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Network parameterizations for the Grassmannian

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Abstract. Deodhar introduced his decomposition of partial flag varieties as a tool for understanding Kazhdan-Lusztig polynomials. The Deodhar decomposition of the Grassmannian is also useful in the context of soliton solutions to the KP equation, as shown by Kodama and the second author. Deodhar components S_D of the Grassmannian are in bijection with certain tableaux D called *Go-diagrams*, and each component is isomorphic to $(\mathbb{K}^*)^a \times (\mathbb{K})^b$ for some non-negative integers a and b. Our main result is an explicit parameterization of each Deodhar component in the Grassmannian in terms of networks. More specifically, from a Go-diagram D we construct a weighted network N_D and its *weight matrix* W_D, whose entries enumerate directed paths in N_D. By letting the weights in the network vary over $\mathbb{K}$ or $\mathbb{K}^*$ as appropriate, one gets a parameterization of the Deodhar component S_D. One application of such a parameterization is that one may immediately determine which Plücker coordinates are vanishing and nonvanishing, by using the Lindstrom-Gessel-Viennot Lemma. We also give a (minimal) characterization of each Deodhar component in terms of Plücker coordinates.

Résumé. Deodhar a introduit une décomposition des variétés drapeaux pour comprendre les polynômes de Kazhdan-Lusztig. La décomposition de Deodhar des Grassmanniennes est aussi utile dans le contexte des solutions solitons de l'équation KP, ce qui a été établi par Kodama et le deuxième auteur. Les composantes de Deodhar S_D sont en bijection avec certains tableaux D appelés *diagrammes de Go*, et chaque composante est isomorphe à $(\mathbb{K}^*)^a \times (\mathbb{K})^b$ où a et b sont des entiers positifs. Notre résultat principal est une paramétrisation explicite de chaque composante de Deodhar des Grassmanniennes en termes de réseaux. Plus précisément, à partir d'un diagramme de Go D, nous construisons un réseau N_D et sa *matrice de poids* W_D, dont les composantes énumèrent les chemins dirigés dans N_D. En faisant varier les poids dans $\mathbb{K}$ ou $\mathbb{K}^*$, nous obtenons une paramétrisation de la composante de Deodhar S_D. Une application de cette paramétrisation est que nous pouvons déterminer quelles coordonnées de Plücker s'annulent, en utilisant le lemme de Lindstrom-Gessel-Viennot. Nous donnons aussi une caractérisation minimale de chaque composante en termes de coordonnées de Plücker.

Keywords: Grassmannian, Deodhar decomposition, networks

1 Introduction

There is a remarkable subset of the real Grassmannian $Gr_{k,n}(\mathbb{R})$ called its *totally non-negative part* $(Gr_{k,n})_{\geq 0}$ [7, 9], which may be defined as the subset of the real Grassmannian where all Plücker coordinates have the same sign. Postnikov showed that $(Gr_{k,n})_{\geq 0}$ has a decomposition into *positroid cells*,

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which are indexed by certain tableaux called Ↄ-diagrams. He also gave explicit parameterizations of each cell. In particular, he showed that from each Ↄ-diagram one can produce a planar *network*, and that one can write down a parameterization of the corresponding cell using the *weight matrix* of that network. This parameterization shows that the cell is isomorphic to $\mathbb{R}^d_{>0}$ for some d. Such a parameterization is convenient, because for example, one may read off formulas for Plücker coordinates from non-intersecting paths in the network, using the Lindstrom-Gessel-Viennot Lemma.

A natural question is whether these network parameterizations for positroid cells can be extended from $(Gr_{k,n})_{\geq 0}$ to the entire real Grassmannian $Gr_{k,n}(\mathbb{R})$. In this paper we give an affirmative answer to this question, by replacing the positroid cell decomposition with the Deodhar decomposition of the Grassmannian $Gr_{k,n}(\mathbb{K})$ (here $\mathbb{K}$ is an arbitrary field).

The components of the Deodhar decomposition are not in general cells, but nevertheless have a simple topology: by [2, 3], each one is isomorphic to $(\mathbb{K}^*)^a \times (\mathbb{K})^b$. The relation of the Deodhar decomposition of $Gr_{k,n}(\mathbb{R})$ to Postnikov's cell decomposition of $(Gr_{k,n})_{\geq 0}$ is as follows: the intersection of a Deodhar component $S_D \cong (\mathbb{R}^*)^a \times (\mathbb{R})^b$ with $(Gr_{k,n})_{\geq 0}$ is precisely one positroid cell isomorphic to $(\mathbb{R}_{>0})^a$ if $b = 0$, and is empty otherwise. In particular, when one intersects the Deodhar decomposition with $(Gr_{k,n})_{\geq 0}$, one obtains the positroid cell decomposition of $(Gr_{k,n})_{\geq 0}$. There is a related *positroid stratification* of the real Grassmannian, and each positroid stratum is a union of Deodhar components.

