Lifted generalized permutahedra and composition polynomials

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Abstract. We introduce a "lifting" construction for generalized permutohedra, which turns an n-dimensional generalized permutahedron into an $(n+1)$-dimensional one. We prove that this construction gives rise to Stasheff's multiplihedron from homotopy theory, and to the more general "nestomultiplihedra," answering two questions of Devadoss and Forcey.

We construct a subdivision of any lifted generalized permutahedron whose pieces are indexed by compositions. The volume of each piece is given by a polynomial whose combinatorial properties we investigate. We show how this "composition polynomial" arises naturally in the polynomial interpolation of an exponential function. We prove that its coefficients are positive integers, and conjecture that they are unimodal.

Résumé. Nous introduisons une construction de "lifting" (redressement) pour permutaèdres généralisés, qui transforme un permutaèdre généralisé de dimension n en un de dimension $n+1$. Nous démontrons que cette construction conduit au multiplièdre de Stasheff à partir de la théorie d'homotopie, et aux "nestomultiplièdres," ce qui répond à deux questions de Devadoss et Forcey.

Nous construisons une subdivision de n'importe quel permutaèdre généralisé dont les pièces sont indexées par compositions. La volume de chaque pièce est donnée par un polynôme dont nous recherchons les propriétés combinatoires. Nous montrons comment ce "polynôme de composition" surgit naturellement dans l'interpolation d'une fonction exponentiel. Nous démontrons que ses coefficients sont strictement positifs, et nous conjecturons qu'ils sont unimodaux.

Keywords: Polytope, permutohedron, associahedron, multiplihedron, nestohedron, subdivision, composition polynomial, polynomial interpolation

1 Introduction

Generalized permutahedra are the polytopes obtained from the permutahedron by changing the edge lengths while preserving the edge directions, possibly identifying vertices along the way. These polytopes, closely related to polymatroids and recently re-introduced by Postnikov [15] have been the subject of great attention due their very rich combinatorial structure. Examples include several remarkable

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polytopes which naturally appear in homotopy theory, in geometric group theory, and in various moduli spaces: permutahedra, matroid polytopes [2], Pitman-Stanley polytopes [14], Stasheff's associahedra [23], Carr and Devadoss's graph associahedra [4], Stasheff's multiplihedra [23], Devadoss and Forcey's multiplihedra [6], and Feichtner and Sturmfels's and Postnikov's nestohedra [15, 8].

We begin in Section 2 by introducing a "lifting" construction which takes a generalized permutahedron P in $\mathbb{R}^n$ into a generalized permutahedron $P(q)$ in $\mathbb{R}^{n+1}$, where $0 \leq q \leq 1$. We show that the lifting construction connects many important generalized permutahedra:

generalized permutahedron P	lifting $P(q)$
permutahedron P_n	permutahedron P_{n+1}
associahedron $\mathcal{K}_n$	multiplihedron $\mathcal{J}_n$
graph associahedron $\mathcal{K}G$	graph multiplihedron $\mathcal{J}G$
nestohedron $\mathcal{K}\mathcal{B}$	nestomultiplihedron $\mathcal{J}\mathcal{B}$
matroid polytope P_M	independent set polytope I_M $(q = 0)$

We provide geometric realizations of these polytopes and concrete descriptions of their face lattices. In particular, in Section 3 we answer two questions of Devadoss and Forcey: we find the Minkowski decomposition of the graph multiplihedra $\mathcal{J}G$ into simplices, and we construct the nestomultiplihedron $\mathcal{J}\mathcal{B}$.

In Section 4 we construct a subdivision of any lifted generalized permutahedron $P(q)$ whose pieces are indexed by compositions c. The volume of each piece is essentially given by a polynomial in q, which we call the *composition polynomial* $g_c(q)$.

Section 5 is devoted to the combinatorial properties of the composition polynomial $g_c(q)$ of a composition $c = (c_1, \ldots, c_k)$. We prove that $g_c(q)$ arises naturally in the polynomial interpolation of an exponential function. We prove that $g_c(q) = (1-q)^k f_c(q)$ where $f_c(q)$ is a polynomial with $f_c(1) \neq 0$. We prove that the coefficients of $f_c(q)$ are positive integers, and conjecture that they are unimodal as well.

In Section 6 we establish a connection between composition polynomials and Stanley's order polytopes. We use this to show that $g_c(q)$ is the generating function for counting linear extensions of a poset P_c.

2 Lifting a generalized permutahedron.

The *permutahedron* P_n is the polytope in $\mathbb{R}^n$ whose $n!$ vertices are the permutations of the vector $(1, 2, \ldots, n)$. A *generalized permutahedron* is a deformation of the permutahedron, obtained by changing the lengths of the edges of P_n in such a way that all edge directions and orientations are preserved, while possibly identifying vertices along the way. [17]. Postnikov showed [15] that every generalized permutahedron can be written in the form:

$$P_n(\{z_I\}) = \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} t_i = z_{[n]}, \sum_{i \in I} t_i \geq z_I \text{ for all } I \subseteq [n] \right\},$$

where z_I is a real number for each $I \subseteq [n] := \{1, \ldots, n\}$, and $z_\emptyset = 0$.

