FPSAC 2012, Nagoya, Japan DMTCS proc. **AR**, 2012, 735–744

A combinatorial formula for fusion coefficients

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Abstract. Using the expansion of the inverse of the Kostka matrix in terms of tabloids as presented by Eğecioğlu and Remmel, we show that the fusion coefficients can be expressed as an alternating sum over cylindric tableaux. Cylindric tableaux are skew tableaux with a certain cyclic symmetry. When the skew shape of the tableau has a cutting point, meaning that the cylindric skew shape is not connected, or if its weight has at most two parts, we give a positive combinatorial formula for the fusion coefficients. The proof uses a slight modification of a sign-reversing involution introduced by Remmel and Shimozono. We discuss how this approach may work in general.

Résumé. En utilisant l'expansion de l'inverse de la matrice Kostka en termes de tabloïdes introduite par Eğecioğlu et Remmel, nous montrons que les coefficients de fusion peuvent être exprimés comme une somme alternée sur les tableaux cylindriques. Les tableaux cylindriques sont des tableaux qui présentent une certaine symétrie cyclique. Lorsque la forme du tableau a un point de coupure, ce qui signifie que la forme cylindrique n'est pas connecté, ou lorsque son poids a au plus deux parts, nous donnons une formule combinatoire positive des coefficients de fusion. La démonstration utilise une légère modification de l'involution qui change le signe introduite par Remmel et Shimozono. Nous discutons comment cette approche pourrait fonctionner en général.

Keywords: fusion coefficients, Gromov–Witten invariants, Littlewood–Richardson coefficients, (inverse) Kostka matrix, crystal graphs, cylindric tableaux, sign-reversing involution

1 Introduction

The famous *Littlewood–Richardson* rule [16] provides a combinatorial expression for the coefficients $c^{\nu}_{\lambda\mu}$ in the expansion of a product of Schur functions

$$s_\lambda s_\mu = \sum_{\nu} c^{\nu}_{\lambda\mu} s_\nu. \tag{1}$$

It states that $c^{\nu}_{\lambda\mu}$ is equal to the number of column-strict tableaux of skew shape ν/λ and content μ whose column reading word is lattice. Here λ, μ, ν are partitions and a column-strict tableau of shape ν/λ is a filling of the skew shape which is weakly increasing across rows and strictly increasing across columns. The content of a tableau or word is $\mu = (\mu_1, \mu_2, \ldots)$, where μ_i counts the number of i in the tableau or word. Furthermore, a word is lattice if all right subwords have partition content.

†Partially supported by the NSF grants DMS–0652641, DMS–0638625, DMS–1001898.
‡Partially supported by the NSF grants DMS–0652641, DMS–0652652, DMS–1001256.

In this paper, we consider the analogous problem for *fusion coefficients*, which first appeared in the literature as the structure constants of the Verlinde fusion algebra for the $\widehat{\mathfrak{sl}}_n$ Wess–Zumino–Witten models of level ℓ [25, 27]. Kac and Walton [10, 28, 29] provided an efficient algorithm for computing fusion coefficients for any type. In type A_{n-1} of level ℓ, their formula is expressed as an alternating sum of the Littlewood–Richardson coefficients

$$c_{\lambda\mu}^{\nu,\ell,n} = \sum_{\sigma \in \widehat{S}_n} \mathrm{sign}(\sigma)\ c_{\lambda\mu}^{\sigma(\nu+\rho)-\rho}\ , \tag{2}$$

where the sum is over the affine symmetric group $\widehat{S}_n$ generated by $\langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \rangle$, $\rho = (n-1, n-2, \ldots, 1, 0)$, the symmetric group acts on compositions by permuting their entries, and $\sigma_0(\lambda) = (\lambda_n + \ell + n, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1 - \ell - n)$.