As for the combinatorics, components of the Deodhar decomposition are indexed by *distinguished subexpressions* [2, 3], or equivalently, by certain tableaux called *Go-diagrams* [6], which generalize Ↄ-diagrams. In this paper we associate a network to each Go-diagram, and write down a parameterization of the corresponding Deodhar component using the weight matrix of that network. Our construction generalizes Postnikov's, but our networks are no longer planar in general.

Our main results can be summed up as follows. See Theorems 3.15 and 4.3 and the constructions preceding them for complete details.

Theorem. *Let $\mathbb{K}$ be an arbitrary field.*

- *Every point in $Gr_{k,n}(\mathbb{K})$ can be realized as the weight matrix of a unique network associated to a Go-diagram, and we can explicitly construct the corresponding network. The networks corresponding to points in the same Deodhar component have the same underlying graph, but different weights.*

- *Every Deodhar component may be characterized by the vanishing and nonvanishing of certain Plücker coordinates. Using this characterization, we can also explicitly construct the network associated to a point given either by a matrix repsresentative or by a list of Plücker coordinates.*

To illustrate the main results, we provide a small example here. More complicated examples may be seen throughout the rest of the paper.

Example 1.1. *Consider the Grassmannian $Gr_{2,4}$. The large Schubert cell in this Grassmannian can be characterized as*

$$\Omega_\lambda = \{A \in Gr_{2,4} \mid \Delta_{1,2}(A) \neq 0\},$$

where Δ_J denotes the Plücker coordinate corresponding to the column set J in a matrix representative of a point in $Gr_{2,4}$. This Schubert cell contains multiple positroid strata, including $S_{\mathcal{I}}$, where $\mathcal{I}$ is the Grassmann necklace $\mathcal{I} = (12, 23, 34, 14)$. This positroid stratum can also be characterized by the nonvanishing

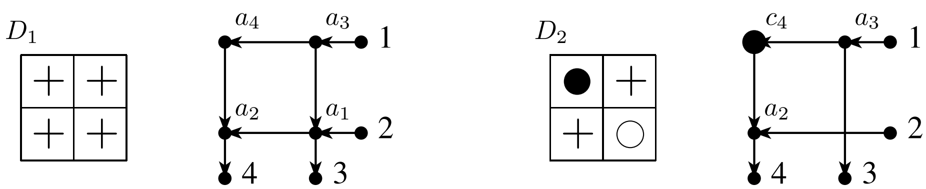

Fig. 1: The diagrams and networks associated to S_{D_1} and S_{D_2} in Example 1.1.

of certain Plücker coordinates:

$$S_{\mathcal{I}} = \{A \in Gr_{2,4} \mid \Delta_{1,2}(A) \neq 0,\ \Delta_{2,3}(A) \neq 0,\ \Delta_{3,4}(A) \neq 0,\ \Delta_{1,4}(A) \neq 0\}.$$

Figure 1 shows two Go-diagrams D_1 and D_2 and their associated networks. Note that the network on the right is not planar. The weight matrices associated to these diagrams are

$$\begin{pmatrix} 1 & 0 & -a_3 & -(a_3a_4 + a_3a_2) \\ 0 & 1 & a_1 & a_1a_2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & -a_3 & -a_3c_4 \\ 0 & 1 & 0 & a_2 \end{pmatrix}.$$

The positroid stratum $S_{\mathcal{I}}$ is the disjoint union of the two corresponding Deodhar components S_{D_1} and S_{D_2}, which can be characterized in terms of vanishing and nonvanishing of minors as:

$$S_{D_1} = \{A \in S_{\mathcal{I}} \mid \Delta_{1,3} \neq 0\} \text{ and } S_{D_2} = \{A \in S_{\mathcal{I}} \mid \Delta_{1,3} = 0\}.$$

Note that if one lets the a_i's range over $\mathbb{K}^*$ and lets c_4 range over $\mathbb{K}$, then we see that $S_{D_1} \cong (\mathbb{K}^*)^4$ and $S_{D_2} \cong (\mathbb{K}^*)^2 \times \mathbb{K}$.

There are several applications of our construction. First, as a special case of our theorem, one may parameterize all $k \times n$ matrices using networks. Second, by applying the Lindstrom-Gessel-Viennot Lemma to a given network, one may write down explicit formulas for Plücker coordinates in terms of collections of non-intersecting paths in the network. Third, building upon work of [6], we obtain (minimal) descriptions of Deodhar components in the Grassmannian, in terms of vanishing and nonvanishing of Plücker coordinates. It follows that each Deodhar component is a union of matroid strata.