We now introduce *lifting*, a procedure which converts a generalized permutahedron in $\mathbb{R}^n$ into a *lifted generalized permutahedron* in $\mathbb{R}^{n+1}$.

Definition 2.1 *Given a generalized permutahedron $P = P_n(\{z_I\})$ in $\mathbb{R}^n$ and a number $0 \leq q \leq 1$, define the q-*lifting *of P to be the polytope $P(q)$ given by the inequalities*

$$\sum_{i=1}^{n+1} t_i = z_{[n]}, \qquad \sum_{i \in I} t_i \geq q z_I \text{ for } I \subseteq [n], \qquad \sum_{i \in I \cup \{n+1\}} t_i \geq z_I \text{ for } I \subseteq [n].$$

In other words, $P(q) := P_{n+1}(\{z'_I\})$ where $z'_J = qz_J$ and $z'_{J \cup \{n+1\}} = z_J$ for $J \subseteq [n]$. The polytope $P(q)$ is called a lifted generalized permutahedron.

We will let the lifting *of P refer to any q-lifting with $0 < q < 1$. We will see in Corollary 2.4 that all such q-liftings are combinatorially isomorphic.*

Proposition 2.2 *If P is a generalized permutahedron, its q-lifting $P(q)$ is a generalized permutahedron.*

Notice that the 1-lifting $P(1)$ is the natural embedding of P in the hyperplane $x_{n+1} = 0$ of $\mathbb{R}^{n+1}$. The 0-lifting $P(0) = P_{n+1}(\{z'_I\})$ is the generalized permutahedron in $\mathbb{R}^{n+1}$ defined by $z'_J = 0$ and $z'_{J \cup \{n+1\}} = z'_J$ for all $J \subseteq [n]$. These are shown in Figure 1.

Recall that the *Minkowski sum* of two polytopes P and Q in $\mathbb{R}^n$ is defined to be $P + Q := \{p + q : p \in P, q \in Q\}$. Since the hyperplane parameters $\{z_I\}$ of generalized permutahedra are additive with respect to Minkowski sums [2, 15], we have:

Proposition 2.3 *For $0 \leq q \leq 1$, the q-lifting of any generalized permutahedron P satisfies that $P(q) = qP(1) + (1-q)P(0)$.*

Corollary 2.4 *All q-liftings of P with $0 < q < 1$ are combinatorially isomorphic.*

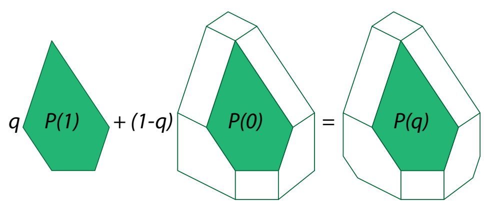

Fig. 1: The q-lifting of a generalized permutahedron $P_n(\{y_I\})$.

For each $I \subseteq [n]$, consider the simplex $\Delta_I = \operatorname{conv}\{e_i : i \in I\}$. Any generalized permutahedron $P = P_n(\{z_I\})$ can be written uniquely as a signed Minkowski sum of simplices in the form $P = P_n(\{y_I\}) := \sum y_I \Delta_I$ for $y_I \in \mathbb{R}$. [(i)] [2, 15] The z-parameters and the y-parameters of P are linearly related by the equations

$$z_I = \sum_{J \subseteq I} y_J, \qquad \text{for all } I \subseteq [n].$$

Proposition 2.5 *The q-lifting of the generalized permutahedron $P = \sum_I y_I \Delta_I$ is*

$$P(q) = q \sum_I y_I \Delta_I + (1-q) \sum_I y_I \Delta_{I \cup \{n+1\}}.$$

[(i)] An equation like $P - Q = R$ should be interpreted as $P = Q + R$.

From these observations it follows that the face of $P(q)$ maximized in the direction $(1,\dots,1,0)$ is a copy of P, while the face maximized in the opposite direction is a copy of P scaled by q. The vertices of $P(q)$ will come from vertices of P, with a factor of q applied to certain specific coordinates. We describe them in Section 4.

3 Nestohedra and nestomultiplihedra.

In his work on homotopy associativity for A_∞ spaces, Stasheff [23] defined the *multiplihedron* $\mathcal{J}_n$, a cell complex which has since been realized in different geometric contexts by Fukaya, Oh, Ohta, and Ono [10], by Mau and Woodward [13], and others. It was first realized as a polytope by Forcey [7].

More generally, Devadoss and Forcey [6] defined, for each graph G, the *graph multiplihedron* $\mathcal{J}G$. [3, 4] When G has no edges, they gave a description of $\mathcal{J}G$ as a Minkowski sum. They asked for a Minkowski sum description of $\mathcal{J}G$ for arbitrary G.

In a different direction, Postnikov [15] defined the *nestohedron* $\mathcal{KB}$, an extension of graph associahedra to the more general context of building sets $\mathcal{B}$. Devadoss and Forcey asked whether there is a notion of *nestomultiplihedron* $\mathcal{JB}$, which extends the graph multiplihedra to this context.

