inv}_\mu(T)$. We say that a cell (i, j) *defends* cells in the set $\{(a, b) : a = i - 1 \text{ and } b > j\} \cup \{(a, b) : a = i - 2 \text{ and } b < j\}$ as seen in Fig. 6. A *defense inversion* occurs when cell c defends cell d and $T(c) > T(d)$. We define $\mathrm{DefInv}(T) = \{(T(c), T(d)) : c \text{ defends } d \text{ and } T(c) > T(d)\}$. We are now ready to define a map $f : \mathcal{F}_\mu|_{\mathrm{maj}_\mu = 0} \to \mathcal{P}_n|_{\mathrm{area}=0}$ with the form $f(T) = (g(T), p(T))$.

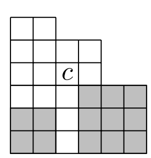


Fig. 6: Cells defended by c

i	$w^{(i)}$	
1	15362874	1 is not defended by any w_j
2	15362874	5 is not defended by any w_j
3	15362874	3 is not defended by any w_j
4	15362874	6 is defended by 1, but $1 < 6$
5	153**2**6874	2 is defended by 5 and $5 > 2$
6	15326874	8 is defended by 5, but $5 < 8$
7	15326874	7 is defended by 1 and 3, but $1 < 7$ and $3 < 7$
8	15326**4**87	4 is defended by 5 and 6 and $6 > 4$, $5 > 4$

Tab. 1: Calculating $w^{(i)}$

Definition 3 *Let $T \in \mathcal{F}_\mu|_{\mathrm{maj}_\mu=0}$ and set $w = \mathrm{RW}(T)$. For each i, $1 < i \leq n$, let c_i be the number of j, $1 \leq j < i$ such that w_j and w_i form a defense inversion. Define $w^{(i)}$ to be $w^{(i-1)}$ with w_i shifted left c_i places. Define $f : \mathcal{F}_\mu|_{\mathrm{maj}_\mu=0} \to \mathcal{P}_n|_{\mathrm{area}=0}$ by $f(T) = (g(T), p(T))$ where $g(T) = (0, 0, \ldots, 0)$ and $p(T) = (w^{(n)})^R$.*

For T in Fig. 5 we compute $f(T)$ using Def. 3. We start by calculating $w^{(n)}$ in Table 1. Then $p(T) = (7, 8, 4, 6, 2, 3, 5, 1)$ and thus $f(T) = ((0, 0, 0, 0, 0, 0, 0, 0), (7, 8, 4, 6, 2, 3, 5, 1))$. By construction, $\mathrm{area}(f(T)) = 0$. One can compute $\mathrm{dinv}(f(T)) = 6 = \mathrm{inv}_\mu(T)$ and $\mathrm{IDes}(T) = \{2, 4, 7\} = \mathrm{IDes}(f(T))$.

The construction of $w^{(n)}$ from the reading word of T is designed to undo the additional inversions $\mathrm{RW}(T)$ has due to defense inversions in T.

Lemma 4 *Given $T \in \mathcal{F}_\mu|_{\mathrm{maj}_\mu=0}$,*

$$\mathrm{inv}(w^{(n)}) = \mathrm{inv}(\mathrm{RW}(T)) - \sum_{i=1}^{n} c_i. \tag{10}$$

Proof: We will show that two consecutive entries in $\mathrm{RW}(T)$ with $w_i < w_j$ where $i < j$ cannot appear as $\ldots w_j \ldots w_i \ldots$ in $w^{(n)}$. For each i, let

$$U_i = \{w_k \in \mathrm{RW}(T) : w_k \text{ defends } w_i \text{ and } w_k > w_i\}.$$

Then $c_i = |U_i|$ and it is sufficient to show that $c_i + j - i > c_j$, that is, that w_i will move farther to the left than w_j.

Suppose w_i and w_j are in the same row. The possible entries in U_i and U_j are shaded in Fig. 7. If a cell in a dark gray region forms an defense inversion with w_j, then it also forms a defense inversion with w_i. Cells in the light gray shaded region can only be in U_i. Cells in the black region can only be in U_j. Note that there are $j - i - 1$ cells in the black region. Thus $|U_i| \geq |U_j| - (j - i - 1)$.