Although less well known than the Schubert decomposition and matroid stratification, the Deodhar decomposition is very interesting in its own right. Deodhar's original motivation for introducing his decomposition was the desire to understand Kazhdan-Lusztig polynomials. In the flag variety, one may intersect two opposite Schubert cells, obtaining a Richardson variety, which Deodhar showed is a union of Deodhar components. Each Richardson variety $\mathcal{R}_{v,w}(q)$ may be defined over a finite field $\mathbb{K} = \mathbb{F}_q$, and in this case, the number of points determines the R-polynomials $R_{v,w}(q) = \#(\mathcal{R}_{v,w}(\mathbb{F}_q))$, introduced by Kazhdan and Lusztig [4] to give a recursive formula for the Kazhdan-Lusztig polynomials. Since each Deodhar component is isomorphic to $(\mathbb{F}_q^*)^a \times (\mathbb{F}_q)^b$ for some a and b, if one understands the decomposition of a Richardson variety into Deodhar components, then in principle one may compute the R-polynonomials and hence Kazhdan-Lusztig polynomials.

Another reason for our interest in the Deodhar decomposition is its relation to soliton solutions of the KP equation. It is well-known that from each point A in the real Grassmannian, one may construct a soliton solution $u_A(x,y,t)$ of the KP equation. It was shown in recent work of Kodama and the second author [6] that when the time variable t tends to $-\infty$, the combinatorics of the solution $u_A(x, y, t)$ depends precisely on which Deodhar component A lies in.

The outline of this paper is as follows. In Section 2, we give some background on the Grassmannian and its decompositions, including the Schubert decomposition, the positroid stratification, and the matroid stratification. In Section 3, we present our main construction: we explain how to construct a network from each diagram, then use that network to write down a parameterization of a subset of the Grassmannian that we call a network component. Our main result is that this network component coincides with the corresponding Deodhar component in the Grassmannian. Finally in Section 4 we give a characterization of Deodhar components in terms of the vanishing and nonvanishing of certain Plücker coordinates.

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2 Background on the Grassmannian

The *Grassmannian* $Gr_{k,n}$ is the space of all k-dimensional subspaces of an n-dimensional vector space $\mathbb{K}^n$. In this paper we will usually let $\mathbb{K}$ be an arbitrary field, though we will often think of it as $\mathbb{R}$ or $\mathbb{C}$. An element of $Gr_{k,n}$ can be viewed as a full-rank $k \times n$ matrix modulo left multiplication by nonsingular $k \times k$ matrices. In other words, two $k \times n$ matrices represent the same point in $Gr_{k,n}$ if and only if they can be obtained from each other by row operations. Let $\binom{[n]}{k}$ be the set of all k-element subsets of $[n] := \{1, \dots, n\}$. For $I \in \binom{[n]}{k}$, let $\Delta_I(A)$ be the *Plücker coordinate*, that is, the maximal minor of the $k \times n$ matrix A located in the column set I. The map $A \mapsto (\Delta_I(A))$, where I ranges over $\binom{[n]}{k}$, induces the *Plücker embedding* $Gr_{k,n} \hookrightarrow \mathbb{KP}^{\binom{n}{k}-1}$ into projective space.

We now describe several useful decompositions of the Grassmannian: the Schubert decomposition, the positroid stratification, and the matroid stratification. Note that the matroid stratification refines the positroid stratification, which refines the Schubert decomposition. The main subject of this paper is the *Deodhar* decomposition of the Grassmannian, which refines the positroid stratification, and is refined by the matroid stratification (as we prove in Corollary 4.4).

2.1 The Schubert decomposition of $Gr_{k,n}$

Throughout this paper, we identify partitions with their Young diagrams. Recall that the partitions λ contained in a $k \times (n-k)$ rectangle are in bijection with k-element subset $I \subset [n]$. The boundary of the Young diagram of such a partition λ forms a lattice path from the upper-right corner to the lower-left corner of the rectangle. Let us label the n steps in this path by the numbers $1, \dots, n$, and define $I = I(\lambda)$ as the set of labels on the k vertical steps in the path. Conversely, we let $\lambda(I)$ denote the partition corresponding to the subset I.

Definition 2.1. *For each partition λ contained in a $k \times (n-k)$ rectangle, we define the* Schubert cell

$$\Omega_\lambda = \{A \in Gr_{k,n} \mid I(\lambda) \text{ is the lexicographically minimal subset such that } \Delta_{I(\lambda)}(A) \neq 0\}.$$

As λ ranges over the partitions contained in a $k \times (n-k)$ rectangle, this gives the Schubert decomposition *of the Grassmannian $Gr_{k,n}$, i.e.*

$$Gr_{k,n} = \bigsqcup_{\lambda \subset (n-k)^k} \Omega_\lambda.$$

We now define the *shifted linear order* $<_i$ (for $i \in [n]$) to be the total order on $[n]$ defined by

$$i <_i i+1 <_i i+2 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1.$$

One can then define *cyclically shifted Schubert cells* as follows.

Definition 2.2. *For each partition λ contained in a $k \times (n-k)$ rectangle, and each $i \in [n]$, we define the* cyclically shifted Schubert cell

$\Omega^i_\lambda = \{A \in Gr_{k,n} \mid I(\lambda)$ *is the lexicographically minimal subset with respect to* $<_i$ *such that* $\Delta_{I(\lambda)} \neq 0\}$.