In this section we answer these questions affirmatively in a unified way, by showing that the q-lifting of the graph associahedron $\mathcal{K}G$ is the graph multiplihedron $\mathcal{J}G$ and, more generally, the q-lifting of the nestohedron $\mathcal{KB}$ is the desired nestomultiplihedron $\mathcal{JB}$.

3.1 Nestohedra and $\mathcal{B}$-forests.

Definition 3.2 *[8, 15] A* building set $\mathcal{B}$ *on a ground set* $[n]$ *is a collection of subsets of* $[n]$ *such that:*
(B1) If $I, J \in \mathcal{B}$ *and* $I \cap J \neq \emptyset$ *then* $I \cup J \in \mathcal{B}$.
(B2) For every $e \in [n]$, $\{e\} \in \mathcal{B}$.

For a building set $\mathcal{B}$ the *nestohedron* $\mathcal{KB}$ is the Minkowski sum of simplices $\mathcal{K}B := \sum_{B\in\mathcal{B}} \Delta_B$.

An important example of a building set is the following: given a graph G on a vertex set $[n]$, the associated building set $\mathcal{B}(G)$ consists of the subsets $I \subseteq [n]$ for which the induced subgraph $G|_I$ is connected. Such subsets are sometimes called the *tubes* of G.

If $\mathcal{B}$ is a building set on $[n]$ and $A \subseteq [n]$, we define the *induced building set* of $\mathcal{B}$ on A to be $\mathcal{B}|_A := \{I \in \mathcal{B} : I \subseteq A\}$. Also let $\mathcal{B}_{\max}$ be the set of containment-maximal elements of $\mathcal{B}$.

Definition 3.3 *[8, 15] A* nested set $\mathcal{N}$ *for a building set* $\mathcal{B}$ *is a subset* $\mathcal{N} \subseteq \mathcal{B}$ *such that:*
(N1) If $I, J \in \mathcal{N}$ *then* $I \subseteq J$ *or* $J \subseteq I$ *or* $I \cap J = \emptyset$.
(N2) If $J_1, \dots, J_k \in \mathcal{N}$ *are pairwise incomparable and* $k \geq 2$ *then* $J_1 \cup \cdots \cup J_k \notin \mathcal{B}$.
(N3) $\mathcal{B}_{\max} \subseteq \mathcal{N}$.
The nested set complex $\mathcal{N}(\mathcal{B})$ *of* $\mathcal{B}$ *is the simplicial complex on* $\mathcal{B}$ *whose faces are the nested sets of* $\mathcal{B}$.

When $\mathcal{B}(G)$ is the building set of tubes of a graph, the nested sets are called the *tubings* of G. If G is the graph shown in Figure 2(a), an example of a nested set or tubing is $\mathcal{N} = \{3, 4, 6, 7, 379, 48, 135679, 123456789\}$, shown in Figure 2(b).(ii)

The sets in a nested set $\mathcal{N}$ form a poset by containment. This poset is a forest rooted at $\mathcal{B}_{\max}$ by (N1). Relabelling each node N with the set $\widehat{N} := N \setminus \bigcup_{M\in\mathcal{N} : M<N} M$, we obtain a $\mathcal{B}$-forest:

(ii) We omit the brackets from the sets in $\mathcal{N}$ for clarity.

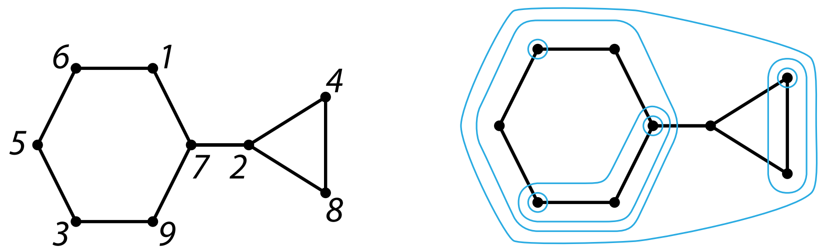

Fig. 2: (a) A graph G. (b) A nested set or tubing of G.

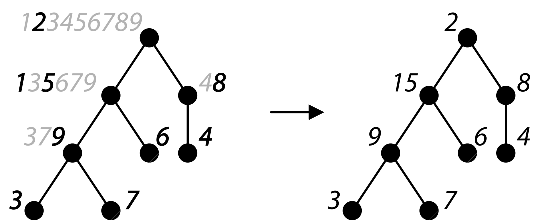

Fig. 3: The poset and the $\mathcal{B}$-forest for the nested set $\mathcal{N} = \{3, 4, 6, 7, 379, 48, 135679, 123456789\}$ of Figure 2(b).