A notorious problem has been to find a direct positive or combinatorial formula for the fusion coefficients as opposed to an alternating expression as in (2). Many attempts have been made along these lines. Tudose [24] in her thesis gave a combinatorial interpretation when λ or μ has at most two columns. For $n = 2, 3$, positive formulas are known [3] as well as when λ and μ are rectangles [23]. Knutson formulated a conjecture for the quantum Littlewood–Richardson coefficients as presented in [4] in terms of puzzles [14]. It is known that the quantum cohomology structure coefficients are related to fusion coefficients [1, 2]. Coskun [5] gave a positive geometric rule to compute the structure constants of the cohomology ring of two-step flag varieties in terms of Mondrian tableaux.

There are many other interpretations and appearances of fusion coefficients. For example, Goodman and Wenzl [8] showed that the fusion coefficients are related to the structure coefficients of Hecke algebras at roots of unity. This was used in [15] to show that they are special cases of the structure coefficients of the k-Schur functions. As mentioned earlier, they are also related to the quantum cohomology structure coefficients [1, 2] and intertwiners in vertex operator algebras [30]. Postnikov [19] formulated the quantum cohomology ring in terms of the affine nilTemperley–Lieb algebra, and Korff and Stroppel [11, 13] provided an analogous construction of the fusion ring in terms of the affine local plactic algebra.

The main result of this paper is a simple proof of a combinatorial formula for the fusion coefficients for a general class of partitions, which includes all previously known cases (that is, the two-column case of Tudose [24], $n = 2, 3$ of [3], and cases considered by Postnikov [19]). The proof uses the fusion Pieri rule to obtain an expression of the fusion coefficients in terms of cylindric tableaux (see Sections 2 and 3). We then amend a sign-reversing involution of Remmel and Shimozono [20] to cancel all negative terms (see Section 4). We finish with a discussion of how this method could lead to a formula for fusion coefficients in general.

To state the main result, several definitions are needed. A partition λ is of rank n if it has at most n parts. It is of level ℓ if $\lambda_1 - \lambda_n \leq \ell$. Let $\mathcal{P}^{\ell,n}$ be the set of level ℓ and rank n partitions. In addition, let $\mathcal{R}^{\ell,n}$ be the set of partitions of rank n with first part not exceeding ℓ, that is, the set of partitions contained in a rectangle of width ℓ and height n.

Theorem 1.1 *Let ℓ, n be two positive integers, $\lambda, \mu \in \mathcal{R}^{\ell,n}$ and $\nu \in \mathcal{P}^{\ell,n}$ such that $|\lambda| + |\mu| = |\nu|$.*

1. *If $\nu/\lambda + \mathrm{shift}_{-\ell,n}(\nu/\lambda)$ is not connected (see Section 3 for the definition of* shift*), then the level ℓ fusion coefficients $c_{\lambda\mu}^{\nu,\ell,n}$ for type A_{n-1} are given in terms of usual Littlewood–Richardson coefficients for transformed partitions*
$$c_{\lambda\mu}^{\nu,\ell,n} = c_{\tilde{\lambda}\mu}^{\tilde{\nu}}, \tag{3}$$

with $\tilde{\lambda}$ and $\tilde{\nu}$ as in Definition 4.2.

2. *If μ has at most two parts, then* (3) *holds with $\tilde{\lambda}$ and $\tilde{\nu}$ as in Definition 4.4.*

The fusion coefficients enjoy many symmetries. For example:

- **Columns of height n:** If λ contains a column of height n, then $c_{\lambda\mu}^{\nu,\ell,n} = c_{\tilde{\lambda}\mu}^{\tilde{\nu},\ell,n}$ where $\tilde{\lambda}$ (resp. $\tilde{\nu}$) is obtained from λ (resp. ν) by removing or adding a column of height n.
- **Level-rank duality:** Denoting by λ^t the transpose partition of λ, we have $c_{\lambda\mu}^{\nu,\ell,n} = c_{\lambda^t\mu^t}^{\nu^t,n,\ell}$. Here ν^t should be identified with its cyclic analogue of attaching all parts ν_i^t for $i > \ell$ to $\nu_{i-\ell}^t$.
- **Strange duality:** For $\lambda \in \mathcal{R}^{\ell,n}$ denote by $\lambda^\vee$ the complement of λ in the rectangle of size $\ell \times n$. Then the fusion coefficient labeled by the complement partitions is related to $c_{\lambda\mu}^{\nu,\ell,n}$. This is best described using toric shapes, see [19].
- S_3 **symmetry:** The fusion coefficients $c_{\lambda\mu}^{\nu,\ell,n}$ are symmetric with respect to any permutation of the partitions λ, μ, ν (up to certain transformations on ν to put all partitions on the same footing). For more details see [19].