Now suppose w_i is in a row higher than w_j. In Fig. 8 the possible elements of U_i and U_j are shaded using the same scheme as before. If a cell in a dark gray region forms a defense inversion with w_j, then it must also form a defense inversion with w_i since $w_j > w_i$. The cells in the light gray region can only be in U_i, while cells shaded black can only be in U_j. There are at most $j - i - 1$ cells in the black region. Thus $|U_i| \geq |U_j| - (j - i - 1)$.

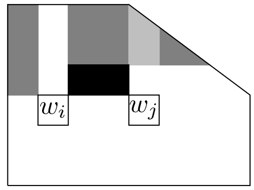


Fig. 7: w_i and w_j in same row

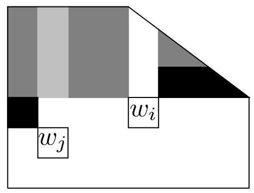


Fig. 8: A possible configuration of w_i, w_j

Therefore $|U_i| \geq |U_j| - (j-i-1)$, so $c_i + j - i > c_j$. Since w_i cannot be moved past a smaller entry of $w^{(i-1)}$, w_i must move past c_i entries of $w^{(i-1)}$ that are greater than w_i and hence

$$\operatorname{inv}(w^{(i)}) = \operatorname{inv}(w^{(i-1)}) - c_i.$$

Note that $\operatorname{inv}(w^{(1)}) = \operatorname{inv}(\mathrm{RW}(T))$ and thus, by induction,

$$\operatorname{inv}(w^{(n)}) = \operatorname{inv}(\mathrm{RW}(T)) - \sum_{i=1}^{n} c_i.$$

□

We now show that f preserves weights and inverse descent sets between standard fillings and parking functions.

Theorem 5 *The map* $f : \mathcal{F}_\mu|_{\mathrm{maj}_\mu=0} \to \mathcal{P}_n|_{\mathrm{area}=0}$ *has the following properties:*

(a) $\mathrm{maj}_\mu(T) = \mathrm{area}(f(T))$*,*

(b) $\mathrm{inv}_\mu(T) = \mathrm{dinv}(f(T))$*, and*

(c) $\mathrm{IDes}(\mathrm{RW}(T)) = \mathrm{IDes}(\mathrm{RW}(f(T)))$.

Proof: Let $T \in \mathcal{F}_\mu|_{\mathrm{maj}_\mu=0}$. By definition of f, $\mathrm{maj}_\mu(T) = \mathrm{area}(f(T))$. To show that $\mathrm{inv}_\mu(T) = \mathrm{dinv}(f(T))$ we first note that $\mathrm{inv}(\mathrm{RW}(T)) = \mathrm{inv}_\mu(T) + |\operatorname{DefInv}(T)|$. Since $\mathrm{area}(f(T)) = 0$, $\mathrm{dinv}(f(T)) = \mathrm{inv}(p(T)^R) = \mathrm{inv}(w^{(n)})$. Thus, it is sufficient to show that $\mathrm{inv}_\mu(T) = \mathrm{inv}(w^{(n)})$ since $w^{(n)} = p(T)^R$.

By Lemma 4,

$$\begin{aligned}\operatorname{inv}(w^{(n)}) &= \operatorname{inv}(\mathrm{RW}(T)) - \sum c_i \\ &= \operatorname{inv}(\mathrm{RW}(T)) - |\operatorname{DefInv}(T)| \\ &= \operatorname{inv}_\mu(T).\end{aligned}$$

Finally, to show that $\mathrm{IDes}(\mathrm{RW}(T)) = \mathrm{IDes}(\mathrm{RW}(f(T))$ it suffices to show that i and $i+1$ can never switch places during the construction of $p(T)$ since when $\operatorname{area}(f(T)) = 0$, $\mathrm{RW}(f(T)) = p(T)^R$. From the proof of Lemma 4 we know that if i and $i+1$ are in positions w_k and w_{k+j} respectively, they will not exchange positions since $i < i+1$.

Suppose $i+1 = w_k$ and $i = w_{k+j}$ for some $k, j > 0$. Suppose $c_k = m$. Then $c_{k+j} \leq m + j$ since any entry that forms a defense inversion with w_k must also form a defense inversion with w_{k+j} and there are at most j additional entries of T that could form a defense inversion with w_{k+j}. □

It remains to show that f is indeed an injection.