Definition 3.4 *[8, 15] Given a building set $\mathcal{B}$ on $[n]$, a* $\mathcal{B}$-forest *$\mathcal{N}$ is a rooted forest whose vertices are labeled with non-empty sets partitioning $[n]$ such that:*
(F1) For any node S, $\mathcal{N}_{\leq S} \in \mathcal{B}$.
(F2) If $S_1, \dots, S_k$ are incomparable and $k \geq 2$, $\bigcup_{i=1}^{k} \mathcal{N}_{\leq S_i} \notin \mathcal{B}$.
(F3) If $R_1, \dots, R_r$ are the roots of F, then the sets $\mathcal{N}_{\leq R_1}, \dots, \mathcal{N}_{\leq R_r}$ are precisely the maximal elements of $\mathcal{B}$.

Here $\mathcal{N}_{\leq S} := \bigcup_{T \leq S} T$. It is clear that nested sets for $\mathcal{B}$ are in bijection with $\mathcal{B}$-forests. As the notation suggests, we will make no distinction between a nested set and its corresponding $\mathcal{B}$-forest.

Theorem 3.5 *[8, 15] The* nestohedron

$$\mathcal{KB} = \sum_{B \in \mathcal{B}} \Delta_B$$

is isomorphic to the opposite of the poset of $\mathcal{B}$-forests.

It is worth remarking that the *graph associahedron $\mathcal{J}G$* is the nestohedron for the building set $\mathcal{B}(G)$ of the graph G. Figure 4 shows the correspondence between faces of the associahedron and $\mathcal{B}$-forests.

Fig. 4: The associahedron with faces labeled with corresponding $\mathcal{B}$-forests.

3.6 Nestomultiplihedra and painted $\mathcal{B}$-forests.

For a building set $\mathcal{B}$ on $[n]$ the *nestomultiplihedron* $\mathcal{J}\mathcal{B} \subseteq \mathbb{R}^{n+1}$ is the Minkowski sum of simplices

$$\mathcal{J}\mathcal{B} := \sum_{B\in\mathcal{B}} \Delta_B + \sum_{B\in\mathcal{B}} \Delta_{B\cup\{n+1\}}.$$

The face poset of the nestomultiplihedron is related the the poset of *painted $\mathcal{B}$-forests*.

Definition 3.7 *A* painted $\mathcal{B}$-forest $\overline{\mathcal{N}} = (\mathcal{N}^-, \mathcal{N}^0, \mathcal{N}^+)$ *is a $\mathcal{B}$-forest $\mathcal{N}$ together with a partition of the vertices into a downset $\mathcal{N}^-$, an antichain $\mathcal{N}^0$, and an upset $\mathcal{N}^+$ such that $\mathcal{N}^- \cup \mathcal{N}^0$ is a downset (and hence $\mathcal{N}^0 \cup \mathcal{N}^+$ is an upset). The vertices of $\mathcal{N}^-, \mathcal{N}^0$, and $\mathcal{N}^+$ are colored white, gray, and black, respectively.*

Figure 5 shows a painted $\mathcal{B}$-forest for the building set of the graph in Figure 2(a). Here $\mathcal{N}^- = \{3,4,6,7\}$, $\mathcal{N}^0 = \{8,9\}$, and $\mathcal{N}^+ = \{15,2\}$.

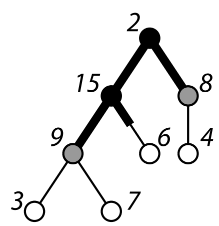

Fig. 5: A painted $\mathcal{B}$-forest. The vertices in $\mathcal{N}^-, \mathcal{N}^0$, and $\mathcal{N}^+$ are shaded white, gray, and black, respectively.

The painted $\mathcal{B}$-forests form a partial order. We go down the poset by successively

- converting a white (W) or black (B) vertex into a gray (G) vertex,
- contracting a BB, WW, or GW edge, or
- contracting all the BG edges below a black vertex.

Figure 6 shows the multiplihedron $\mathcal{J}_3$ (which is also the graph multiplihedron $\mathcal{J}K_3$, as well as the nestomultiplihedron $\mathcal{J}\mathcal{B}(K_3)$ for the building set of K_3), whose faces are in order-preserving bijective correspondence with the painted trees on $[3]$. Our next theorem describes the face poset of the nestomultiplihedron, which plays the analogous role for an arbitrary building set $\mathcal{B}$.

Theorem 3.8 *The face poset of the nestomultiplihedron*

$$\mathcal{J}\mathcal{B} = \sum_{B\in\mathcal{B}} \Delta_B + \sum_{B\in\mathcal{B}} \Delta_{B\cup\{n+1\}}$$

is isomorphic to the opposite of the poset of painted $\mathcal{B}$-forests.