If under any of the above symmetries, one of the cases of Theorem 1.1 holds, a combinatorial formula for the corresponding fusion coefficient follows. In particular, Theorem 1.1 (2) under the level-rank duality is equivalent to the case when μ has at most two columns. This corresponds to the case studied by Tudose [24]. We would like to point out that the proof given here (see Section 4.2) is much simpler than the proof in [24] which involves many case checks.

Acknowledgments

We thank Jason Bandlow for collaboration at the beginning of this project. AS would also like to thank Catharina Stroppel for discussions during the Special Trimester "On the Interaction of Representation Theory with Geometry and Combinatorics" at the Hausdorff Institut in Bonn, January - April 2011. Thanks also to Christian Korff, Chris Manon, Alexander Postnikov, and Peter Tingley for helpful discussions, and Anders Buch and Nicolas Thiéry for their help in getting `lrcalc` integrated into `Sage` [21, 22], which helped with computer experiments.

2 Tabloids

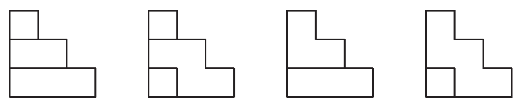
To derive a formula for the fusion coefficients, we will use the well-known relation between Schur functions s_μ and the homogeneous symmetric functions h_α (see for example [17, §I.6, Table 1])

$$s_\mu = \sum_\alpha K_{\alpha\mu}^{-1}\, h_\alpha, \tag{4}$$

where $K_{\alpha\mu}^{-1}$ is the inverse of the Kostka matrix. Eğecioğlu and Remmel [6] gave an interpretation for the entries in the inverse Kostka matrix using a combinatorial structure called tabloids. Note that these tabloids are different from the ones used in the representation theory of the symmetric group.

The definition of a tabloid is a filling of a partition μ with certain shapes called ribbons. A *ribbon* is a connected skew shape which does not contain any 2×2 squares. The *height* of a ribbon is one less than the number of occupied rows. A *tabloid* of shape μ is then a tiling of μ by ribbons such that each ribbon contains a cell in the first column. The *weight* of a tabloid is $\beta = (\beta_1, \beta_2, \ldots)$, where β_i is the length of the ribbon starting in the i-th cell from the bottom in the first column of μ. Here we use French notation for the shape μ placing the longest part of μ at the bottom. The *sign* of a tabloid T is $(-1)^{\mathrm{height}(T)}$, where the height of T is the sum of the heights of all ribbons it contains. The *type* of a tabloid T with weight β is the partition α obtained by rearranging β into non-increasing order.

Example 2.1 The four tabloids T of shape $\mu = (3, 2, 1)$ are



with

$$\mathrm{sign}(T)\mathrm{weight}(T) = (3,2,1), -(1,4,1), -(3,0,3), (1,0,5),$$

and

$$\mathrm{type}(T) = (3,2,1), (4,1,1), (3,3), (5,1),$$

respectively.

Eğecioğlu and Remmel [6] proved that

$$K^{-1}_{\alpha\mu} = \sum_T \mathrm{sign}(T), \tag{5}$$

where the sum is over all tabloids T of type α and shape μ.