2.2 The positroid stratification of $Gr_{k,n}$

The *positroid stratification* of the Grassmannian $Gr_{k,n}$ is obtained by taking the simultaneous refinement of the n Schubert decompositions with respect to the n shifted linear orders $<_i$. This stratification was first considered by Postnikov [9], who showed that the strata are conveniently described in terms of *Grassmann necklaces*, as well as *decorated permutations* and Ⅎ*-diagrams*. Postnikov coined the terminology *positroid* because the intersection of the positroid stratification of the real Grassmannian with the *totally non-negative part of the Grassmannian* $(Gr_{k,n})_{\geq 0}$ gives a cell decomposition of $(Gr_{k,n})_{\geq 0}$ (whose cells are called *positroid cells*).

Definition 2.3. *[9, Definition 16.1] A* Grassmann necklace *is a sequence $\mathcal{I} = (I_1, \ldots, I_n)$ of subsets $I_r \subset [n]$ such that, for $i \in [n]$, if $i \in I_i$ then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, for some $j \in [n]$; and if $i \notin I_i$ then $I_{i+1} = I_i$. (Here indices i are taken modulo n.) In particular, we have $|I_1| = \cdots = |I_n|$, which is equal to some $k \in [n]$. We then say that $\mathcal{I}$ is a Grassmann necklace of* type (k, n).

Example 2.4. $\mathcal{I} = (1345, 3456, 3456, 4567, 4567, 1467, 1478, 1348)$ *is an example of a Grassmann necklace of type* $(4, 8)$.

Lemma 2.5. *[9, Lemma 16.3] Given $A \in Gr_{k,n}$, let $\mathcal{I}(A) = (I_1, \ldots, I_n)$ be the sequence of subsets in $[n]$ such that, for $i \in [n]$, I_i is the lexicographically minimal subset of $\binom{[n]}{k}$ with respect to the shifted linear order $<_i$ such that $\Delta_{I_i}(A) \neq 0$. Then $\mathcal{I}(A)$ is a Grassmann necklace of type (k, n).*

The *positroid stratification* of $Gr_{k,n}$ is defined as follows.

Definition 2.6. *Let $\mathcal{I} = (I_1, \ldots, I_n)$ be a Grassmann necklace of type (k, n). The* positroid stratum $S_{\mathcal{I}}$ *is defined to be*

$$S_{\mathcal{I}} = \{A \in Gr_{k,n} \mid \mathcal{I}(A) = \mathcal{I}\}.$$

Equivalently, each positroid stratum is an intersection of n cyclically shifted Schubert cells, that is,

$$S_{\mathcal{I}} = \bigcap_{i=1}^{n} \Omega^i_{\lambda(I_i)} .$$

Grassmann necklaces are in bijection with tableaux called Ⅎ*-diagrams*.

Definition 2.7. *[9, Definition 6.1] Fix k, n. A* Ⅎ*-diagram $(\lambda, D)_{k,n}$ of type (k, n) is a partition λ contained in a $k \times (n-k)$ rectangle together with a filling $D : \lambda \to \{0, +\}$ of its boxes which has the* Ⅎ*-property: there is no 0 which has a $+$ above it and a $+$ to its left.*(i) *(Here, "above" means above and in the same column, and "to its left" means to the left and in the same row.)*

In Figure 2 we give an example of a Ⅎ-diagram.

(i) This forbidden pattern is in the shape of a backwards L, and hence is denoted Ⅎ and pronounced "Le."

+	0	+	+
+	0	+	
0	0	+	
+	0		

Fig. 2: A Le-diagram $L = (\lambda, D)_{k,n}$.

2.3 The matroid stratification of $Gr_{k,n}$

Definition 2.8. *A* matroid *of* rank k *on the set* $[n]$ *is a nonempty collection* $\mathcal{M} \subset \binom{[n]}{k}$ *of* k*-element subsets in* $[n]$*, called* bases *of* $\mathcal{M}$*, that satisfies the* exchange axiom*:*
For any $I, J \in \mathcal{M}$ *and* $i \in I$ *there exists* $j \in J$ *such that* $(I \setminus \{i\}) \cup \{j\} \in \mathcal{M}$.

Given an element $A \in Gr_{k,n}$, there is an associated matroid $\mathcal{M}_A$ whose bases are the k-subsets $I \subset [n]$ such that $\Delta_I(A) \neq 0$.

Definition 2.9. *Let* $\mathcal{M} \subset \binom{[n]}{k}$ *be a matroid. The* matroid stratum $S_{\mathcal{M}}$ *is defined to be*

$$S_{\mathcal{M}} = \{A \in Gr_{k,n} \mid \Delta_I(A) \neq 0 \textit{ if and only if } I \in \mathcal{M}\}.$$

This gives a stratification of $Gr_{k,n}$ *called the* matroid stratification*, or* Gelfand-Serganova stratification.