Theorem 6 *The function* $f : \mathcal{F}_\mu|_{\mathrm{maj}_\mu = 0} \to \mathcal{P}_n|_{\mathrm{area}=0}$ *is an injection.*

Proof:

Suppose, to the contrary, that there exist $T_1, T_2 \in \mathcal{F}_\mu|_{\mathrm{maj}_\mu=0}$ such that $T_1 \neq T_2$ but $f(T_1) = f(T_2)$. Set $w = \mathrm{RW}(T_1)$ and $v = \mathrm{RW}(T_2)$. Since $f(T_1) = f(T_2)$, $w^{(n)} = v^{(n)} = u$. Let i be the smallest index such that $w_i \neq v_i$. We will assume that $w_i < v_i$. As in the definition of f, let $c_k = |\operatorname{DefInv}(w_k)|$ for all k. Since w and v are both permutations of $\{1, \ldots, n\}$, there exists some $j > i$ such that $v_j = w_i$. We will show that v_j cannot end up in the same location as w_i in u. In order for v_j to be in the same location in u as w_i, we need $|\operatorname{DefInv}(v_j)| = c_i + j - i$.

First note that if v_j is in a lower row and to the right of v_i or more than one row below v_i and v_j has some entry $v_k < v_j$ directly above it, then by the proof of Lemma 4, v_j cannot be moved past v_k and hence cannot ever achieve the same position as w_i in u. This leaves three potential configurations for v_i and v_j in T_2.

1. We look first at what happens if v_j is just one row below v_i and to the left of v_i. This can be seen in Fig. 9. The dark gray in the picture indicates the entries that could be in $\operatorname{DefInv}(v_i)$, while the dark and light gray indicate the entries that could be in $\operatorname{DefInv}(v_j)$. Note that $\operatorname{DefInv}(v_i) \subseteq \operatorname{DefInv}(v_j)$ since $v_j < v_i$. The farthest left that v_j can be moved during the construction of $v^{(n)}$ is $|\operatorname{DefInv}(v_j)| \leq |\operatorname{DefInv}(w_i)| + j - i - 1$, but this is not far enough to move v_j to the same location as w_i in u.

2. If v_i and v_j are in the same row (with v_j to the right of v_i), there are at most $j - i - 1$ possible entries of T_2 in $\operatorname{DefInv}(v_j)$ that are not also in $\operatorname{DefInv}(w_i)$. Thus, there is no way for v_j to move far enough left in $v^{(n)}$ to end in the same position as w_i in u.

3. Similarly, if v_j is in a lower row and to the right of v_i, with no entries in the column above v_j, there are still at most $j - i - 1$ possible entries of T_2 in $\operatorname{DefInv}(v_j)$ that are not also in $\operatorname{DefInv}(w_i)$.

Therefore, $f : \mathcal{F}_\mu|_{\mathrm{maj}_\mu=0} \to \mathcal{P}_n|_{\mathrm{area}=0}$ is an injection. □

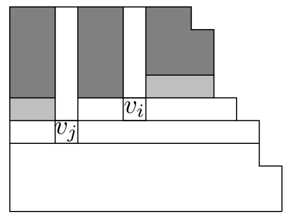


Fig. 9: One possible configuration of v_i and v_j

4.2 Injection when μ is a hook shape

In this section we describe an injection from $\mathcal{F}_\mu$ to $\mathcal{P}_n$ when $\mu = (1^{n-k}k)$, that is, the diagram of μ has a hook shape. Let $T \in \mathcal{F}_{(1^{n-k}k)}$. Create a new filling T_g of shape $(1^{n-k}k)$ by filling all cells $(i, 1)$ in the bottom row with 0. Compute T_g recursively by letting $T_g(i, j) = T_g(i-1, j)$ if $T(i, j) < T(i-1, j)$ or $T_g(i, j) = T_g(i-1, j) + 1$ if $T(i, j) > T(i-1, j)$. Note that T_g can be calculated in this manner for T of any partition shape and is not restricted to hook shapes.

$$T = \begin{array}{|c|ccc} \hline 7 \\ \cline{1-1} 8 \\ \cline{1-1} 1 \\ \cline{1-1} 5 \\ \cline{1-1} 4 \\ \hline 6 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad T_g = \begin{array}{|c|ccc} \hline 2 \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1} 1 \\ \cline{1-1} 0 \\ \hline 0 & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} \\ \hline \end{array}$$

Fig. 10: Calculating T_g from a given T

Lemma 7 *Given $T \in \mathcal{F}_\mu$,*

$$\mathrm{maj}_\mu(T) = \sum_{c \in T_g} c. \tag{11}$$

We are now ready to define the function $f : \mathcal{F}_{(1^{n-k}k)} \to \mathcal{P}_n$.