3 Cyclic symmetry

To compute the fusion coefficients, it suffices to calculate $s_\lambda s_\mu$ in the fusion ring. Using (4) we obtain

$$s_\lambda s_\mu = \sum_\alpha K^{-1}_{\alpha\mu}\ h_\alpha s_\lambda\ . \tag{6}$$

Note that h_α is multiplicative with $h_\alpha = h_{\alpha_1} h_{\alpha_2} \cdots$. This enables us to compute the product $h_\alpha s_\lambda$ using the fusion Pieri rule [8, Proposition 2.6]: for $1 \le r \le \ell$ and $\lambda \in \mathcal{P}^{\ell,n}$

$$h_r s_\lambda = \sum_\nu s_\nu\ , \tag{7}$$

where the sum is over all $\nu \in \mathcal{P}^{\ell,n}$ such that ν/λ is a horizontal r-strip and $\nu_1 - \lambda_n \le \ell$.

3.1 Cylindric tableaux

This leads us to the definition of *cylindric tableaux*. See also [7, 12, 18, 19]. For the precise definition we use a notion of shifting (skew) partitions. View a skew partition ν/λ as being placed at the origin so that the bottom leftmost cells of ν and λ are placed at $(0, 0)$. We then define $\mathrm{shift}_{a,b}(\nu/\lambda)$ to be the skew partition where the bottom leftmost cells of ν and λ are placed at position (a, b) in the plane. We denote the superposition of a skew partition ν/λ and its shift by $\nu/\lambda + \mathrm{shift}_{a,b}(\nu/\lambda)$. We can similarly shift skew tableaux, which are just fillings of skew shapes.

Note that when $\nu, \lambda \in \mathcal{P}^{\ell,n}$ such that ν/λ is a skew shape, then $\nu/\lambda + \mathrm{shift}_{-\ell,n}(\nu/\lambda)$ can be viewed as a skew shape inside the quadrant $x \geq -\ell$ and $y \geq 0$.

Definition 3.1 *For two positive integers ℓ and n and $\lambda \subseteq \nu \in \mathcal{P}^{\ell,n}$, a cylindric tableau t of shape ν/λ is a column-strict filling of the shape ν/λ such that t+$\mathrm{shift}_{-\ell,n}(t)$ is still column-strict.*

We denote the set of all cylindric tableaux of shape ν/λ and content μ by $\mathcal{T}^{\mathrm{cyc}}_{\nu/\lambda,\mu}$ where, as usual, the content μ of a tableau t is the tuple such that μ_i is the number of letters i in t.

Example 3.2 Let $\lambda = (1,1)$, $\mu = (2,2,2)$, $\nu = (4,2,2)$ and $\ell = n = 3$. Then there are two cylindric tableaux in $\mathcal{T}^{\mathrm{cyc}}_{\nu/\lambda,\mu}$:

$$\begin{array}{|c|c|cc} \hline 1 & 3 & \multicolumn{2}{c}{} \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{2}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \cline{2-4} \end{array} \quad \text{and} \quad \begin{array}{|c|c|cc} \hline 2 & 3 & \multicolumn{2}{c}{} \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{2}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \cline{2-4} \end{array}\,. \quad \text{Note that} \quad \begin{array}{|c|c|cc} \hline 3 & 3 & \multicolumn{2}{c}{} \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{2}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \cline{2-4} \end{array}$$

is not cylindric since after shifting by $(-3,3)$ the rightmost 2 would sit above the leftmost 3, which is not column-strict any longer.

3.2 Fusion coefficients

By iteration of (7), we derive that

$$h_\alpha s_\lambda = \sum_\nu K^{\mathrm{cyc}}_{\nu/\lambda,\alpha} s_\nu \;, \tag{8}$$

where $K^{\mathrm{cyc}}_{\nu/\lambda,\alpha} = |\mathcal{T}^{\mathrm{cyc}}_{\nu/\lambda,\alpha}|$ is the cardinality of the set of cylindric tableaux of skew shape ν/λ and content α. Combining this with (6) we obtain