Remark 2.10. *Clearly the matroid stratification refines the positroid stratification, which in turn refines the Schubert decomposition.*

3 The main result: network parameterizations from Go-diagrams

In this section we define certain tableaux called *Go-diagrams*, then explain how to parameterize the Grassmannian using networks associated to Go-diagrams. First we will define more general tableaux called *diagrams*.

3.1 Diagrams and networks

Definition 3.1. *Let* λ *be a partition contained in a* $k \times (n-k)$ *rectangle. A* diagram *in* λ *is an arbitrary filling of the boxes of* λ *with pluses* $+$*, black stones* ●*, and white stones* ○.

To each diagram D we associate a network N_D as follows.

Definition 3.2. *Let* λ *be a partition with* ℓ *boxes contained in a* $k \times (n-k)$ *rectangle, and let* D *be a diagram in* λ*. Label the boxes of* λ *from* 1 *to* ℓ*, starting from the rightmost box in the bottom row, then reading right to left across the bottom row, then right to left across the row above that, etc. The* (weighted) network N_D *associated to* D *is a directed graph obtained as follows:*

- *Associate an* internal vertex *to each* $+$ *and each* ●*;*
- *After labeling the southeast border of the Young diagram with the numbers* $1, 2, \ldots, n$ *(from northeast to southwest), associate a* boundary vertex *to each number;*

- *From each internal vertex, draw an edge right to the nearest $+$-vertex or boundary vertex;*
- *From each internal vertex, draw an edge down to the nearest $+$-vertex or boundary vertex;*
- *Direct all edges left and down. After doing so, k of the boundary vertices become* sources *and the remaining $n-k$ boundary vertices become* sinks*.*
- *If e is a horizontal edge whose left vertex is a $+$-vertex (respectively a ●-vertex) in box b, assign e the weight a_b (respectively c_b). We think of a_b and c_b as indeterminates, but later they will be elements of $\mathbb{K}^*$ and $\mathbb{K}$ respectively.*
- *If e is a vertical edge, assign e the weight 1.*

Note that in general such a directed graph is not planar, as two edges may cross over each other without meeting at a vertex. See Figure 3 for an example of a diagram and its associated network.

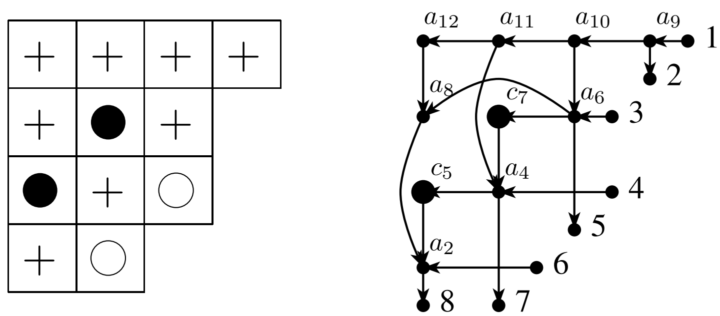

Fig. 3: An example of a diagram and its associated network.

We now explain how to associate a *weight matrix* to such a network.

Definition 3.3. *Let N_D be a network as in Definition 3.2. Let $I = \{i_1 < i_2 < \cdots < i_k\} \subset [n]$ denote the sources. If P is a directed path in the network, let $w(P)$ denote the product of all weights along P. If P is the empty path which starts and ends at the same boundary vertex, we let $w(P) = 1$. If r is a source and s is any boundary vertex, define*

$$W_{rs} = \pm \sum_P w(P),$$

where the sum is over all paths P from r to s. The sign is chosen (uniquely) so that

$$\Delta_{I\setminus\{r\}\cup\{s\}}(W_D) = \sum_P w(P), \text{ where}$$

$$W_D = (W_{rs})$$

is the $k \times (n-k)$ weight matrix*. We make the convention that the rows of W_D are indexed by the sources $i_1, \ldots, i_k$ from top to bottom, and its columns are indexed by $1, 2, \ldots, n$ from left to right.*

Example 3.4. *The weight matrix associated to the network in Figure 3 is*

$$\begin{pmatrix} 1 & a_9 & 0 & 0 & a_9a_{10} & 0 & -a_9a_{10}(a_{11}+c_7) & -a_9a_{10}(a_{11}a_{12}+a_{11}c_5+a_8+c_7c_5) \\ 0 & 0 & 1 & 0 & -a_6 & 0 & a_6c_7 & a_6a_8+a_6c_7c_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & a_4 & -a_4c_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & a_2 \end{pmatrix}$$

3.2 Distinguished expressions

We now review the notion of distinguished subexpressions, as in [2] and [8]. This definition will be essential for defining Go-diagrams. We assume the reader is familiar with the (strong) Bruhat order $<$ on $W = \mathfrak{S}_n$, and the basics of reduced expressions, as in [1].