Corollary 3.9 *The lifting of the nestohedron $\mathcal{K}\mathcal{B}$ is the nestomultiplihedron $\mathcal{J}\mathcal{B}$.*

Remark 3.10 *In [6], Devadoss and Forcey asked for a nice Minkowski decomposition of the* graph multiplihedron $\mathcal{K}G$. *By definition, $\mathcal{K}G$ is combinatorially isomorphic to the nestomultiplihedron for the building set $\mathcal{B}(G)$ of the graph G. Therefore Theorem 3.8 offers a satisfactory answer to their question.*

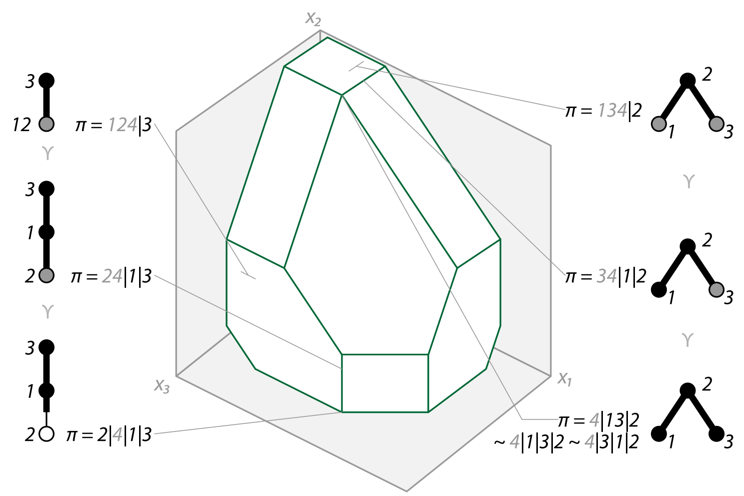

Fig. 6: The multiplihedron $\mathcal{J}_3 = \mathcal{J}K_3$ whose faces correspond to the painted trees on $[3]$, projected onto the hyperplane $x_4 = 0$. (Not all forests are pictured.)

4 Face q-liftings and volumes.

We will now modify the q-lifting operator $P(q)$ and define the *face q-lifting operator* $P^\pi(q)$, which acts on a specific face P_π of a generalized permutahedron instead of acting on P as a whole. This operator is useful in that it subdivides the polytope $P(q)$ into pieces whose volumes are easy to compute, i.e.

$$P(q) = \bigcup_{\pi \in \mathcal{P}^n} P^\pi(q),$$

and

$$\mathrm{Vol}_n(P(q)) = \sum_{\pi \in \mathcal{P}^n} \mathrm{Vol}_n(P^\pi(q)),$$

where $\mathrm{Vol}_n(P^\pi(q))$ is a degree-n polynomial in q. The family of polynomials described in this volume formula are defined in terms of compositions of n, and we explore them in greater depth in Section 5.

For the sake of visualizations and the cleanliness of formulas, for this section let us treat $P(q)$ as a full-dimensional polytope in $\mathbb{R}^n$ via projection onto the hyperplane $x_{n+1} = 0$, rather than as a codimension-1 polytope in $\mathbb{R}^{n+1}$. Thus if $P = P_n(\{z_I\})$ then it follows from Definition 2.1 that $P(q)$ will have hyperplane description

$$P(q) = \left\{ x \in \mathbb{R}^n : qz_I \le \sum_{i \in I} x_i \le z_{[n]} - z_{[n]\setminus I} \text{ for all } I \subseteq [n] \right\}.$$

Consider the linear functional $f(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n$. Partition $[n]$ into blocks $A_1, \ldots, A_k$ so $a_i = a_j$ if and only if i and j both belong to the same block A_s, and $a_i < a_j$ if and only if $i \in A_s$ and $j \in A_t$ for some $s < t$. If we let $\pi = A_1|\cdots|A_k$ then we say that the functional f is of *type* π. The f-maximal face P_f of P depends only on the type π.

Definition 4.1 *Define the face q-lifting of P as follows. Let $\pi = B_1|\cdots|B_k$ be an ordered partition of $[n]$ and let P_π be the face of P that maximizes a linear functional of type π. Now for $i = 0, \ldots, k$ construct a modified copy of P_π by applying a factor of q to the coordinates of the vertices of P_π whose indices belong to the first i blocks of π, $B_1 \cup \cdots \cup B_i$. The convex hull of all of these modified copies of P_π is the* face q-lifting *of P_π, and we denote it as $P^\pi(q)$.*

Example 4.2 *Consider the associahedron $\mathcal{K}(4)$. We have $\mathcal{K}(4)_{1|3|2} = conv\{(1,4,1)\}$. Then*

$$\mathcal{K}(4)^{1|3|2}(q) = conv\{(1,4,1),(q,4,1),(q,4,q),(q,4q,q)\}.$$

This and other face q-liftings are pictured in Figure 7.