Definition 8 *Define $f : \mathcal{F}_{(1^{n-k}k)} \to \mathcal{P}_n$ by $f(T) = (g(T), p(T))$. To find $g(T)$, compute T_g (see Fig. 10) and then set $g(T) = \mathrm{RW}^R(T_g)$. To find $p(T)$, first note that $\mathrm{RW}(T) = wv$ where $w = w_1 \ldots w_m$ is the subword containing those entries of T whose corresponding entries in T_g area nonzero and $v = v_1 \ldots v_{n-m}$ is the subword containing those entries of T whose corresponding entries in T_g are zero. For $1 < i \leq n - m$ let c_i be the number of v_j forming defense inversions with v_i where $i > j$. Let $v^{(i)}$ be defined recursively as $v^{(i-1)}$ with v_i shifted left c_i places. Then $p(T) = (v^{(n-m)})^R w^R$.*

Note that the construction of $f(T)$ means that the only possible contributors to $\mathrm{dinv}(f(T))$ are those that are on the same diagonal.

For $\mu = (1^5 4)$ and T as in Fig. 10, we find that $g(T) = (0, 0, 0, 0, 0, 1, 1, 2, 2)$ and $\mathrm{RW}(T) = 781546293$. Then $w = 7815$ and $v = 46293$. See Table 2 to see how $v^{(5)} = 42639$ is calculated. Then $p(T) = (9, 3, 6, 2, 4, 5, 1, 8, 7)$. See Fig. 11 for the associated labeled Dyck path. Note that $\mathrm{maj}_\mu(T) = 6 = \mathrm{area}(f(T))$, $\mathrm{inv}_\mu(T) = 3 = \mathrm{dinv}(f(T))$, and $\mathrm{IDes}(T) = \{4, 6, 3\} = \mathrm{IDes}(f(T))$.

i	$v^{(i)}$	
1	46293	4 is not defended by any v_j
2	46293	6 is not defended by any v_j
3	42693	2 is defended by 4 and $4 > 2$
4	42693	9 is defended by 4, but $9 > 4$
5	42639	3 is defended by 4 and $4 > 3$

Tab. 2: Computing $v^{(i)}$

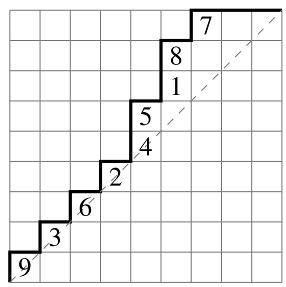


Fig. 11: Labeled Dyck path with $g = (0, 0, 0, 0, 0, 1, 1, 2, 2)$ and $p = (9, 3, 6, 2, 4, 5, 1, 8, 7)$

Proposition 9 *Let $\mu = (1^{n-k}k)$. Then $f : \mathcal{F}_{(1^{n-k}k)} \to \mathcal{P}_n$ is an injection and*

(a) $\mathrm{maj}_\mu(T) = \mathrm{area}(f(T))$,

(b) $\mathrm{inv}_\mu(T) = \mathrm{dinv}(f(T))$, *and*

(c) $\mathrm{IDes}(T) = \mathrm{IDes}(f(T))$.

Proof: Let $T \in \mathcal{F}_\mu$ for $\mu = (1^{n-k}k)$. From Lemma 7 and (7) we know that $\mathrm{maj}_\mu(T) = \mathrm{area}(f(T))$. Properties (b), (c), and injectivity follow quickly from the proof of Theorem 5 and the fact that the order of the entries in T that correspond to nonzero entries in T_g do not get reordered and do not contribute either to $\mathrm{inv}_\mu(T)$ or $\mathrm{dinv}(f(T))$. $\square$

5 Conclusion

In this paper we have shown a new injection from the standard fillings of Ferrers diagrams that generate the Hilbert series of the Garsia-Haiman module to parking functions, which are conjectured to generate the Hilbert series of the module of diagonal harmonics, for two special cases. It is desirable to have an injection that works for any partition μ. Maps similar to those in this paper appear to give the desired result in fillings with $\sum_{c \in \mathrm{Des}(T)} |\,\mathrm{ARM}(c)| = 0$, but difficulties arise when $\sum_{c \in \mathrm{Des}(T)} |\,\mathrm{ARM}(c)| \neq 0$.

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