Let $\mathbf{w} := s_{i_1} \dots s_{i_m}$ be a reduced expression for $w \in W$. A *subexpression* $\mathbf{v}$ of $\mathbf{w}$ is a word obtained from the reduced expression $\mathbf{w}$ by replacing some of the factors with 1. For example, consider a reduced expression in $\mathfrak{S}_4$, say $s_3s_2s_1s_3s_2s_3$. Then $s_3s_2\,1\,s_3s_2\,1$ is a subexpression of $s_3s_2s_1s_3s_2s_3$. Given a subexpression $\mathbf{v}$, we set $v_{(k)}$ to be the product of the leftmost k factors of $\mathbf{v}$, if $k \geq 1$, and $v_{(0)} = 1$.

Definition 3.5. *[8, 2] Given a subexpression* $\mathbf{v}$ *of a reduced expression* $\mathbf{w} = s_{i_1}s_{i_2}\dots s_{i_m}$, *we define*

$$J^{\circ}_{\mathbf{v}} := \{k \in \{1,\dots,m\} \mid v_{(k-1)} < v_{(k)}\},$$
$$J^{+}_{\mathbf{v}} := \{k \in \{1,\dots,m\} \mid v_{(k-1)} = v_{(k)}\},$$
$$J^{\bullet}_{\mathbf{v}} := \{k \in \{1,\dots,m\} \mid v_{(k-1)} > v_{(k)}\}.$$

The expression $\mathbf{v}$ *is called non-decreasing if* $v_{(j-1)} \leq v_{(j)}$ *for all* $j = 1,\dots,m$*, e.g.* $J^{\bullet}_{\mathbf{v}} = \emptyset$.

Definition 3.6 (Distinguished subexpressions)**.** *[2, Definition 2.3] A subexpression* $\mathbf{v}$ *of* $\mathbf{w}$ *is called distinguished if we have*

$$v_{(j)} \leq v_{(j-1)}\; s_{i_j} \qquad \textit{for all } \; j \in \{1,\dots,m\}. \tag{1}$$

In other words, if right multiplication by s_{i_j} *decreases the length of* $v_{(j-1)}$*, then in a distinguished subexpression we must have* $v_{(j)} = v_{(j-1)}s_{i_j}$.

We write $\mathbf{v} \prec \mathbf{w}$ *if* $\mathbf{v}$ *is a distinguished subexpression of* $\mathbf{w}$.

Definition 3.7 (Positive distinguished subexpressions)**.** *We call a subexpression* $\mathbf{v}$ *of* $\mathbf{w}$ *a positive distinguished subexpression (or a PDS for short) if*

$$v_{(j-1)} < v_{(j-1)}s_{i_j} \qquad \textit{for all } \; j \in \{1,\dots,m\}. \tag{2}$$

In other words, it is distinguished and non-decreasing.

Lemma 3.8. *[8] Given* $v \leq w$ *and a reduced expression* $\mathbf{w}$ *for* w*, there is a unique PDS* $\mathbf{v}_+$ *for* v *in* $\mathbf{w}$.

3.3 Go-diagrams

In this section we explain how to index distinguished subexpressions by certain tableaux called *Go-diagrams*, which were introduced in [6]. Go-diagrams are fillings of Young diagrams by pluses $+$, *black stones* ●, and *white stones* ○.[ii]

[ii] In KW2, we used a slightly different convention and used blank boxes in place of $+$'s.

s_5	s_4	s_3	s_2	s_1
s_6	s_5	s_4	s_3	s_2
s_7	s_6	s_5	s_4	s_3

15	14	13	12	11
10	9	8	7	6
5	4	3	2	1

15	12	9	6	3
14	11	8	5	2
13	10	7	4	1

Fig. 4: The labeling of a the boxes of a partition by simple generators s_i, and two reading orders.

Fix k and n. Let $W_k = \langle s_1, s_2, \ldots, \hat{s}_{n-k}, \ldots, s_{n-1}\rangle$ be a parabolic subgroup of $W = \mathfrak{S}_n$. Let W^k denote the set of minimal-length coset representatives of W/W_k. Recall that a *descent* of a permutation π is a position j such that $\pi(j) > \pi(j+1)$. Then W^k is the subset of permutations of $\mathfrak{S}_n$ which have at most one descent; and that descent must be in position $n-k$.

It follows from [11] and [10] that elements w of W^k can be identified with partitions λ_w contained in a $k \times (n-k)$ rectangle. More specifically, let Q^k be the poset whose elements are the boxes of a $k \times (n-k)$ rectangle; if b_1 and b_2 are two adjacent boxes such that b_2 is immediately to the left or immediately above b_1, we have a cover relation $b_1 \lessdot b_2$ in Q^k. The partial order on Q^k is the transitive closure of $\lessdot$. Now label the boxes of the rectangle with simple generators s_i as in the figure below. If b is a box of the rectangle, then let s_b denote its label by a simple generator. Let $w_0^k \in W^k$ denote the longest element in W^k. Then the set of reduced expressions of w_0^k can be obtained by choosing a linear extension of Q^k and writing down the corresponding word in the s_i's. We call such a linear extension a *reading order*; two linear extensions are shown in the figure below. Additionally, given a partition λ contained in the $k \times (n-k)$ rectangle (chosen so that the upper-left corner of its Young diagram is aligned with the upper-left corner of the rectangle), and a linear extension of the sub-poset of Q^k comprised of the boxes of λ, the corresponding word in s_i's is a reduced expression of a minimal length coset representatives $w \in W^k$. The element $w \in W^k$ depends only on the partition, not the linear extension, and all reduced expressions of w can be obtained by varying the linear extension. Finally, this correspondence is a bijection between partitions λ_w contained in the $k \times (n-k)$ rectangle and elements $w \in W^k$.