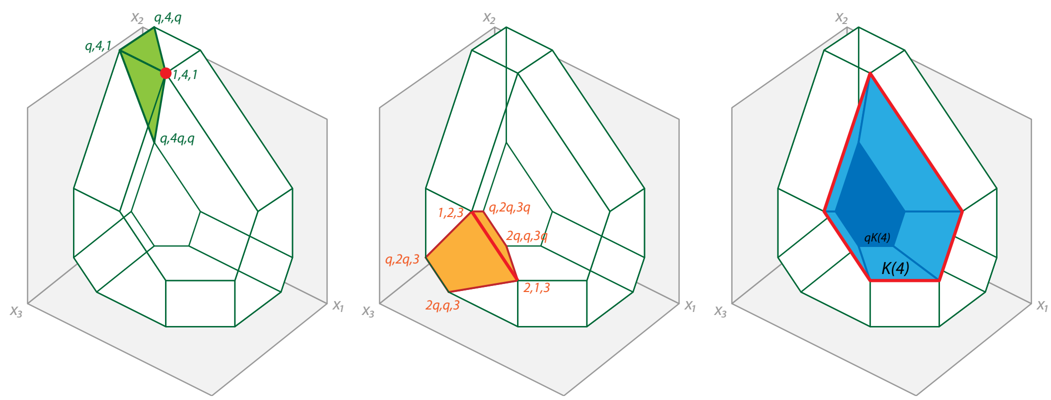

Fig. 7: Three face q-liftings of the associahedron $\mathcal{K}(4)$: $\mathcal{K}(4)^{1|3|2}(q)$, $\mathcal{K}(4)^{12|3}(q)$, and $\mathcal{K}(4)^{123}(q)$. The red regions represent the faces $\mathcal{K}(4)_\pi$.

Definition 4.3 *For a subset $I \subseteq [n]$ define $x_I := \sum_{i \in I} x_i$. For a generalized permutahedron $P = P_n(\{z_I\})$ and an ordered partition $\pi = B_1|\cdots|B_k$ define*

$$z^{B_i} := z_{B_1 \cup \cdots \cup B_i} - z_{B_1 \cup \cdots \cup B_{i-1}}.$$

Theorem 4.4 *The set of face q-liftings $\{P^\pi(q) : \pi$ an ordered partition of $[n]\}$ forms a subdivision of the q-lifted polytope $P(q)$.*

Proof: (Sketch.) Let $x \in P(q)$. We consider the 2^n points $v_I = (z_I, \sum_{i \in I} x_i) \in \mathbb{R}^2$ for $I \subseteq [n]$. Let the lower hull of this set of points consist of $v_{A_0}, v_{A_1}, v_{A_2}, \ldots, v_{A_k}$ in that order. We prove that $A_{i-1} \subset A_i$ for each i, and define $B_i = A_i \setminus A_{i-1}$ and $\pi = B_1|\cdots|B_k$. Then we show that $x \in P^\pi(q)$. □

Theorem 4.5 *Let P be a generalized permutahedron in $\mathbb{R}^n$. Let $\pi = B_1|\cdots|B_k$ be an ordered partition of $[n]$. Then the volume of the face q-lifting $P^\pi(q)$ is a polynomial in q given by*

$$\mathrm{Vol}_n(P^\pi(q)) = z^{B_1} \cdots z^{B_k} \mathrm{Vol}_{n-k}(P_\pi) \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} t_1^{|B_1|-1} \cdots t_k^{|B_k|-1} dt_1 \cdots dt_k.$$

The proof of this theorem relies on the observation that the face q-lifting $P^\pi(q)$ is combinatorially the product of P_π and a k-dimensional simplex Δ. We consider the projection to Δ whose fibers are combinatorially isomorphic to P_π; their volumes differ from the volume of P_π by a predictable factor. We then integrate these fibers along Δ. Figure 8 depicts this construction.

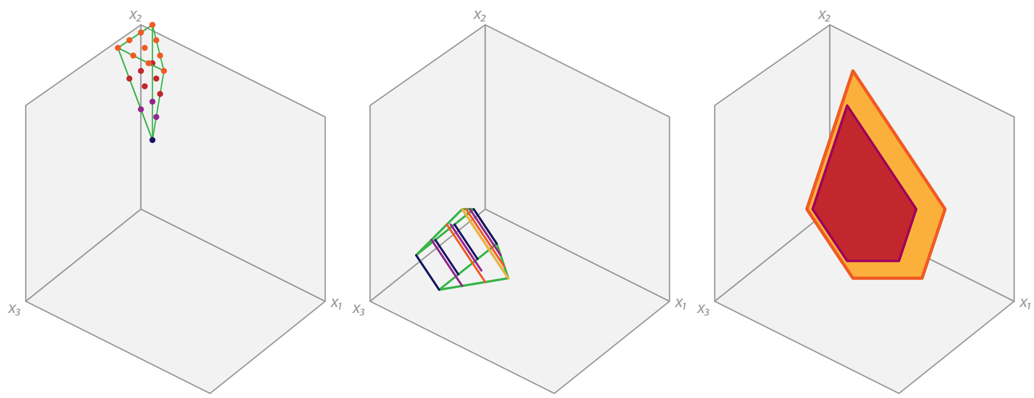

Fig. 8: Computing the volumes of face q-liftings. These correspond to the face q-liftings shown in Figure 7.

5 Composition polynomials.

Here we introduce the *composition polynomial* $g_c(q)$ of an ordered composition $c = (c_1, \ldots, c_k)$ of n and the *reduced composition polynomial* $f_c(q) = (1-q)^{-k} g_c(q)$. We present Theorem 5.3, which summarizes our main results on these polynomials. We show that $f_c(q)$ is indeed a polynomial. We prove that its coefficients are positive integers, and interpret them combinatorially. We ask whether the coefficients of $f_c(q)$ are also unimodal, or even log-concave.