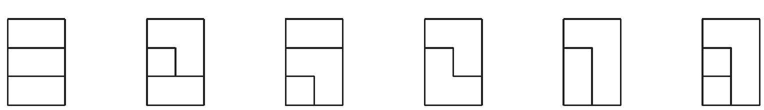
$$s_\lambda s_\mu = \sum_{\alpha,\nu} K^{-1}_{\alpha\mu}\, K^{\mathrm{cyc}}_{\nu/\lambda,\alpha} s_\nu \;, \tag{9}$$

which shows that the fusion coefficient is given by the formula

$$c^{\nu,\ell,n}_{\lambda\mu} = \sum_\alpha K^{\mathrm{cyc}}_{\nu/\lambda,\alpha}\, K^{-1}_{\alpha\mu} \;. \tag{10}$$

As we can see from (5), the inverse of the Kostka matrix contains negative signs, so this formula is an alternating sum. In the next section we will discuss a sign-reversing involution to cancel terms in certain cases; the number of fixed points under this involution will precisely amount to the fusion coefficient.

Example 3.3 As in Example 3.2 consider $\lambda = (1,1)$, $\mu = (2,2,2)$, $\nu = (4,2,2)$ and $\ell = n = 3$. The tabloids of shape μ are



with $\mathrm{sign}(T)\mathrm{weight}(T) = (2,2,2), -(2,1,3), -(1,3,2), (0,3,3), -(0,2,4), (1,1,4)$, respectively. There are no cylindric tableaux of skew shape ν/λ and weight $(0,3,3), (0,2,4)$ or $(1,1,4)$. The cylindric tableaux of skew shape ν/λ and weights $(2,2,2), (2,1,3)$, and $(1,3,2)$ are

$$\begin{array}{cccc} 1 & 3 & & \\ & 2 & & \\ & 1 & 2 & 3 \end{array} \qquad \begin{array}{cccc} 2 & 3 & & \\ & 2 & & \\ & 1 & 1 & 3 \end{array} \qquad \begin{array}{cccc} 1 & 3 & & \\ & 2 & & \\ & 1 & 3 & 3 \end{array} \qquad \begin{array}{cccc} 2 & 3 & & \\ & 2 & & \\ & 1 & 2 & 3 \end{array} .$$

Since two of them come with a positive sign and two with a negative sign, Equation (10) shows that the fusion coefficient $c_{(11),(222)}^{(422),3,3} = 0$.

Note that if instead of the fusion Pieri rule as in (7) one uses the usual Pieri rule for $h_r s_\lambda$, one obtains the following expression for the Littlewood–Richardson coefficients

$$c_{\lambda\mu}^{\nu} = \sum_{\alpha} K_{\nu/\lambda,\alpha}\, K_{\alpha\mu}^{-1}\ , \tag{11}$$

where $K_{\nu/\lambda,\alpha}$ is the skew Kostka matrix.

4 Sign-reversing involution

Remmel and Shimozono [20] proved the Littlewood–Richardson rule using a sign-reversing involution. Let us explain their approach first as we will use a modification of it in Section 4.2 to prove Theorem 1.1.

4.1 The Littlewood–Richardson case

Note that each tabloid T as defined in Section 2 is in one-to-one correspondence with its weight. Clearly a given T yields $\mathrm{weight}(T)$. Conversely, given a weight $(\alpha_1, \alpha_2, \ldots)$, start with the bottommost cell in the first column and draw a ribbon of length α_1; there is a unique way of doing so. Then proceed to the second cell in the first column and draw a ribbon of length α_2 and so on. It is not hard to see that either there is no way of doing so or there is a unique way of drawing the ribbons such that at each step the resulting shape is a partition.

Under this correspondence between tabloids and weights, (11) can be rewritten as (see also [20, Eq. (1.14)])

$$c_{\lambda\mu}^{\nu} = \sum_{(\sigma,t)} \mathrm{sign}(\sigma)\ , \tag{12}$$

where the sum is over all pairs (σ, t) with $\sigma \in S_n$ and $t \in \mathcal{T}_{\nu/\lambda,\alpha}$ a column-strict skew tableau of shape ν/λ and weight $\alpha = \sigma(\mu+\rho) - \rho$. Here $\rho = (n-1, n-2, \ldots, 1, 0)$ and σ acts on an n-tuple by permuting its entries.