Definition 3.9. *[6, Section 4] Fix k and n. Let $w \in W^k$, let $\mathbf{w}$ be a reduced expression for w, and let $\mathbf{v}$ be a distinguished subexpression of $\mathbf{w}$. Then w and $\mathbf{w}$ determine a partition λ_w contained in a $k \times (n-k)$ rectangle together with a reading order of its boxes. The* Go-diagram *associated to $\mathbf{v}$ and $\mathbf{w}$ is a filling of λ_w with pluses and black and white stones, such that: for each $k \in J_{\mathbf{v}}^{\circ}$ we place a white stone in the corresponding box; for each $k \in J_{\mathbf{v}}^{\bullet}$ we place a black stone in the corresponding box of λ_w; and for each $k \in J_{\mathbf{v}}^{+}$ we place a plus in the corresponding box of λ_w.*

Remark 3.10. *By [6, Section 4], the Go-diagram associated to $\mathbf{v}$ and $\mathbf{w}$ does not depend on $\mathbf{w}$, only on w. Moreover, whether or not such a filling of a partition λ_w is a Go-diagram does not depend on the choice of reading order of the boxes of λ_w.*

Definition 3.11. *We define the* standard reading order *of the boxes of a partition to be the reading order which starts at the rightmost box in the bottom row, then reads right to left across the bottom row, then right to left across the row above that, then right to left across the row above that, etc. This reading order is illustrated at the right of the figure below.*

By default, we will use the standard reading order in this paper.

Example 3.12. *Let $k = 3$ and $n = 7$, and let $\lambda = (4,3,1)$. The standard reading order is shown at the right of the figure below.*

s_4	s_3	s_2	s_1
s_5	s_4	s_3	
s_6			

8	7	6	5
4	3	2	
1			

Then the following diagrams are Go-diagrams of shape λ.

○	○	○	○
○	○	○	
○			

+	○	○	+
+	○	+	
○			

●	+	+	○
+	○	○	
+			

They correspond to the expressions $s_6s_3s_4s_5s_1s_2s_3s_4$, $s_61s_411s_2s_31$, and $1s_3s_41s_111s_4$. The first and second are positive distinguished subexpressions (PDS's), and the third one is a distinguished subexpression (but not a PDS).

Remark 3.13. *The Go-diagrams associated to PDS's are in bijection with ⅃-diagrams, see [6, Section 4]. Note that the Go-diagram associated to a PDS contains only pluses and white stones. This is precisely a ⅃-diagram.*

3.4 The main result

To state the main result, we now consider Go-diagrams (not arbitrary diagrams), the corresponding networks (*Go-networks*), and the corresponding weight matrices.

Definition 3.14. *Let D be a Go-diagram contained in a $k \times (n-k)$ rectangle. We define a subset $\mathcal{R}_D$ of the Grassmannian $Gr_{k,n}$ by letting each variable a_i of the weight matrix (Definition 3.3) range over all nonzero elements $\mathbb{K}^*$, and letting each variable c_i of the weight matrix range over all elements $\mathbb{K}$. We call $\mathcal{R}_D$ the* network component *associated to D.*

We will not define the Deodhar decomposition of the Grassmannian, but refer to [2, 3, 8] for details.

Theorem 3.15. *Let D be a Go-diagram contained in a $k \times (n-k)$ rectangle. Suppose that D has t pluses and u black stones. Then $\mathcal{R}_D$ is isomorphic to the corresponding Deodhar component, and in particular is isomorphic to $(\mathbb{K}^*)^t \times \mathbb{K}^u$. Furthermore, $Gr_{k,n}$ is the disjoint union of the network components $\mathcal{R}_D$, as D ranges over all Go-diagrams contained in a $k \times (n-k)$ rectangle. In other words, each point in the Grassmannian $Gr_{k,n}$ can be represented uniquely by a weighted network associated to a Go-diagram.*

Corollary 3.16. *Every matrix can be represented by a unique weighted network associated to a Go-diagram.*

4 A characterization of Deodhar components by minors

In this section we characterize Deodhar components in the Grassmannian by a list of vanishing and nonvanishing Plücker coordinates.