- $f_{(1,1,1,1)}(q) = \frac{1}{24}$.
- $f_{(2,2,2,2)}(q) = \frac{1}{384}(1+q)^4$.
- $f_{(1,2,2)}(q) = \frac{1}{120}(8+9q+3q^2)$.
- $f_{(2,2,1)}(q) = \frac{1}{120}(3+9q+8q^2)$.
- $f_{(5,3)}(q) = \frac{1}{120}(5+10q+15q^2+12q^3+9q^4+6q^5+3q^6)$.
- $f_{(a,b)}(q) = \frac{1}{ab(a+b)}(b + 2bq + \cdots + (a-2)bq^{a-3} + (a-1)bq^{a-2} + abq^{a-1} +$
  $+ a(b-1)q^a + a(b-2)q^{a+1} + \cdots + 2aq^{a+b-3} + aq^{a+b-2})$

Let us begin by reviewing the notion of a composition of an integer.

Definition 5.1 *A* composition *c is a finite ordered tuple of positive integers, denoted $c = (c_1, \ldots, c_k)$. We call the c_i the* parts *of c, and the sum $c_1 + \cdots + c_k$ the* size *of c. If $c = (c_1, \ldots, c_k)$ has size n, we say that c is a composition of n into k parts. The* reverse *of the composition c is defined as $\bar{c} = (c_k, \ldots, c_1)$.*

Definition 5.2 *For a composition $c = (c_1, \ldots, c_k)$ let $\mathbf{t}^{\mathbf{c}-\mathbf{1}} := t_1^{c_1-1} \cdots t_k^{c_k-1}$, where $t = (t_1, \ldots, t_k)$. Then define the* composition polynomial $g_c(q)$ *by*

$$g_c(q) := \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} \mathbf{t}^{\mathbf{c}-\mathbf{1}} dt_1 \cdots dt_k.$$

This polynomial has degree n and belongs to $\mathbb{Q}[q]$.

The main goal for this section is to prove the following theorem about composition polynomials.

Theorem 5.3 *Let $c = (c_1, \ldots, c_k)$ be a composition of n. Then:*

0. *Let $\beta_i = c_1 + \cdots + c_i$, and let $h(t) = a_0 + \cdots + t^k a_k$ be the polynomial of smallest degree that passes through the $k+1$ points (β_i, q^{β_i}) for $i = 0, \ldots, k$. Here the coefficients a_i are functions of q. Then $a_k = \pm g_c(q)$.*
1. *$g_c(q)$ factors as $g_c(q) = (1-q)^k f_c(q)$, where deg$(f_c(q)) = n - k$ and $f_c(1) \neq 0$.*
2. *The coefficients of $f_c(q)$ are strictly positive.*
3. *$f_c(1) = 1/k!$.*
4. *$f_{\bar{c}}(q) = q^{n-k} f_c(1/q)$.*
5. *$f_{\alpha c}(q) = \frac{1}{\alpha^k}(1 + q + \cdots + q^{\alpha-1})^k f_c(q^\alpha)$ for any positive integer α.*

Definition 5.4 *We will refer to the polynomial $f_c(q)$ as the* reduced composition polynomial *for the composition c.*

Our proof of the remaining parts of Theorem 5.3 relies on a recursive construction of the polynomial $g_c(q)$.

Definition 5.5 *Define the* truncated compositions $c^L := (c_2, \ldots, c_k)$ *and* $c^R := (c_1, \ldots, c_{k-1})$. *For $m \in \{1, \ldots, k-1\}$ we define the* merged composition c^m *as the composition formed by combining the parts c_m and c_{m+1} into a single part:*

$$c^m := (c_1, \ldots, c_{m-1}, c_m + c_{m+1}, c_{m+2}, \ldots, c_k).$$

Proposition 5.6 *Let $c = (c_1, \ldots, c_k)$ be a composition of n into k parts. Let c^m be the merged composition $(c_1, \ldots, c_m + c_{m+1}, \ldots, c_k)$, and let $c^L = (c_1, \ldots, c_k)$ and $c^R = (c_0, \ldots, c_{k-1})$ be the truncated compositions. Then the composition polynomial $g_{c^m}(q)$ follows the recursion*

$$g_{c^m}(q) = \frac{\beta_m}{n} g_{c^R}(q) + \left(1 - \frac{\beta_m}{n}\right) q^{c_1} g_{c^L}(q), \tag{1}$$

where β_m denotes the partial sum $\beta_m = c_1 + \cdots + c_m$.

This result is significant because every composition c except for the trivial composition $(1, \ldots, 1)$ can be thought of as a merged composition. Also notice that the sizes of c^L and c^R are each strictly less than the size of c^m, though the number of parts remains constant. This means we have actually produced a recursive expression for an arbitrary nontrivial composition polynomial in terms of composition polynomials of strictly smaller degree. Theorem 5.3 follows swiftly from this result.

6 Connection with order polytopes

Consider the poset P_c consisting of a chain $p_0 < p_1 < \cdots < p_k$ together with a chain of size $c_i - 1$ below p_i for $1 \leq i \leq k$. The *order polytope* $\mathcal{O}(P_c)$ is the polytope of points $x \in \mathbb{R}^{P_c}$ such that $0 \leq x_i \leq x_j \leq 1$ whenever $i \leq j \in P$.