The set of column-strict skew tableaux of given shape ν/λ over the alphabet $\{1,2,\ldots,n\}$, denoted $\mathcal{T}_{\nu/\lambda}$, is endowed with *crystal operators* $\tilde{e}_i, \tilde{f}_i, \tilde{s}_i$ for $1 \le i < n$. For $t \in \mathcal{T}_{\nu/\lambda}$, let $\mathrm{word}(t)$ be the column reading word of t. That is, read the columns of t top to bottom, left to right. On a word w, the crystal operators $\tilde{e}_i, \tilde{f}_i$, and $\tilde{s}_i$ only act on the letters i and $i+1$. In the subword of w consisting of the letters i and $i+1$, successively bracket pairs $(i+1)\ i$. Then $\tilde{f}_i$ makes the rightmost unbracketed i into an $i+1$; if no such i exists, $\tilde{f}_i$ annihilates the word. Similarly, $\tilde{e}_i$ changes the leftmost unbracketed $i+1$ into an i; if no such $i+1$ exists, $\tilde{e}_i$ annihilates the word. Finally, if the subword of w of unbracketed letters i and $i+1$ is $i^a(i+1)^b$, then in $\tilde{s}_i(w)$ this subword is replaced by $i^b(i+1)^a$.

Example 4.1 Let $n = 4$ and $w = 4123322341214223$. The bracketing for $i = 2$ yields

$$\begin{array}{cccccccccccccccc} 4 & 1 & 2 & 3 & 3 & 2 & 2 & 3 & 4 & 1 & 2 & 1 & 4 & 2 & 2 & 3 \\ & & \cdot & (& (&) &) & (& & &) & & & \cdot & \cdot & \cdot \end{array}$$

so that

$$\begin{array}{rcccccccccccccccc} \tilde{e}_2(w) = & 4 & 1 & 2 & 3 & 3 & 2 & 2 & 3 & 4 & 1 & 2 & 1 & 4 & 2 & 2 & 2 \\ \tilde{f}_2(w) = & 4 & 1 & 2 & 3 & 3 & 2 & 2 & 3 & 4 & 1 & 2 & 1 & 4 & 2 & 3 & 3 \\ \tilde{s}_2(w) = & 4 & 1 & 2 & 3 & 3 & 2 & 2 & 3 & 4 & 1 & 2 & 1 & 4 & 3 & 3 & 3 \end{array} .$$

The action of the crystal operators on column-strict skew tableaux is determined by the action on their column-words. It is known that after the application of $\tilde{e}_i, \tilde{f}_i$, and $\tilde{s}_i$, the resulting skew tableau is still column-strict. A tableau $t \in \mathcal{T}_{\nu/\lambda}$ is called *highest weight* if $\tilde{e}_i(t) = 0$ for all $1 \leq i < n$. Notice that t is highest weight if and only if $\mathrm{word}(t)$ is lattice.

To prove the Littlewood–Richardson rule, Remmel and Shimozono [20] introduced the following sign-reversing involution θ:

1. If t is highest weight, then $\theta(\sigma, t) = (\sigma, t)$.

2. Otherwise, let $r + 1$ be the rightmost letter in $\mathrm{word}(t)$ that violates the lattice condition. Define $\theta(\sigma, t) = (\sigma_r \sigma, \tilde{s}_r \tilde{e}_r(t))$.

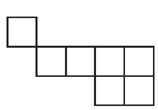
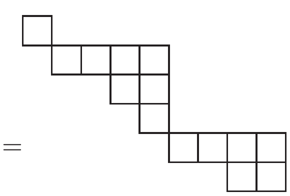
Since the highest weight elements $t \in \mathcal{T}_{\nu/\lambda}$ yield the fixed-points of θ, their weight must be a partition. It was shown in [20], that this implies that $\sigma = \mathrm{id}$ so that $t \in \mathcal{T}_{\nu/\lambda,\mu}$. Hence indeed $c^{\nu}_{\lambda\mu}$ counts the tableaux in $\mathcal{T}_{\nu/\lambda,\mu}$ that are lattice (or equivalently highest weight).