Definition 4.1. *[6, Definition 5.4] Let $W = \mathfrak{S}_n$, let $\mathbf{w} = s_{i_1} \dots s_{i_m}$ be a reduced expression for $w \in W^k$ and choose $\mathbf{v} \prec \mathbf{w}$. This determines a Go-diagram D of shape $\lambda = \lambda_w$. Let $I = I(\lambda)$. It is not hard to check that $I = w\{n, n-1, \dots, n-k+1\}$.*

Let b be any box of D. Note that the set of all boxes of D which are weakly southeast of b forms a Young diagram λ_b^{in}; also the complement of λ_b^{in} in λ is a Young diagram which we call λ_b^{out} (see Example 4.2 below). By looking at the restriction of $\mathbf{w}$ to the positions corresponding to boxes of λ_b^{in}, we obtained a reduced expression $\mathbf{w}_b^{\text{in}}$ for some permutation w_b^{in}, together with a distinguished subexpression $\mathbf{v}_b^{\text{in}}$ for some permutation v_b^{in}. Similarly, by using the positions corresponding to boxes of λ_b^{out}, we obtained $\mathbf{w}_b^{\text{out}}$, w_b^{out}, $\mathbf{v}_b^{\text{out}}$, and v_b^{out}. When the box b is understood, we will often omit the subscript b.

If b contains a $+$, define $I_b = v^{\text{in}}(w^{\text{in}})^{-1} I \in \binom{[n]}{k}$. If b contains a white or black stone, define $I_b = v^{\text{in}} s_b (w^{\text{in}})^{-1} I \in \binom{[n]}{k}$.

Example 4.2. *Let $W = \mathfrak{S}_7$ and $\mathbf{w} = s_4 s_5 s_2 s_3 s_4 s_6 s_5 s_1 s_2 s_3 s_4$ be a reduced expression for $w \in W^3$. Let $\mathbf{v} = s_4 s_5 1 1 s_4 1 s_5 s_1 1 1 s_4$ be a distinguished subexpression. So $w = (3,5,6,7,1,2,4)$ and $v = (2,1,3,4,6,5,7)$. We can represent this data by the poset λ_w and the corresponding Go-diagram:*

s_4	s_3	s_2	s_1
s_5	s_4	s_3	s_2
s_6	s_5	s_4	

●	$+$	$+$	○
●	○	$+$	$+$
$+$	○	○	

Let b be the box of the Young diagram which is in the second row and the second column (counting from left to right). Then the diagram below shows: the boxes of λ^{in} and λ^{out}; a reading order which puts the boxes of λ^{out} after those of λ^{in}; and the corresponding labeled Go-diagram. Using this reading order, $\mathbf{w}^{\text{in}} = s_4 s_5 s_2 s_3 s_4$, $\mathbf{w}^{\text{out}} = s_6 s_5 s_1 s_2 s_3 s_4$, $\mathbf{v}^{\text{in}} = s_4 s_5 1 1 s_4$, and $\mathbf{v}^{\text{out}} = 1 s_5 s_1 1 1 s_4$.

out	*out*	*out*	*out*
out	*in*	*in*	*in*
out	*in*	*in*	

11	10	9	8
7	5	4	3
6	2	1	

Theorem 4.3. *Let D be a Go-diagram of shape λ contained in a $k \times (n-k)$ rectangle. Let $A \in Gr_{k,n}$. Then A lies in the Deodhar component S_D if and only if the following conditions are satisfied:*

1. *$\Delta_{I_b}(A) = 0$ for all boxes in D containing a white stone.*
2. *$\Delta_{I_b}(A) \neq 0$ for all boxes in D containing a $+$.*
3. *$\Delta_{I(\lambda)}(A) \neq 0$.*
4. *$\Delta_J(A) = 0$ for all k-subsets J which are lexicographically smaller than $I(\lambda)$.*

Corollary 4.4. *The Deodhar decomposition of the Grassmannian is a coarsening of the matroid stratification: in other words, each Deodhar component is a union of matroid strata.*

Remark 4.5. *Theorem 4.3 implicitly gives an algorithm for determining the Deodhar component and corresponding network of a point of the Grassmannian, given by a matrix representative or by a list of its Plücker coordinates. The steps are as follows.*

1. *Find the lexicographically minimal nonzero Plücker coordinate Δ_I. Then the Go-diagram has shape $\lambda(I)$. Fix a reading order for this shape.*
2. *We determine how to fill each box, working in the reading order, as follows. First check whether the box b is forced to contain a black stone. If not, b must contain a white stone if $\Delta_{I(b)} = 0$, and b must contain a plus if $\Delta_{I(b)} \neq 0$. This process will completely determine the Go-diagram.*
3. *Given the Go-diagram, we know the underlying graph of the network. To determine the weights on horizontal edges, work in the reading order again. The Plücker coordinate $\Delta_{I(b)}$ will only use edge weights a_b (when b contains a $+$) or c_b (when b contains a black stone) and weights $a_{b'}$ and $c_{b'}$ corresponding to boxes b' which are earlier than b in the reading order. Thus, we may use the Lindström-Gessel-Viennot Lemma recursively to determine each weight a_b or c_b.*

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