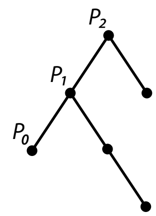

Fig. 9: The poset P_{32}.

Proposition 6.1 *Let $H \in \mathbb{R}^{P_c}$ be the hyperplane $x_{p_0} = q$. Then*

$$\mathrm{Vol}\,(\mathcal{O}(P_c) \cap H) = \frac{g_c(q)}{(c_1 - 1)! \cdots (c_k - 1)!}.$$

Proof: For any $0 \leq q \leq t_1 \leq \cdots \leq \cdots t_k \leq 1$, the intersection of $\mathcal{O}(P_c)$ with $x_{p_0} = q$ and $x_{p_i} = t_i$ for $1 \leq i \leq k$ is a product of k simplices having volume $\prod_{i=1}^{k} \frac{t_i^{c_i - 1}}{(c_i - 1)!}$. Now integrate over all such values. □

Corollary 6.2 *The composition polynomial is given by*

$$g_c(q) = \frac{(c_1 - 1)! \cdots (c_k - 1)!}{n!} \sum_{i=0}^{n} N_{i+1} \binom{n}{i} q^i (1 - q)^{n-i}$$

where N_j is the number of linear extensions of P_c such that x_0 has height j. We have $N_j^2 \geq N_{j-1} N_{j+1}$ for $2 \leq j \leq n$.

Proof: This follows from Stanley's work on order polytopes, namely Proposition 6.1 and (15) of [19]. □

References

[1] M. Aguiar, F. Ardila. *The Hopf monoid of generalized permutahedra*. Preprint, 2011.

[2] F. Ardila, C. Benedetti, J. Doker. *Matroid Polytopes and their Volumes.* Discrete & Computational Geometry **43** (2010)

[3] F. Ardila, V. Reiner, and L. Williams. *Bergman complexes, Coxeter arrangements, and graph associahedra.* Seminaire Lotharingien de Combinatoire, 54A (2006), Article B54Aj.

[4] M. Carr, S. Devadoss, *Coxeter complexes and graph associahedra.* Topology and its Applications 153 (2006) 2155-2168

[5] S. Devadoss. *A Realization of Graph Associahedra.* Discrete Mathematics **309** (2009), 271-276.

[6] Devadoss and Forcey. *Marked tubes and the graph multiplihedron.* Algebraic and Geometric Topology, 8(4) 2081-2108, 2008.

[7] S. Forcey, *Convex hull realizations of the multiplihedra.*Topology and its Applications, 156, 326-347, 2008.

[8] Feichtner and Sturmfels, *Matroid polytopes, nested sets and Bergman fans.* Port. Math. (N.S.) 62 (2005), 437-468.

[9] S. Forcey. *Convex Hull Realizations of the Multiplihedra.* Topology and Its Applications **156**, no. 2 (2008), 326–347.

[10] K. Fukaya, Y. Oh, H. Ohta, and K. Ono. Lagrangian Intersection Floer Theory: Anomaly and Obstruction. AMS/IP Studies in Advanced Mathematics, Vol. 46, 2009

[11] M. Haiman, *Constructing the associahedron*, Unpublished manuscript, MIT, 1984, 11 pages.

[12] J.L. Loday. *Realization of the Stasheff polytope*. Archiv der Mathematik **83** (2004), 267-278.

[13] S. Mau and C. Woodward, Geometric realizations of the multiplihedron and its complexification. Preprint, 2008. `arXiv:0802.2120`.

[14] J. Pitman and R. Stanley. *A polytope related to empirical dis- tribution, plane trees, parking functions, and the associahedron.* Discrete Comput. Geom. **27** 2002 603-632.

[15] A. Postnikov. *Permutohedra, associahedra and beyond.* Int. Math. Res. Notices **2009** (6) (2009), 1026–1106.

[16] A. Postnikov. Personal communication. 2007.

[17] A. Postnikov, V. Reiner, L. Williams. *Faces of generalized permutahedra.* Documenta Mathematica, **13** (2008), 207–273.

[18] R. P. Stanley. *Enumerative Combinatorics, Vol. I*. Cambridge University, London, 1997.

[19] Richard P. Stanley. *Two poset polytopes*. Discrete Comput. Geom. **1** (1986) 9-23.

[20] R. P. Stanley. *Eulerian partition of a unit hypercube*. Higher combinatorics, M. Aigner ed. Reidel Dordrecht, Boston (1977).

[21] R. P. Stanley. *Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry.* Annals of the New York Academy of Sciences, **576** (1989), 500-535.

[22] R. P. Stanley. *Two Combinatorial Applications of the Aleksandrov-Fenchel Inequalities.* J. Comb. Theory, Ser. A (1981), 56–65.

[23] J. Stasheff. H-spaces from a homotopy point of view. Lecture Notes in Mathematics, Vol. 161. Springer- Verlag, Berlin, 1970.