4.2 Proof of Theorem 1.1

Before giving the proof of Theorem 1.1, we need to provide the definition of *cutting points* and $\tilde{\nu}$ and $\tilde{\lambda}$ in the statement of the theorem.

Definition 4.2 *Given two positive integers ℓ and n, consider $\lambda \in \mathcal{R}^{\ell,n}$ and $\nu \in \mathcal{P}^{\ell,n}$ such that ν/λ is a skew shape where $\nu/\lambda + \mathrm{shift}_{-\ell,n}(\nu/\lambda)$ is not connected. A cutting point c is the index of the rightmost column in $\nu/\lambda + \mathrm{shift}_{-\ell,n}(\nu/\lambda)$ such that the columns with x-coordinate c and $c+1$ do not share a common edge. Note that such a c must exist since the skew shape is not connected. Let $\tilde{\nu}$ and $\tilde{\lambda}$ denote the partitions where $\tilde{\nu}/\tilde{\lambda}$ is the skew shape in the window from column $c - \ell$ to column c.*

Example 4.3 Let $\ell = 4$, $n = 3$, $\lambda = (3, 1)$, and $\nu = (5, 5, 1)$, so that

$\nu/\lambda =$  and $\nu/\lambda + \mathrm{shift}_{-4,3}(\nu/\lambda) =$  .

Then $c = 1$ is a cutting point and

$$\tilde{\nu}/\tilde{\lambda} = \begin{array}{cccc} \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ & & \boxed{} & \boxed{} \\ & & & \boxed{} \end{array}\,.$$

Indeed with $\mu = (4,2,1)$ we have that $c^{(551),4,3}_{(31),(421)} = c^{(444)}_{(32),(421)} = 1$ verifying Theorem 1.1 (1).

For the proof of Theorem 1.1 (1), one may use the same arguments as in the derivation of (12) for the Littlewood–Richardson coefficients to rewrite (10) for the fusion coefficients as

$$c^{\nu,\ell,n}_{\lambda\mu} = \sum_{(\sigma,t)} \operatorname{sign}(\sigma)\ , \tag{13}$$

where now the sum is over all pairs (σ,t) with $\sigma \in S_n$ and $t \in \mathcal{T}^{\mathrm{cyc}}_{\nu/\lambda,\alpha}$ a cylindric tableau of shape ν/λ and weight $\alpha = \sigma(\mu+\rho) - \rho$.

Due to the cylindric symmetry of the tableaux in $\mathcal{T}^{\mathrm{cyc}}_{\nu/\lambda,\alpha}$, we have $|\mathcal{T}^{\mathrm{cyc}}_{\nu/\lambda,\alpha}| = |\mathcal{T}^{\mathrm{cyc}}_{\tilde{\nu}/\tilde{\lambda},\alpha}|$, where recall the definition of $\tilde{\nu}$ and $\tilde{\lambda}$ from Definition 4.2. Since c (as in Definition 4.2) is a cutting point, that is, the adjacent columns do not share an edge, there is no cylindric column-strict condition imposed on the elements in $\mathcal{T}^{\mathrm{cyc}}_{\tilde{\nu}/\tilde{\lambda},\alpha}$. Hence $\mathcal{T}^{\mathrm{cyc}}_{\tilde{\nu}/\tilde{\lambda},\alpha} = \mathcal{T}_{\tilde{\nu}/\tilde{\lambda},\alpha}$, so that the arguments from Section 4.1 apply. This proves Theorem 1.1 (1).

Definition 4.4 *Given two positive integers ℓ and $n \geq 2$, let $\lambda, \mu \in \mathcal{R}^{\ell,n}$ and $\nu \in \mathcal{P}^{\ell,n}$ be such that μ has at most two parts. Then either ν/λ has a cutting point or it contains at least one column of height two. Let c be the rightmost such column. Then $\tilde{\nu}$ and $\tilde{\lambda}$ denote the partitions such that $\tilde{\nu}/\tilde{\lambda}$ is the skew shape of $\nu/\lambda + \operatorname{shift}_{-\ell,n}(\nu/\lambda)$ in the window from column $c-\ell$ to column c.*

By the same arguments as in the proof of part (1) of Theorem 1.1 we have $|\mathcal{T}^{\mathrm{cyc}}_{\nu/\lambda,\alpha}| = |\mathcal{T}^{\mathrm{cyc}}_{\tilde{\nu}/\tilde{\lambda},\alpha}|$ by cyclic symmetry. Since μ has only two parts, the only crystal operators that apply in this case are $\tilde{s}_1\tilde{e}_1$. The column of height 2 (which after the cyclic shift is the rightmost column) contains the letters 21 by column-strictness. The crystal operators cannot change this column due to the crystal bracketing rules. Hence the cylindric column-strict conditions are always guaranteed which proves Theorem 1.1 (2). Note that this case is related to the cyclic $\mathfrak{sl}_2$ crystals introduced in [9].

5 Beyond the cutting point

The cylindric tableaux of Example 3.3 do not have a cutting point and hence Theorem 1.1 does not apply. Under the Remmel–Shimozono involution we have

$$\tilde{s}_2\tilde{e}_2 \begin{array}{cccc} \boxed{1} & \boxed{3} & & \\ & \boxed{2} & & \\ & \boxed{1} & \boxed{2} & \boxed{3} \end{array} = \begin{array}{cccc} \boxed{1} & \boxed{3} & & \\ & \boxed{2} & & \\ & \boxed{1} & \boxed{3} & \boxed{3} \end{array}\,.$$

However, the action of the Remmel–Shimozono involution on the other two cylindric tableaux yields non-cylindric tableaux:

$$\tilde{s}_2\tilde{e}_2 \begin{array}{cccc} \boxed{2} & \boxed{3} & & \\ & \boxed{2} & & \\ & \boxed{1} & \boxed{1} & \boxed{3} \end{array} = \begin{array}{cccc} \boxed{3} & \boxed{3} & & \\ & \boxed{2} & & \\ & \boxed{1} & \boxed{1} & \boxed{3} \end{array} \quad \text{and} \quad \tilde{s}_2\tilde{e}_2 \begin{array}{cccc} \boxed{2} & \boxed{3} & & \\ & \boxed{2} & & \\ & \boxed{1} & \boxed{2} & \boxed{3} \end{array} = \begin{array}{cccc} \boxed{3} & \boxed{3} & & \\ & \boxed{2} & & \\ & \boxed{1} & \boxed{3} & \boxed{3} \end{array}$$

and hence does not yield a cancelation within the set $\mathcal{T}^{\mathrm{cyc}}_{\nu/\lambda,\cdot}$.

Note, however, that it is possible to amend the operators used by Remmel and Shimozono by conjugating the action of $\tilde{s}_i\tilde{e}_i$ by a cyclic shift. In the above example, moving the 2 in the leftmost column down we obtain

$$\begin{array}{|c|cc} \cline{1-1} 3 & & \\ \cline{1-1} 2 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{c}{} & & \multicolumn{1}{|c|}{2} \\ \cline{3-3} \end{array} \quad \text{and} \quad \begin{array}{|c|cc} \cline{1-1} 3 & & \\ \cline{1-1} 2 & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{c}{} & & \multicolumn{1}{|c|}{2} \\ \cline{3-3} \end{array}$$

which cancel under the action of $\tilde{s}_1\tilde{e}_1$.

We conjecture that such cyclic cancelations are always possible. In fact, computer experiments using `Sage` [21, 22] suggest that the resulting fusion lattice tableaux (that is, the skew tableaux that are fixed points under the involution) correspond to the 2d puzzles conjectured by Knutson [4] to yield the quantum Littlewood–Richardson coefficients or equivalently fusion coefficients by a bijection between puzzles and tableaux similar to [26, Figure 11].

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